HOOLEY'S Δ_r-FUNCTIONS WHEN r IS LARGE

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1. Introduction. In an important paper [3], Hooley introduced the function

$$\Delta_r(n) := \max_{u_1,\dots,u_{r-1}} \operatorname{card}\{d_1,\dots,d_{r-1} : d_1\dots d_{r-1} \mid n,\ u_i < d_i \le eu_i\ \forall i\}.$$

I am interested here in upper bounds for the sum

$$S_r(x,y) := \sum_{n\le x} \Delta_r(n) y^{\omega(n)}, \quad y>0,$$

where $\omega(n)$ denotes the number of distinct prime factors of n. I follow Hall and Tenenbaum [2] in denoting by $\alpha(r,y)$ the infimum of the numbers ξ for which $S_r(x,y) \ll_\xi x(\log x)^\xi$ and in setting $A_r := \alpha(r,1)$. The function $\alpha(r,y)$ is known precisely, for $y\in \mathbf{R}^+\setminus(\frac{1}{2},2)$ and in Theorem 1 I shorten the excluded interval. In Theorem 2 I improve the known upper bounds for A_r for $r\ge 4$: in particular $A_r<\sqrt{r}-1$ $(4\le r\le 18)$, and at least in this range, Theorem 1B [3] is reinstated (cf. [2] concerning this theorem). For $r\ge 19$, the result is better than $A_r < 33(r+7)/244$.

The applications of Hooley's "new technique" set out in [3] required upper estimates for A_r only. Not only does the more general function $\alpha(r,y)$ seem interesting, particularly since for certain y there is a simple formula for it, but in the cases $r=2,3,4$ the upper best bounds, viz

$$A_2 < .21969, \quad A_3 < .55153, \quad A_4 < .92752,$$

have depended on estimates for $\alpha(r,y)$, $y\ne 1$ [2]. Such information can be applied to the study of A_r in two ways: by virtue of the fact that $\alpha(r,y)$ is a convex function of $\log y$, (by Hölder's inequality applied to $S_r(x,y)$), and through the Iteration Inequality of Hall and Tenenbaum [2]: for $r\ge s\ge 1$, $y,z>0$,

$$2\alpha(r,y) \le \alpha(r,z) + (s-1)\max(sz-1,0) + \alpha(r-s+1, y^2/z)$$

($\alpha(1,y) := y-1$). The following information is available from [2]:

THEOREM A. *We have* $A_r \le r/4$, $(r\ge 5)$.

THEOREM B. *For* $r\ge 2$ *and* $y\notin(\frac{1}{2},2)$ *we have* $\alpha(r,y) = y-1+(r-1)\chi(y-1)$, *where* $\chi(u) := \max(0,u)$. *Moreover* $\alpha(r,y)\ge y-1+(r-1)\chi(y-1)$ *for all* y.

The second part of Theorem B follows directly from

$$\Delta_r(n) \gg 1+\tau_r(n)/(\log n)^{r-1}.$$

The first part suggests the definitions:

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$$\Lambda_r^- := \sup\{y : \alpha(r,z) = z-1,\ 0<z<y\},$$
$$\Lambda_r^+ := \inf\{y : \alpha(r,z) = rz-r,\ y<z<\infty\},$$

so that $1/2 \le \Lambda_r^- \le 1 \le \Lambda_r^+ \le 2$ for every r. By the convexity, $\alpha(r,y)$ is continuous and the supremum and infimum are attained. I shall prove the following.

THEOREM 1. *We have* $\limsup \Lambda_r^+ \le (17+7\sqrt{7})/27 = 1.31556\ldots$; *more precisely, for* $r \ge 4$,

$$\Lambda_r^+ \le \frac{17r^3-45r^2+27r+(7r^2-15r+9)^{3/2}}{27r(r-1)(r-2)}.$$

Thus $\Lambda_r^+ < 4/3$ for $r \ge 12$, $\Lambda_r^+ < 3/2$ for $r \ge 4$. It is known (cf. [2]) that $\Lambda_2^+ \le \pi/(2\pi-4)$, $\Lambda_3^+ \le 1+1/\sqrt{3}$.

