ON THE REFLEXIVITY OF ALGEBRAS AND LINEAR SPACES OF OPERATORS

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This paper is dedicated to our good friend George Piranian on the occasion of his retirement

Let $\mathcal{H}$ be a complex Hilbert space (of arbitrary dimension), and let $\mathcal{L}(\mathcal{H})$ denote the algebra of bounded linear operators on $\mathcal{H}$. Among the useful topologies on $\mathcal{L}(\mathcal{H})$ are the weak* topology (sometimes called the ultraweak operator topology) and the weak operator topology. If $\mathfrak{M}$ is any linear manifold in $\mathcal{L}(\mathcal{H})$, then $\mathfrak{M}$ inherits these two topologies. A linear functional on $\mathfrak{M}$ that is continuous in the weak* [resp., weak operator] topology will be called a *weak** [resp., *weakly*] *continuous functional.* If $\mathfrak{M}$ is closed in the weak operator topology, we will call $\mathfrak{M}$ a *weakly closed subspace.* One knows from the Hahn–Banach theorem that every weak* [resp., weakly] continuous functional on $\mathfrak{M}$ has the form $[\phi]=\phi\mid\mathfrak{M}$ where ϕ is a weak* [resp., weakly] continuous functional on $\mathcal{L}(\mathcal{H})$. In this paper we will be concerned mostly with weakly continuous functionals, and therefore we remind the reader that every such functional on $\mathcal{L}(\mathcal{H})$ is a finite sum of functionals of the form $x\otimes y$ with $x, y\in\mathcal{H}$, where

$$(x\otimes y)(A)=\langle Ax, y\rangle,\quad A\in\mathcal{L}(\mathcal{H}).$$

(Weak* continuous functionals on $\mathcal{L}(\mathcal{H})$ have the form $\sum_{n=1}^{\infty} x_n\otimes y_n$, but this fact will not be needed herein.)

Let $\mathfrak{M}$ be a linear manifold in $\mathcal{L}(\mathcal{H})$. As in [11], we will use the notation $\operatorname{Ref}(\mathfrak{M})$ for the set of all operators X in $\mathcal{L}(\mathcal{H})$ such that $Xy\in(\mathfrak{M}y)^-$ for every y in $\mathcal{H}$. The subspace $(\mathfrak{M}y)^-$ will be referred to (somewhat improperly) as the *cyclic space* for $\mathfrak{M}$ *generated* by y. The following concept of reflexivity was introduced by Loginov and Sulman in [4].

DEFINITION 1. A linear manifold $\mathfrak{M}\subset\mathcal{L}(\mathcal{H})$ is said to be *reflexive* if $\operatorname{Ref}(\mathfrak{M})=\mathfrak{M}$.

It is easy to verify that $\operatorname{Ref}(\mathfrak{M})=\operatorname{Alg}\operatorname{Lat}(\mathfrak{M})$ if $\mathfrak{M}$ is an algebra containing $1_{\mathcal{H}}$, and for such algebras the above definition gives the usual one of reflexive algebras. Note, however, that $\mathfrak{M}=\{0\}$ is reflexive as a subspace but not as an algebra.

In this paper we study the relationship between the reflexivity of a linear manifold $\mathfrak{M}$ in $\mathcal{L}(\mathcal{H})$ and the structure of the weakly continuous functionals on $\mathfrak{M}$. The following definition is pertinent to the kind of structure we have in mind.

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DEFINITION 2. Let $\mathfrak{M}$ be a linear manifold in $\mathfrak{L}(\mathfrak{H})$ and let p and q be cardinal numbers satisfying $1 \leq p, q \leq \aleph_0$. We say that $\mathfrak{M}$ has property $(\mathbf{A}_{p,q})$ [resp., $(\mathbf{B}_{p,q})$] provided that for every family $\{\phi_{ij}: 0 \leq i < p,\ 0 \leq j < q\}$ of weak* [resp., weakly] continuous functionals on $\mathfrak{M}$, there exist sequences $\{x_i: 0 \leq i < p\}$ and $\{y_j: 0 \leq j < q\}$ of vectors in $\mathfrak{H}$ such that

