REMARKS ON THE APPROXIMATION TO AN ALGEBRAIC NUMBER BY ALGEBRAIC NUMBERS

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1. Introduction. Let α be a real algebraic number and let k be a real algebraic number field, $\alpha \notin k$. The celebrated theorem of K. F. Roth ([12], [13]) asserts that α cannot be approximated too well by elements of k and, more precisely, for every $\epsilon > 0$ there is $c(\alpha, \epsilon) > 0$ such that for every $\beta \in k$

$$|\alpha - \beta| > c(\alpha, \epsilon) H_k(\beta)^{-2-\epsilon}, \tag{1}$$

where $H_k(\beta)$ is the field height, that is the largest coefficient of the integral polynomial with roots $\sigma(\beta)$, counted with multiplicity, for all distinct embeddings $\sigma: k \to \mathbf{C}$. Since it is known (see e.g. Schmidt [14, Ch. VIII, Th. 2A]) that there are infinitely many β's in the field k such that

$$|\alpha - \beta| < c(\alpha) H_k(\beta)^{-2}, \tag{2}$$

the above result of Roth is clearly best possible.

It is a well-known feature of Roth's theorem that inequality (1) is ineffective, in the sense that the proof yields the existence of the constant $c(\alpha, \epsilon)$ in (1) but does not allow the calculation of a lower bound for it. If we ask for effective lower bounds for $|\alpha - \beta|$ then our knowledge about approximations is much weaker than that given by (1). Let us define $\mu_{\text{eff}}(\alpha, k)$ to be the infimum of all μ's for which an inequality $|\alpha - \beta| > c(\alpha) H_k(\beta)^{-\mu}$ holds for every $\beta \in k$ and some effectively computable $c(\alpha) > 0$.

The first general improvement on the elementary bound $\mu_{\text{eff}}(\alpha, k) \leqq [k(\alpha):k]$ was obtained by Baker [1] using his theory of linear forms in logarithms, and eventually Feldman [11] proved (at least in the case $k = \mathbf{Q}$) that $\mu_{\text{eff}}(\alpha, k) \leqq \deg \alpha - \eta$, where $\eta = \eta(\alpha, k) > 0$ is a positive very small constant. For an account of this theory see Baker's monograph [2].

The Baker–Feldman theorem is the only non-trivial effective result available today valid for every algebraic number α and every number field k. On the other hand, for special numbers α better effective results are known, in particular: $\alpha = \xi^{1/r}$ with $\xi \in k$ ([3], [4], [6], [7]); α a cubic number ([7], [10]); some special algebraic numbers, such as $\alpha^r + m\alpha - 1 = 0$ ([5]); the typical situation here is the case in which $k = \mathbf{Q}$, while α is restricted in various ways.

In this paper we show that for any given α one can find algebraic number fields k for which precise information about effective approximation can be obtained.

THEOREM 1. *Let α be a real algebraic number of degree $r \geqq 3$ and let $\eta > 0$ be any positive constant. Then one can find infinitely many real algebraic number fields k of degree $r-1$ such that $\mu_{\text{eff}}(\alpha, k) \leqq 2 + \eta$.*

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At the end of this paper we illustrate our theorem with the example of $\alpha=\sqrt[3]{2}$ and $k=\mathbf{Q}(\sqrt{D})$ for suitable very large discriminants D. Finally we should mention that our method of proof yields the stronger result $\mu_{\text{eff}}(\alpha', k) \leqq 2+\eta$ for every $\alpha' \in k(\alpha)$, $\alpha' \notin k$; if $\deg \alpha > 3$, this statement appears to be a non-trivial reinforcement of the conclusion of our theorem.

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2. The Thue principle. Thue's method depends on the comparison of two distinct approximations β_1, β_2 to the algebraic number α, and is used to show that it is not possible, under appropriate auxiliary conditions, for both approximations to be exceptionally good. If the first approximation β_1 to α is so good that Thue's method implies a non-trivial lower bound for approximations to α, then we call (α, β_1) an anchor pair. The papers [5] and [6] provide the first explicit formulation of Thue's method which was actually used to produce explicit examples of anchor pairs. We state it in simplified form as follows:

THUE'S PRINCIPLE. *Let k be a real number field and let α, $|\alpha|<1$, be real algebraic of degree $r \geqq 2$ over k. Let $h(\xi)$ denote the absolute height of the algebraic number ξ.*