We can deduce an upper bound for A_r from knowledge about Λ_r^-, Λ_r^+. Let $t \in (0,1)$ be such that

$$O = (1-t)\log\Lambda_r^- + t\log\Lambda_r^+.$$

Then

$$\alpha(r,1) \le (1-t)\alpha(r,\Lambda_r^-) + t\alpha(r,\Lambda_r^+)$$
$$\le (1-t)(\Lambda_r^- -1) + rt(\Lambda_r^+ -1);$$

that is,

$$A_r \le \frac{(\Lambda_r^- -1)\log\Lambda_r^+ - r(\Lambda_r^+ -1)\log\Lambda_r^-}{\log(\Lambda_r^+/\Lambda_r^-)}.$$

Thus either $\Lambda_r^+ \to 1$ or $\Lambda_r^- \to 1$ as $r \to \infty$ is sufficient for $A_r = o(r)$. Combining Theorem 1 with $\Lambda_r^- \ge 1/2$ yields $\limsup(A_r/r) < 5/22$.

We improve on this in Theorem 2 below. It seems very difficult to make any advance on the result $\Lambda_r^- \ge 1/2$ which was obtained in [2] from the iteration method; in its present form this gives Lemma 4 [2] immediately to the right of 1/2: the upper bound achieved exceeds $y-1$ and deteriorates as r increases. Nevertheless it is this result, combined with other techniques in the "hybrid method" which leads to the upper bounds for A_2 and A_3 quoted above.

To deal with Λ_r^+, or what is the same thing, large y, the technique involving Fourier transforms initiated by Hooley [3] is appropriate, although in its basic form this suffers from the drawback that any upper bound achieved for $\alpha(r,y)$ must be at least as large as $h_r'y-1$, where

$$h_r' := \frac{1}{(2\pi)^{r-1}}\int_0^{2\pi}\cdots\int_0^{2\pi}|1+e^{i\theta_1}+\cdots+e^{i\theta_{r-1}}|\,d\theta_1\cdots d\theta_{r-1}$$
$$= r\int_0^\infty J_0(t)^{r-1}J_1(t)\frac{dt}{t} \sim \frac{1}{2}(\pi r)^{1/2}, \quad (r\to\infty).$$

The single integral formula is due to Hooley [4]. Now $A_2 < h_2'-1$ $(=(4/\pi)-1)$, $A_3 < h_3'-1$ $(=.57\ldots)$ so that it appears that for intermediate values of y, such as

$y=1$, to be most effective the Fourier transform technique requires the kind of refinement begun in Lemma 3 [2], and then to be combined with the iteration method in some way. However, it seems well suited to the study of Λ_r^+; we also use it in the present paper to obtain the following.

THEOREM 2. *There exist constants* $c_0 < .9303$ *and* $c_1 > 1.0655$ *such that for* $r \geq 14$, *we have that*

$$A_r < \frac{r}{8}\{3\sqrt{3}-5+2(3\sqrt{3}-5)^{1/2}\}+c_0-c_1\left(\frac{2r-1}{r^2}\right).$$

For $4 \leq r \leq 13$ *we have*

$$A_r \leq \frac{(r-1)^{3/2}}{r+\sqrt{(r-1)}}.$$

In particular, $A_r < \sqrt{r}-1$ for $3 \leq r \leq 18$: for such r Theorem 1B [3] follows.