$$\phi_{ij} = [x_i \otimes y_j], \quad 0 \leq i < p, \ 0 \leq j < q.$$

Furthermore, we say that $\mathfrak{M}$ has property $(\mathbf{A}_{p,q}^{\sim})$ [resp., $(\mathbf{B}_{p,q}^{\sim})$] if for every $\epsilon > 0$ there exists $\delta > 0$ such that, given any family $\{\phi_{ij}: 0 \leq i < p,\ 0 \leq j < q\}$ of weak* [resp. weakly] continuous functionals on $\mathfrak{M}$ and sequences $\{x_i': 0 \leq i < p\}$ and $\{y_j': 0 \leq j < q\}$ in $\mathfrak{H}$ satisfying the inequalities

$$\|\phi_{ij} - [x_i' \otimes y_j']\| < \delta, \quad 0 \leq i < p, \ 0 \leq j < q,$$

there exist sequences $\{x_i: 0 \leq i < p\}$ and $\{y_j: 0 \leq j < q\}$ in $\mathfrak{H}$ such that

$$\phi_{ij} = [x_i \otimes y_j], \quad 0 \leq i < p, \ 0 \leq j < q,$$

and

$$\|x_i' - x_i\| < \epsilon, \quad \|y_i' - y_j\| < \epsilon, \quad 0 \leq i < p, \ 0 \leq j < q.$$

We begin by making some remarks concerning Definition 2. Since every weakly continuous functional on a linear manifold $\mathfrak{M} \subset L(\mathfrak{H})$ is also weak* continuous, it is obvious that if $\mathfrak{M}$ has property $(\mathbf{A}_{p,q})$, then it has property $(\mathbf{B}_{p,q})$. Quite interestingly, as was pointed out to us by C. Apostol, $\mathfrak{M}$ has property $(\mathbf{A}_{p,q}^{\sim})$ if and only if it has property $(\mathbf{B}_{p,q}^{\sim})$. We leave the proof of this fact to the interested reader, since it will not be needed herein.

In this paper we will be concerned only with properties $(\mathbf{B}_{p,q})$ and $(\mathbf{B}_{p,q}^{\sim})$ for finite values of p and q. It is worthwhile to note that there are few linear manifolds that enjoy property $(\mathbf{B}_{p,q}^{\sim})$ if p or q equals $\aleph_0$. We also point out that for $p = q = n < \aleph_0$, property $(\mathbf{A}_{p,q})$ [resp., $(\mathbf{B}_{p,q})$, $(\mathbf{A}_{p,q}^{\sim})$, $(\mathbf{B}_{p,q}^{\sim})$] is exactly property $(\mathbf{A}_n)$ [resp., $(\mathbf{B}_n)$, $(\mathbf{A}_n^{\sim})$, $(\mathbf{B}_n^{\sim})$] as defined in [5] and [2].

The main result of this paper (Theorem 15) shows that any weakly closed subspace $\mathfrak{M} \subset \mathfrak{L}(\mathfrak{H})$ which has property $(\mathbf{B}_{2,3}^{\sim})$ is reflexive. This improves the result from [2] to the effect that any weakly closed linear manifold $\mathfrak{M} \subset \mathfrak{L}(\mathfrak{H})$ which has property $(\mathbf{B}_n^{\sim})$ for every positive integer n is reflexive. An earlier result along these lines is [4, Theorem 1], which can be reformulated as follows: Suppose that $\mathfrak{M} \subset \mathfrak{L}(\mathfrak{H})$ is a weak* closed subalgebra of $\mathfrak{L}(\mathfrak{H})$ which is isometrically isomorphic and weak*-homeomorphic to the algebra $H^\infty(\mathbf{D})$ (notation: $\mathfrak{M} \simeq H^\infty(\mathbf{D})$) of bounded analytic functions on the open unit disc $\mathbf{D}$. If $\mathfrak{M}$ has property $(\mathbf{A}_n^{\sim})$ for every positive integer n, then $\mathfrak{M}$ is reflexive. This result was extended in [13] to algebras $\mathfrak{M} \simeq H^\infty(G)$ where G is a "nice" multiply connected domain. A variation of this result appears in [3], where it was shown that any algebra $\mathfrak{M} \simeq H^\infty(\mathbf{D})$ which has property $(\mathbf{A}_{\aleph_0})$ is reflexive. We also mention a related result of Olin and Thomson [12] which says that the weak*-closed algebra generated by a subnormal operator is reflexive. The proof given in [12] used property

$(\mathbf{A}_{1,\aleph_0})$. Finally we mention that by a result in [11], every reflexive linear manifold $\mathfrak{M} \subset \mathcal{L}(\mathcal{H})$ that has property $(\mathbf{B}_{1,1})$ is, in fact, *hereditarily reflexive,* in the sense that every weakly closed subspace of $\mathfrak{M}$ is reflexive.