2. The integral $K_r(\sigma; b, c)$. Hooley showed that (in the particular case $y=1$), $\alpha(r, y) \leq h_r(y)-1$ where $h_r(y)$ denotes the infimum of the numbers ξ for which, as $X \to \infty$,

$$\int_{-1}^{1}\cdots\int_{-1}^{1}\exp\left\{y\int_1^X |e^{i\theta_0 t}+e^{i\theta_1 t}+\cdots+e^{i\theta_{r-1}t}|\frac{dt}{t}\right\}d\theta_0\cdots d\theta_{r-1} \ll_\xi X^\xi,$$

where we have introduced the extra variable θ_0 for the sake of symmetry—only the differences between the θ's matter. Our idea is to estimate the inner integral by means of an inequality

$$x \leq a+bx^2-cx^4, \quad (0 \leq x \leq r),$$

using optimal values of a, b, c. A moment's consideration suggests that the polynomial $P(x)=a-x+bx^2-cx^4$ should have a root at $x=r$ and a double root at some point $\lambda \in [0, r]$. Since the coefficients of x and x^3 are known, the polynomial is determined completely by λ, indeed

$$\text{(1)} \qquad a=\frac{\lambda}{2}\left(1-\frac{\lambda^2}{(r+\lambda)^2}\right), \quad b=\frac{1}{2\lambda}\left(1+\frac{2\lambda^2}{(r+\lambda)^2}\right), \quad c=\frac{1}{2\lambda(r+\lambda)^2}.$$

A natural development would be to consider higher degree polynomials—however, the term x^6 already leads to technical difficulties which are as yet unresolved. This will be explained when we come to deal with the integral $K_r(\sigma; b, c)$ below. Even so, it is worth remarking at this point that one possible snag which will have occurred to the reader simply does not arise: if we suppose that

$$\begin{aligned} P_k(x) &= a_0-x+a_1x^2-a_2x^4+\cdots-a_{2k}x^{4k} \\ &= (x-\lambda_1)^2(x-\lambda_2)^2\cdots(x-\lambda_k)^2(r-x)(p_0+p_1x+\cdots+p_{2k-1}x^{2k-1}) \end{aligned}$$

and solve for $p_0, p_1, \ldots, p_{2k-1}$ by equating coefficients of $x, x^3, \ldots, x^{4k-1}$, we may rest assured that the coefficients a_m are positive (i.e., the signs are as above), and that $P_k(x)$ has no other root in $(0, r)$ (i.e., it is non-negative). Both assertions follow from Descartes' rule of signs. Negative terms in $P(x)$ present complications but no serious difficulty and so we should always choose the degree to be a multiple of 4, with negative leading term.

In [2], the infimum over ξ in question was denoted by $h_r(y)$ and we now estimate this. We write $\sigma=1/X$ and we note that

$$\int_1^X e^{i\theta t}\frac{dt}{t}=\log\left(\frac{1}{\sigma+|\theta|}\right)+O(1)$$

uniformly on any fixed range $|\theta|\le H$. We apply our inequality to the innermost integrand above and find that the integral is

$$\ll_{r,y} X^{(a+br-c(2r^2-r))y}K_r(\sigma; by, cy),$$

where

$$K_r(\sigma; d, c):=\int_{-1}^{1}\cdots\int_{-1}^{1}\frac{\Pi(\sigma+|\theta_i+\theta_j-\theta_k-\theta_l|)^c}{\Pi(\sigma+|\theta_p-\theta_q|)^b}\,d\theta_0\cdots d\theta_{r-1},$$

and the products run over all choices $0\le i, j, k, l, p, q<r$ except those involving $p=q$, $(i,j)=(k,l)$ or (l,k).

We may restrict our attention, by symmetry, to the range $\theta_0\le\theta_1\le\theta_2\le\cdots\le\theta_{r-1}$ and we put $x_m=\theta_m-\theta_{m-1}$, $1\le m<r$. We simplify the integral above by supposing $p>q$ in the denominator, and absorbing such factors as $(\sigma+|\theta_p+\theta_j-\theta_q-\theta_j|)$ and $(\sigma+|2\theta_p-2\theta_q|)$ in the numerator into the denominator. Of course in the latter case, there is a factor of at most 2 lost. So we have $K_r(\sigma; b, c)\ll K_r'(\sigma; d, c)$, where

$$K_r'(\sigma; d, c):=\int_{-1}^{1}\cdots\int_{\theta_{r-2}}^{1}\frac{\Pi'(\sigma+|\theta_i+\theta_j-\theta_k-\theta_l|)^{2c}}{\Pi'(\sigma+\theta_p-\theta_q)^{2d}}\,d\theta_0\cdots d\theta_{r-1},$$

and where the products are now restricted to $p>q$, $\max(i,j)>\max(k,l)$, neither i nor j is equal to either k or l, and either $i\ne j$ or $k\ne l$, and $d=b-(4r-3)c$.

In the denominator we write

$$\sigma+\theta_p-\theta_q\ge\sigma+x_m,\quad x_m=\max(x_{q+1}, x_{q+2},\ldots,x_p).$$

Next, we have

$$\theta_i+\theta_j-\theta_k-\theta_l=\sum_{n<r}\delta_n x_n,\quad -1\le\delta_n\le 2,$$

and we write

$$\sigma+|\theta_i+\theta_j-\theta_k-\theta_l|\ll_r\sigma+x_m,\quad x_m=\max\{x_n:\delta_n\ne 0\}.$$

We say that x_m appears in a factor of the numerator or denominator if $\delta_m\ne 0$ or $q<m\le p$ respectively—it is then a candidate to be maximum. Now let $1\le m_1<m_2<\cdots<m_s<r$ and consider the variables $x_{m_1}, x_{m_2},\ldots,x_{m_s}$. We need a formula for the number of factors in the numerator and denominator in which at least one of these variables appears. In the denominator this is simply

$$\binom{r}{2}-\sum_{u=0}^{s}\binom{m_{u+1}-m_u}{2},$$

where it is understood that $m_0=0$, $m_{s+1}=r$ when we evaluate the sum. We write $m_{u+1}-m_u=t_u$, $0\le u\le s$, and since $\sum t_u=r$ this becomes

$$\frac{1}{2}r^2-\frac{1}{2}\sum_{u=0}^{s}t_u^2.$$

If none of the variables appears in $\sigma+\theta_p-\theta_q$ there exists $u\le s$ such that $m_u\le q<p<m_{u+1}$, which accounts for $\binom{t_u}{2}$ factors, hence the result.

It is helpful in counting to split the product in the numerator into three parts. The first part comprises factors for which $\min(i,j)>\max(k,l)$. If none of the variables $x_{m_1},\dots,x_{m_s}$ appears in such a factor, we must have $m_u\le\min(k,l)<\max(i,j)<m_{u+1}$ for some $u\le s$. The number of factors counted in the first part is therefore

$$4\binom{r+1}{4}-4\sum_{u=0}^{s}\binom{t_u+1}{4}.$$

The second part of the product comprises factors in which $\max(k,l)>\min(i,j)>\min(k,l)$. If none of the variables appears in such a factor, either there is a u such that $m_u\le\min(k,l)<\max(i,j)<m_{u+1}$, or there exist u, v with $u<v$ and

$$m_u\le\min(k,l)<\min(i,j)<m_{u+1},\qquad m_v\le\max(k,l)<\max(i,j)<m_{v+1}.$$

These possibilities account for

$$4\sum_{u=0}^{s}\binom{t_u}{4}+4\sum_{0\le u<v\le s}\binom{t_u}{2}\binom{t_v}{2}$$

factors, hence the number to be counted is

$$4\binom{r}{4}-r\sum_{u=0}^{s}\binom{t_u}{4}-2\left\{\sum_{u=0}^{s}\binom{t_u}{2}\right\}^2+2\sum_{u=0}^{s}\binom{t_u}{2}^2.$$