We will use the notation $\mathfrak{M}^{(n)} = \{A^{(n)} : A \in \mathfrak{M}\}$, where $A^{(n)} = A \oplus A \oplus \cdots \oplus A \in \mathcal{L}(\mathcal{H}^{(n)})$ is the direct sum of n copies of A.

LEMMA 3. *Assume that* $\mathfrak{M}$ *is a linear manifold in* $\mathcal{L}(\mathcal{H})$ *and* k *is a positive integer. Then* $\operatorname{Ref}(\mathfrak{M}^{(k)})$ *consists of all operators of the form* $X^{(k)}$, *where* $X \in \mathcal{L}(\mathcal{H})$ *has the property that* $\sum_{j=1}^{k} \langle Xx_j, y_j\rangle = 0$ *whenever vectors* $\{x_j, y_j \in \mathcal{H} : 1 \le j \le k\}$ *satisfy the relation* $\sum_{j=1}^{k} [x_j \otimes y_j] = 0$.

Proof. Let $Z \in \operatorname{Ref}(\mathfrak{M}^{(k)})$. It is immediately seen, using vectors of the form $0 \oplus \cdots \oplus 0 \oplus x \oplus 0 \oplus \cdots \oplus 0$, $x \in \mathcal{H}$, that $Z = X_1 \oplus X_2 \oplus \cdots \oplus X_k$, $X_1, X_2, \ldots, X_k \in \mathcal{L}(\mathcal{H})$. Considering next cyclic subspaces generated by $x \oplus x \oplus \cdots \oplus x$, $x \in \mathcal{H}$, we conclude that $X_1 = X_2 = \cdots = X_k = X$. Now, the relation $\sum_{j=1}^{k} [x_j \otimes y_j] = 0$ means that

$$\langle A^{(k)}(x_1 \oplus x_2 \oplus \cdots \oplus x_k), y_1 \oplus y_2 \oplus \cdots \oplus y_k\rangle = \sum_{j=1}^{k} \langle Ax_j, y_j\rangle = 0, \quad A \in \mathfrak{M},$$

or, equivalently, that $y_1 \oplus y_2 \oplus \cdots \oplus y_n$ is orthogonal to $(\mathfrak{M}^{(k)}(x_1 \oplus x_2 \oplus \cdots \oplus x_k))^{-}$. The relation $\sum_{j=1}^{k} \langle Xx_j, y_j\rangle = 0$ follows then from the fact that

$$X^{(k)}(x_1 \oplus x_2 \oplus \cdots \oplus x_k) \in (\mathfrak{M}^{(k)}(x_1 \oplus x_2 \oplus \cdots \oplus x_k))^{-}. \qquad \square$$

The considerations above can be reversed to show that $X^{(k)} \in \operatorname{Ref}(\mathfrak{M}^{(k)})$ if the implication $\sum_{j=1}^{k} [x_j \otimes y_j] = 0 \Rightarrow \sum_{j=1}^{k} \langle Xx_j, y_j\rangle = 0$ holds.

We have the following obvious consequence of Lemma 3 and the Hahn–Banach theorem.

COROLLARY 4. *Assume that* $\mathfrak{M}$ *is closed in the weak operator topology of* $\mathcal{L}(\mathcal{H})$, *and* $X \in \mathcal{L}(\mathcal{H})$. *Then* $X \in \mathfrak{M}$ *if and only if* $X^{(k)} \in \operatorname{Ref}(\mathfrak{M}^{(k)})$ *for all integers* $k \ge 1$.

We begin now with our reflexivity results. We include for the sake of completeness a proof of the following result of Larson [3].