The third part of the product comprises factors in which $\min(i,j)<\min(k,l)$. A similar calculation to the above shows that at least one of the variables appears in

$$\frac{2}{3}\binom{r}{2}\binom{r-1}{2}-\frac{2}{3}\sum_{u=0}^{s}\binom{t_u}{2}\binom{t_u-1}{2}-2\left\{\sum_{u=0}^{s}\binom{t_u}{2}\right\}^2+2\sum_{u=0}^{s}\binom{t_u}{2}^2$$

factors in this part of the product, and hence in

$$2\binom{r}{2}\binom{r-1}{2}-2\sum_{u=0}^{s}\binom{t_u}{2}\binom{t_u-1}{2}-4\left\{\sum_{u=0}^{s}\binom{t_u}{2}\right\}^2+4\sum_{u=0}^{s}\binom{t_u}{2}^2$$

factors of the numerator. After simplification, this is

$$2\binom{r}{2}\binom{r-1}{2}+\frac{1}{2}\sum_{u=0}^{s}t_u^4+\frac{1}{2}(4r-3)\sum_{u=0}^{s}t_u^2-\left\{\sum_{u=0}^{s}t_u^2\right\}^2+r-r^2.$$

Now consider the integral $K_r'(\sigma;dy,cy)$; we split the range of integration with respect to $x_1,\dots,x_{r-1}$ into $(r-1)!$ parts; in each of which there exists a permutation ρ on $r-1$ symbols such that

$$x_{\rho(1)}\le x_{\rho(2)}\le\cdots\le x_{\rho(r-1)}.$$

Each integral is majorized (to within $\ll_{r,y}$) by

$$\int_0^2 \frac{dx_{\rho(1)}}{(\sigma+x_{\rho(1)})^{z_1}} \int_{x_{\rho(1)}}^2 \frac{dx_{\rho(2)}}{(\sigma+x_{\rho(2)})^{z_2}} \cdots \int_{x_{\rho(r-2)}}^2 \frac{dx_{\rho(r-1)}}{(\sigma+x_{\rho(r-1)})^{z_{r-1}}}, \tag{2}$$

where the exponents z_m vary with ρ. We always have

$$z_1+z_2+\cdots+z_{r-1}=2dy\binom{r}{2}-4cy\binom{r}{2}\binom{r-1}{2}.$$

In the proof of Theorems 1 and 2 which follow we need a lower bound for $z_{r-s}+z_{r-s+1}+\cdots+z_{r-1}$. We set $m_1=\rho(r-s)$, $m_2=\rho(r-s+1),\ldots,m_s=\rho(r-1)$ so that the variables $x_{m_1}, x_{m_2},\ldots,x_{m_s}$ are the s largest x's. It follows that one or other of them must be chosen as maximum in any factor in which at least one of them appears. From the above, we deduce that

$$\begin{aligned} z_{r-s}+\cdots+z_{r-1} &= dyr^2-by\sum_{u=0}^{s} t_u^2-4cy\binom{r}{2}\binom{r-1}{2} \\ &\quad -cy\sum_{u=0}^{s} t_u^4+2cy\left\{\sum_{u=0}^{s} t_u^2\right\}^2+2cy(r^2-r), \end{aligned} \tag{3}$$

using the relation $b=d+(4r-3)c$. The numbers t_u are positive integers whose sum is r. We minimize the right-hand side by varing the t_u—this depends on the relation between b and c and hence on λ.

If instead of the quartic polynomial we had taken $P_2(x)$ of degree eight, we should have had an integral similar to K_r but more complicated. In particular, a lower bound for factors of the form $\sigma+|\theta_i+\theta_j+\theta_k-\theta_l-\theta_m-\theta_n|$ in the denominator would be needed. I have found no satisfactory way of dealing with this. The numerator would be rather complicated, but no more difficult in principle than the case already considered.

3. Proof of Theorem 1. Suppose that for $1\le s<r$ we have $z_{r-s}+\cdots+z_{r-1}\ge s$. Then the integral (2) is

$$\ll\left(\frac{1}{\sigma}\right)^{z_1+z_2+\cdots+z_{r-1}-r+1}\left(\log\frac{1}{\sigma}\right)^{r-1} \quad (\sigma<1/2).$$

This proposition may be proved by induction on r: we estimate the innermost integral as $\ll(\sigma+x_{\rho(r-2)})^{1-z_{r-1}}\log(1/\sigma)$ and replace the exponents $z_1,\ldots,z_{r-1}$ by $z_1,\ldots,z_{r-3},z'_{r-2}$, where $z'_{r-2}=z_{r-2}+z_{r-1}-1$. Subject to the condition above this yields

$$\begin{aligned} h_r(y) &\le \left\{a+br-c(2r^2-r)+2d\binom{r}{2}-4c\binom{r}{2}\binom{r-1}{2}\right\}y-r+1 \\ &\le (a+br^2-cr^4)y-r+1=ry-r+1, \end{aligned}$$

because $P(r)=0$. We shall therefore have $\alpha(r,y)=ry-r$ for suitable values of y: λ is at our disposal.

Recall the formula for $z_{r-s}+\cdots+z_{r-1}$ given in §2, and put $\sum t_u^2=t^2+s$. Since $t_u\geq 1$ $(0\leq u\leq s)$ and $\sum t_u=r$, we have $t\leq r-s$ and $\sum t_u^4\leq t^4+s$. Hence it would be sufficient to have

$$b(r^2-t^2-s)-c(r^4-t^4-4st^2-2s^2+s)\geq s/y,\quad 1\leq s<r \tag{4}$$

where t is chosen to minimize the left-hand side. We choose λ so that

$$\frac{b}{2c}=\frac{r^2+2r\lambda+3\lambda^2}{2}\geq (r-1)^2+2, \tag{5}$$

and hence the left-hand side of (4) is a decreasing function of t. Substituting $t=r-s$, it is therefore sufficient to have

$$b(2r-s-1)-c\{(4r^3-4r^2+1)-(6r^2-8r+2)s+(4r-4)s^2-s^3\}\geq 1/y,$$

for $1\leq s\leq r$. Since the left-hand side is concave as a function of s, we need only check $s=1$ or $r-1$. Hence we require both

$$2(r-1)\{b-c(2r^2-3r+3)\}\geq 1/y,$$
$$r\{b-c(r^2+r-1)\}\geq 1/y.$$

We choose λ so that $b/c=3r^2-5r+3$ which is consistent with (5) when $r\geq 3$. Both these inequalities become $y\geq 1/2cr(r-1)(r-2)$ and, on solving the equation $3\lambda^2+2r\lambda=(r-1)(2r-3)$ and substituting λ in the formula (1) for c, we obtain

$$\Lambda_r^+\leq\frac{17r^3-45r^2+27r+(7r^2-15r+9)^{3/2}}{27r(r-1)(r-2)},\quad r\geq 3.$$

This is the result stated. □

4. Proof of Theorem 2. We put $y=1$, and we show that for suitable λ the integral (2) is $\ll_r(\log 1/\sigma)^{r-1}$. We shall then have

$$h_r(1)\leq a+br^2-c(2r^2-r)=:\phi(\lambda),$$

say. Substituting for a, b, c from (1) we find that

$$\phi'(\lambda)=\frac{r(r-1)(r+3\lambda)(\lambda^2-r+1)}{2\lambda^2(r+\lambda)^3}.$$

Ideally we should like to choose $\lambda=\sqrt{(r-1)}$ for a minimum, and this is possible when $4\leq r\leq 13$. For larger values of r we have to choose $\lambda>\sqrt{(r-1)}$ to keep b small—otherwise $K_r(\sigma;b,c)$ is too large.