THEOREM 5. *Assume that* $\mathfrak{M}$ *is a weakly closed subspace of* $\mathcal{L}(\mathcal{H})$. *If* $\mathfrak{M}$ *has property* $(\mathbf{B}_{1,1})$, *then* $\mathfrak{M}^{(3)}$ *is reflexive.*

Proof. By Lemma 3 we have to show that an operator X, for which the implication $\sum_{j=1}^{3} [x_i \otimes y_i] = 0 \Rightarrow \sum_{j=1}^{3} \langle Xx_j, y_j\rangle = 0$ holds, necessarily belongs to $\mathfrak{M}$. Let X be one such operator. By Corollary 4, it suffices to show that the implication

$$\sum_{j=1}^{k} [x_j \otimes y_j] = 0 \Rightarrow \sum_{j=1}^{k} \langle Xx_j, y_j\rangle = 0 \tag{6}$$

holds for all integers k. We proceed by induction. We know that (6) is satisfied for $k \le 3$. Assume that (6) has been proved for all $k < n$, $n > 3$, and let $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in \mathcal{H}$ satisfy the relation $\sum_{j=1}^{n} [x_j \otimes y_j] = 0$. Since $\mathfrak{M}$

has $(\mathbf{B}_{1,1})$, there exist vectors $u, v \in \mathcal{H}$ such that $[u \otimes v] = \sum_{j=3}^{n} [x_j \otimes y_j]$ or, equivalently, $[-u \otimes v] + \sum_{j=3}^{n} [x_j \otimes y_j] = 0$. By (6) with $k = n-1$ we deduce that $\langle -Xx, v\rangle + \sum_{j=3}^{n} \langle Xx_j, y_j\rangle = 0$ or, equivalently,

$$\langle Xu, v\rangle = \sum_{j=3}^{n} \langle Xx_j, y_j\rangle. \tag{7}$$

Now we also have $[x_1 \otimes y_1] + [x_2 \otimes y_2] + [u \otimes v] = 0$, so that

$$\langle Xx_1, y_1\rangle + \langle Xx_2, y_2\rangle + \langle Xu, v\rangle = 0 \tag{8}$$

by (6) with $k = 3$. Combining (7) and (8) we get $\sum_{j=1}^{n} \langle Xx_j, y_j\rangle = 0$, and (6) is proved by induction. The theorem follows. □

THEOREM 9. *Assume that $\mathfrak{M}$ is a weakly closed subspace of $\mathcal{L}(\mathcal{H})$. If $\mathfrak{M}$ has property $(\mathbf{B}_{1,2})$, then $\mathfrak{M}^{(2)}$ is reflexive.*

Proof. By Lemma 3, we have to show that every operator X for which (6) holds for $k = 2$ belongs to $\mathfrak{M}$. Assume therefore that X satisfies (6) for $k = 2$. Now $(\mathbf{B}_{1,2})$ implies $(\mathbf{B}_{1,1})$, so by Theorem 5 it will suffice to show that $X^{(3)} \in \operatorname{Ref}(\mathfrak{M}^{(3)})$ or, equivalently, that (6) is satisfied for $k = 3$. Let $x_1, x_2, x_3, y_1, y_2, y_3 \in \mathcal{H}$ satisfy the relation $\sum_{j=1}^{3} [x_j \otimes y_j] = 0$. Since $\mathfrak{M}$ has $(\mathbf{B}_{1,2})$, we can find vectors $u, v_1, v_2 \in \mathcal{H}$ such that $[u \otimes v_1] = [x_1 \otimes y_1]$ and $[u \otimes v_2] = [x_2 \otimes y_2]$. These relations imply, via (6) for $k = 2$, that

$$\langle Xu, v_1\rangle = \langle Xx_1, y_1\rangle \quad \text{and} \quad \langle Xu, v_2\rangle = \langle Xx_2, y_2\rangle. \tag{10}$$

We have

$$[u \otimes (v_1 + v_2)] + [x_3 \otimes y_3] = [u \otimes v_1] + [u \otimes v_2] + [x_3 \otimes y_3] = \sum_{j=1}^{3} [x_j \otimes y_j] = 0$$

and, again by (6) with $k = 2$, we deduce

$$\langle Xu, v_1 + v_2\rangle + \langle Xx_3, y_3\rangle = 0. \tag{11}$$

It is easy now to combine (10) and (11) to get $\sum_{j=1}^{3} \langle Xx_j, y_j\rangle = 0$, thus proving (6) for $k = 3$. The theorem is proved. □

The next result may be regarded as an invariant subspace theorem if $\mathfrak{M}$ is an algebra containing $1_{\mathcal{H}}$.