A necessary and sufficient condition for the integral (2) to be $\ll_r(\log 1/\sigma)^{r-1}$ is that

$$z_1+z_2+\cdots+z_s\leq s,\quad \text{for } 1\leq s<r. \tag{6}$$

This is proved by induction on r; it is clearly true for $r=2$, and we suppose it true for $r-1$, $r\geq 3$. Let (6) hold. We have

$$\int_{x_{\rho(r-2)}}^{2}\frac{dx_{\rho(r-1)}}{(\sigma+x_{\rho(r-1)})^{z_{r-1}}}\ll\frac{\log 1/\sigma}{(\sigma+x_{\rho(r-2)})^{z-1}},$$

where $z=\max(1, z_{r-1})$. This gives us an integral similar to (2), with $r-2$ variables and $z'_{r-2}=z_{r-2}+z-1$. If $z_{r-1}<1$ and $z=1$, then

$$z_1+z_2+\cdots+z'_{r-2}=z_1+z_2+\cdots+z_{r-2}\le r-2.$$

If $z_{r-1}\ge 1$ then

$$z_1+z_2+\cdots+z'_{r-2}=z_1+z_2+\cdots+z_{r-1}-1\le r-2.$$

By the induction hypothesis, the remaining integral is $\ll(\log 1/\sigma)^{r-2}$, and the induction is complete. Now suppose (6) false, that is, suppose there exists $s_0<r$ such that $z_1+z_2+\cdots+z_{s_0}>s_0$. There are two cases: we can find such an $s_0<r-1$ or not. In the first case, (6) is also false for the integral with respect to $r-2$ variables, for $z'_{r-2}=z_{r-2}+\chi(z_{r-1}-1)\ge z_{r-2}$. If the only $s_0=r-1$, then

$$z_1+z_2+\cdots+z_{r-2}\le r-2$$

but $z_1+z_2+\cdots+z_{r-1}>r-1$ and $z_{r-1}>1$. So

$$z'_{r-2}=z_{r-2}+z_{r-1}-1 \quad \text{and} \quad z_1+z_2+\cdots+z'_{r-2}>r-2.$$

Again (6) is false for the reduced integral and the induction is complete.

Obviously (6) is equivalent to

$$z_{r-s}+\cdots+z_{r-1}\ge 2d\binom{r}{2}-4c\binom{r}{2}\binom{r-1}{2}-(r-s-1), \tag{7}$$

and we recall the formula (3) for the left-hand side. We need

$$2c\left\{\sum_{u=0}^{s} t_u^2\right\}^2-c\sum_{u=0}^{s} t_u^4-b\sum_{u=0}^{s} t_u^2\ge -(r-s-1)-dr-2c(r^2-r) \tag{8}$$

for $1\le s<r$ and any integers $t_u\ge 1$ such that $\sum t_u=r$. Once again we put $\sum t_u^2=t^2+s$ so that $t\le r-s$; moreover, $\sum t_u^4\le t^4+s$, and (8) reduces to

$$ct^4+(4cs-b)t^2\ge -(r-s-1)-b(r-s)+c(2r^2-2s^2-r+s). \tag{9}$$

The left-hand side has a minimum when $t^2=(b/2c)-2s$ if this does not exceed $(r-s)^2$, otherwise it is decreasing. We consider two cases.

Case (i), $(r-s)^2+2s\ge b/2c$.

We put $t^2=(b/2c)-s$, and we need

$$c\left(\frac{b}{2c}-2s\right)^2\le r-s-1+b(r-s)-c(r-s)(2r+2s-1).$$

By hypothesis, $(r-s)^2-2(r-s)+1\ge(b/2c)-2r+1$; that is,

$$r-s-1\ge((b/2c)-2r+1)^{1/2}.$$

From (1) we have

$$\frac{b}{2c}=\frac{1}{2}(r^2+2r\lambda+3\lambda^2),$$

and so for $r\ge 4$ we have $b/2c\ge r^2/2>2r-1$, also $b\ge 4rc>(2r+2s-1)c$. Hence it will be sufficient to have

$$c(b/2c)^2 \le ((b/2c)-2r+1)^{1/2}.$$

If we set $\lambda =: \mu r$, $\nu := \mu^2/(1+\mu)^2$, this reduces to

$$\text{(10)} \qquad \frac{(\frac{1}{2}+\nu)^3}{4\nu} \le 1 - \frac{2(2r-1)}{(1+2\mu+3\mu^2)r^2}.$$

In consideration of case (ii), we shall choose $\nu = (3\sqrt{3}-5)/4$ so that the left-hand side is equal to 27/32, and $\mu = .28443\ldots$; the inequality (10) holds for $r \ge 14$.