PROPOSITION 12. *Assume that $\mathfrak{M}$ is a linear manifold in $\mathcal{L}(\mathcal{H})$. If $\mathfrak{M}$ has property $(\mathbf{B}_{2,2})$, then there exists $x \in \mathcal{H}$, $x \neq 0$, such that $(\mathfrak{M}x)^- \neq \mathcal{H}$.*

Proof. The proposition is trivial if $\mathfrak{M} = \{0\}$, so we will assume $\mathfrak{M} \neq \{0\}$. Then there exists a nonzero weakly continuous functional ϕ on $\mathfrak{M}$. Choose by $(\mathbf{B}_{2,2})$ vectors $x_1, x_2, y_1, y_2 \in \mathcal{H}$ satisfying the equations $[x_i \otimes y_j] = \delta_{ij}\phi$, $1 \leq i, j \leq 2$. It is clear that $x_1 \neq 0$ (because $[x_1 \otimes y_1] \neq 0$), $y_2 \neq 0$ (because $[x_2 \otimes y_2] \neq 0$), and $y_2 \perp (\mathfrak{M}x_1)^-$ (because $[x_1 \otimes y_2] = 0$). □

It is interesting to note that the subspace $(\mathfrak{M}x_1)^-$ constructed above is nonzero; this follows from the fact that $[x_1 \otimes y_1] \neq \{0\}$. For our last result we need two additional observations.

LEMMA 13. *A linear manifold $\mathfrak{M} \subset \mathcal{L}(\mathcal{H})$ with property $(\mathbf{B}_{p,q}^{\sim})$ also has property $(\mathbf{B}_{p,q})$.*

Proof. Let δ be provided by the definition of $(\mathbf{B}_{p,q}^{\sim})$ for $\epsilon = 1$, and let

$$\{\phi_{ij} : 1 \le i \le p,\ 1 \le j \le q\}$$

be a system of weakly continuous functionals on $\mathfrak{M}$. Set

$$M = \max\{\|\phi_{ij}\| : 1 \le i \le p,\ 1 \le j \le q\},$$

and $\psi_{ij} = (\delta/2M)\phi_{ij}$, $x_i = 0$, $y_j = 0$ for $1 \le i \le p$, $1 \le j \le q$. Then we clearly have $\|\psi_{ij} - [x_i \otimes y_j]\| < \delta$, $1 \le i \le p$, $1 \le j \le q$, and by Definition 2 we can choose $\{x_i', y_j' \in \mathcal{H} : 1 \le i \le p,\ 1 \le j \le q\}$ satisfying the equations $\psi_{ij} = [x_i' \otimes y_j']$, $1 \le i \le p$, $1 \le j \le q$. We clearly have then $\phi_{ij} = [\xi_i \otimes \eta_j]$, where $\xi_i = (2M/\delta)^{1/2} x_i'$, $\eta_j = (2M/\delta)^{1/2} y_j'$, for $1 \le i \le p$, $1 \le j \le q$, and property $(\mathbf{B}_{p,q})$ follows. □

A similar argument goes into the proof of the following result.

LEMMA 14. *Assume that $\mathfrak{M}$ is a linear manifold in $\mathcal{L}(\mathcal{H})$ and $\mathfrak{M}$ has property $(\mathbf{B}_{2,3}^{\sim})$. If $x_1, x_2, y_1, y_2 \in \mathcal{H}$, and ϵ is a given positive number, there exist vectors $\xi_1, \xi_2, \eta_1, \eta_2, \eta_3 \in \mathcal{H}$ such that*

(i) $[\xi_i \otimes \eta_j] = [x_i \otimes y_j]$, $1 \le i, j \le 2$, $[\xi_1 \otimes \eta_3] = [x_2 \otimes y_2]$, $[\xi_2 \otimes \eta_3] = 0$; *and*
(ii) $\|x_i - \xi_i\| < \epsilon$, $\|y_j - \eta_j\| < \epsilon$, $1 \le i, j \le 2$.

Proof. Let $\delta = \delta(\epsilon)$ be provided by Definition 2, and choose a number $\delta_1 > 0$ so small that $\delta_1 \|[x_2 \otimes y_2]\| < \delta$. If we set now $\phi_{ij} = [x_i \otimes y_j]$, $1 \le i, j \le 2$, $\phi_{1,3} = \delta_1 [x_2 \otimes y_2]$, $\phi_{2,3} = 0$, and $y_3 = 0$, the inequalities

$$\|\phi_{ij} - [x_i \otimes y_j]\| < \delta, \quad 1 \le i \le 2,\ 1 \le j \le 3,$$

are satisfied. Thus property $(\mathbf{B}_{2,3}^{\sim})$ provides vectors $x_1', x_2', y_1', y_2', y_3' \in \mathcal{H}$ such that $[\phi_{ij}] = [x_i' \otimes y_j']$, $\|x_i - x_i'\| < \epsilon$, and $\|y_j - y_j'\| < \epsilon$ for $1 \le i \le 2$, $1 \le j \le 3$. To complete the proof of the lemma it suffices now to define $\xi_i = x_i'$, $\eta_j = y_j'$, $1 \le i, j \le 2$, and $\eta_3 = (1/\delta_1) y_3'$. □

Observe that the only estimate for η_3 that we get from the above proof is $\|\eta_3\| < \epsilon/\delta_1$ and, with a careful choice of δ_1, this can be upgraded to $\|\eta_3\| \le (\epsilon/\delta)\|[x_2 \otimes y_2]\|$. It is also easy to see that the dependence of δ on ϵ is quadratic, i.e., $\delta \le c\epsilon^2$, $c > 0$, so $\|\eta_3\| \le (c/\epsilon)\|[x_2 \otimes y_2]\|$. (We have not been able to make any use of this estimate.)

We are now ready to prove our main result.

THEOREM 15. *Assume that $\mathfrak{M}$ is a weakly closed subspace of $\mathcal{L}(\mathcal{H})$. If $\mathfrak{M}$ has property $(\mathbf{B}_{2,3}^{\sim})$, then $\mathfrak{M}$ is hereditarily reflexive.*

Proof. By the remark made in the introduction, it suffices to show that $\mathfrak{M}$ is reflexive. Let X be an operator in $\mathrm{Ref}(\mathfrak{M})$; thus (6) is satisfied for $k = 1$. By Lemma 13, $\mathfrak{M}$ also has property $(\mathbf{B}_{1,2})$, and in order to prove that $X \in \mathfrak{M}$ it will suffice to show that $X^{(2)} \in \mathrm{Ref}(\mathfrak{M}^{(2)})$. Equivalently, by Lemma 3, we have to show that (6) is true for $k = 2$. Assume therefore that $x_1, x_2, y_1, y_2 \in \mathcal{H}$ and $[x_1 \otimes y_1] + [x_2 \otimes y_2] = 0$. For each $\epsilon > 0$ we choose vectors $\xi_i = \xi_i(\epsilon)$, $\eta_j =$

$\eta_j(\epsilon)$, $1 \le i \le 2$, $1 \le j \le 3$, satisfying conditions (i) and (ii) of Lemma 14. We have then

$$[\xi_1 \otimes (\eta_1 + \eta_3)] = [\xi_1 \otimes \eta_1] + [\xi_1 \otimes \eta_3] = \sum_{j=1}^{2} [x_j \otimes y_j] = 0,$$

and property (6) with $k=1$ implies

$$\langle X\xi_1, \eta_1 \rangle + \langle X\xi_1, \eta_3 \rangle = \langle X\xi_1, \eta_1 + \eta_3 \rangle = 0. \tag{16}$$