Case (ii), $(r-s)^2+2s < b/2c$.

The function of t in (9) is decreasing for $t \le r-s$ and to make it a minimum we put $t = r-s$, so that we need

$$\text{(11)} \qquad c(r-s)^4+(4cs-b)(r-s)^2+(1+b-(2r+2s-1)c)(r-s) \ge 1.$$

There is equality when $s = r-1$. The left-hand side is equal to

$$f(r-s)+c(r-s)\{4s(r-s)-2r-2s+1\}+b(r-s),$$

where $f(x) := cx^4-bx^2+x$. We put $x = \lambda z$ so that $f(x) = \frac{1}{2}x\{\nu z^3-(1+2\nu)z+2\}$ and set $\nu = (3\sqrt{3}-5)/4$ so that f has a double zero which occurs at $z = 1+\sqrt{3}$; indeed,

$$f(x) = \frac{\lambda}{8}(3\sqrt{3}-5)z(z-1-\sqrt{3})^2(z+2+2\sqrt{3}).$$

There is a maximum at $z = 1$, $x = \lambda > 2$ for $\mu > .28$, $r \ge 14$. It follows that for $2 \le x \le \frac{2}{3}(1+\sqrt{3})\lambda$, we have

$$f(x) \ge \min\left\{2-4b, \frac{8}{81}(1+\sqrt{3})\right\} \ge 1.$$

Hence (11) is satisfied for $2 \le r-s \le \frac{2}{3}(1+\sqrt{3})\lambda$ as the left-hand side exceeds $f(r-s)$. For $r-s \ge \frac{2}{3}(1+\sqrt{3})\lambda$ we have

$$b(r-s) \ge \frac{1+\sqrt{3}}{3}(1+2\nu) = 1, \qquad f(r-s) \ge 0.$$

Since $\mu > .28$ we have $s \le r/2$, and for $1 \le s \le r/2$ $(r \ge 14)$, we have

$$4s(r-s)-2r-2s+1 \ge 0.$$

Hence (11) is satisfied on this range also.

We have now proved that (11) is valid for $r \ge 14$ when $\lambda = \mu r$ (μ as above). Hence

$$h_r(1) \le \frac{\mu r}{2}(1-\nu)+\frac{1+2\nu}{2\mu}-\frac{\nu}{2\mu^3}\left(\frac{2r-1}{r^2}\right).$$

We find that

$$\frac{\mu}{2}(1-\nu) = \frac{3\sqrt{3}-5+2(3\sqrt{3}-5)^{1/2}}{8} = .13524\ldots,$$

and we put $c_0 = (1+2\nu)/2\mu-1$, $c_1 = \nu/2\mu^3$. This gives the result stated for $r \ge 14$.

When $4 \leq r \leq 13$ we can choose $\lambda = \sqrt{(r-1)}$ optimally, checking the various inequalities by direct computation.

Added in proof. In accordance with current practice I re-define

$$\Lambda_r^- = \sup\{y : S_r(x,z) \ll x(\log x)^{z-1},\ 0 < z < y\},$$

$$\Lambda_r^+ = \inf\{y : S_r(x,z) \ll x(\log x)^{rz-r},\ y < z < \infty\}.$$

Theorem 1 still holds, and provides a sharp bound for $S_r(x,z)$ for $z > \Lambda_r^+$. Similarly, $\Lambda_r^- \geq 1/2$ and for $z < 1/2$, $S_r(x,z)$ is known to within constants.

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