Let us consider first the particular case in which $[x_1 \otimes y_2] = 0$. In this case we have

$$\begin{aligned}[(\xi_1 - \xi_2) \otimes (\eta_2 + \eta_3)] &= [\xi_1 \otimes \eta_2] + [\xi_1 \otimes \eta_3] - [\xi_2 \otimes \eta_2] - [\xi_2 \otimes \eta_3] \\ &= 0 + [x_2 \otimes y_2] - [x_2 \otimes y_2] - 0 = 0,\end{aligned}$$

from which we infer

$$\langle X(\xi_1 - \xi_2), \eta_2 + \eta_3 \rangle = 0 \tag{17}$$

by (6) with $k=1$. Since we also have $\langle X\xi_1, \eta_2 \rangle = \langle X\xi_2, \eta_3 \rangle = 0$, (17) is easily seen to imply

$$\langle X\xi_1, \eta_3 \rangle - \langle X\xi_2, \eta_2 \rangle = 0. \tag{18}$$

Relations (16) and (18) can now be combined to yield $\langle X\xi_1, \eta_1 \rangle + \langle X\xi_2, \eta_2 \rangle = 0$, from which we infer

$$0 = \lim_{\epsilon \to 0} \sum_{j=1}^{2} \langle X\xi_j(\epsilon), \eta_j(\epsilon) \rangle = \sum_{j=1}^{2} \langle Xx_j, y_j \rangle.$$

Summing up this case, we have proved that

$$[x_1 \otimes y_2] = 0, \; \sum_{j=1}^{2} [x_j \otimes y_j] = 0 \Rightarrow \sum_{j=1}^{2} \langle Xx_j, y_j \rangle = 0. \tag{19}$$

To consider the general case we use (19) with x_1, y_1, x_2, y_2 replaced by $\xi_2, \eta_2, \xi_1, -\eta_3$, respectively. We have indeed

$$[\xi_2 \otimes (-\eta_3)] = 0, \qquad [\xi_2 \otimes \eta_2] + [\xi_1 \otimes (-\eta_3)] = [x_2 \otimes y_2] - [x_2 \otimes y_2] = 0,$$

and (19) allows us to conclude that (18) holds in the general case. As above, (16) and (18) imply that $\sum_{j=1}^{2} \langle Xx_j, y_j \rangle = 0$, thus completing our proof. □

It is quite obvious that Theorems 9 and 15 admit "symmetric" versions. That is, $(\mathbf{B}_{2,1})$ implies that $\mathfrak{M}^{(2)}$ is reflexive, and $(\mathbf{B}_{3,2}^{-})$ implies that $\mathfrak{M}$ is reflexive (provided, of course, that $\mathfrak{M}$ is closed in the weak operator topology). Olin and Thomson proved in [5] that subnormal operators (or rather the weakly closed algebras they generate) have property $(\mathbf{B}_{1,2})$, and in fact, property $(\mathbf{B}_{1,\aleph_0})$. They used this result to show that all subnormal operators are reflexive. However, the conclusion of Theorem 9 cannot be upgraded to say that $\mathfrak{M}$ is reflexive, and there are indeed examples of nonreflexive algebras with property $(\mathbf{B}_{1,\aleph_0})$.

PROPOSITION 16. *The algebra*

$$\mathfrak{M}=\left\{\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \in \mathcal{L}(\mathbf{C}^2): a, b \in \mathbf{C}\right\}$$

has property $(\mathbf{B}_{1,\aleph_0})$, *and yet is not reflexive.*

Proof. We denote by $e_1=(1,0)^T$ and $e_2=(0,1)^T$ the usual basis for $\mathbf{C}^{(2)}$, and show first that if $y=\alpha_1 e_1+\alpha_2 e_2$ is an arbitrary vector in $\mathbf{C}^{(2)}$ such that $[e_2 \otimes y]=0$, then $y=0$. Indeed, set $N=\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$, and note that $\bar{\alpha}_2=\langle e_2, y\rangle=[e_2 \otimes y](I)=0$, while $\bar{\alpha}_1=\langle e_1, y\rangle=\langle N e_2, y\rangle=[e_2 \otimes y](N)=0$. Since the dual space of $\mathfrak{M}$ is 2-dimensional, and we just showed that the mapping $y \rightarrow[e_2 \otimes y]$ is one-to-one on $\mathbf{C}^{(2)}$, it follows that every linear functional on $\mathfrak{M}$ has the form $[e_2 \otimes y]$ for some $y \in \mathbf{C}^{(2)}$. This shows that $\mathfrak{M}$ has property $(\mathbf{B}_{1,\aleph_0})$. That the algebra $\mathfrak{M}$ is not reflexive is well known and is left as an exercise for the reader. □

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