For every $\beta \in k$ with $|\alpha-\beta|<1$ and for every positive a, $0<a<1$, we have

$$\mu_{\text{eff}}(\alpha, k) \leqq \frac{2(r+a^2)[k:\mathbf{Q}]}{(1-a)^2} \frac{\log h(\beta)+(r/a^2)(\log h(\alpha)+1)}{\log(1/|\alpha-\beta|)}.$$

More precisely, the right-hand side is also a bound for

$$\mu_{\text{eff}}(k(\alpha)/k) = \sup_{\alpha' \in k(\alpha)} \mu_{\text{eff}}(\alpha', k).$$

Here $h(\xi)$ denotes the absolute height of ξ, that is,

$$\log h(\xi) = \sum_v \log^+ |\xi|_v,$$

where v runs over all the normalized absolute values of a number field F with $\xi \in F$. We have $h(\xi)^{\deg \xi} = M(\xi)$, where $M(\xi)$ is the Mahler measure and thus

$$2^{-d} H_k(\xi) \leqq h(\xi)^d \leqq (d+1)^{1/2} H_k(\xi),$$

where $d=[k:\mathbf{Q}]$, for any $\xi \in k$. If $P(x)=a_0x^d+\cdots+a_d$ is a defining equation for ξ we have

$$M(\xi) = |a_0| \prod_{i=1}^{d} \max(1, |\xi_i|),$$

where ξ_i runs over the roots of $P(x)$.

In some cases the bound obtained by the Thue principle is better than the trivial Liouville bound r and then we say that (α, β) is an *anchor pair* for the extension K/k.

The result stated above is a corollary of the similar result in [6, p. 179], with the following remarks:

(i) the quantity $c(\vartheta t)$ in [6] can be replaced by log 3, with the same conclusion;
(ii) $A_1 \leqq rt^2/(2-rt^2)(\log h(\alpha_1)+\frac{1}{2})$;
(iii) the alternatives (5A) and (5B) of [6] imply the bound $\mu_{\text{eff}}(\alpha_2, k) \leqq 2\vartheta/(t-\tau)$, provided α_2 generates $k(\alpha_1)$ over k;
(iv) if v is the real place associated to the real field k, then $|\alpha_1-\beta_1|_v^{[k:\mathbf{Q}]} = |\alpha_1-\beta_1|$, where $|\ |$ denotes the ordinary absolute value in $\mathbf{R}$.

First of all, if $r=2$ we have $\mu_{\text{eff}}(\alpha, k)=2$ by the Liouville bound in [6, p. 189] while the upper bound we want to prove in the Thue principle is >2, so that there is nothing to prove.

Hence let us assume $r \geqq 3$. We apply the Thue principle [6, p. 179] with $\alpha_1=\alpha$, $\alpha_2=\alpha'$, $\beta_1=\beta$ and β_2 variable. We choose $t=(2/(r+a^2))^{1/2}$ and δ so small that $\tau=(2a^2/(r+a^2)+(r-1)\delta)^{1/2}<t$. We put $A_i=rt^2/(2-rt^2)(\log h(\alpha_i)+\frac{1}{2})$, $i=1,2$. To choose ϑ, note first that there is $\lambda>0$ such that

$$|\alpha_1-\beta_1|_v=(3e^{A_1}h(\beta_1))^{-2/\lambda(t-\tau)}, \tag{3}$$

where v is the valuation such that $|\alpha_1-\beta_1|_v^d=|\alpha_1-\beta_1|$ for $d=[k:\mathbf{Q}]$. From the Liouville bound of [6, p. 189] we deduce

$$(2h(\alpha_1)h(\beta_1))^{-r} \leqq (3e^{A_1}h(\beta_1))^{-2/\lambda t} < (2h(\alpha_1)h(\beta_1))^{-2/\lambda t},$$

so that $\lambda>t$. Since also $3^{-2/x}<x$ $(x>0)$ we see that

$$|\alpha_1-\beta_1| \leqq |\alpha_1-\beta_1|_v < \lambda(t-\tau). \tag{4}$$

Now take any $\vartheta>\lambda$. We shall verify that

$$\mu_{\text{eff}}(\alpha', k) \leqq 2\vartheta/(t-\tau), \tag{5}$$

provided $k(\alpha')=k(\alpha)$. Translating if necessary by a rational integer, we may assume $|\alpha'| \leqq 1$. A simple calculation from (3) shows that $2\vartheta/(t-\tau)$ approaches the upper bound of Thue's principle as δ approaches 0.

To prove (5) note first that if $\vartheta \geqq (r/2)t$ then $2\vartheta/(t-\tau)>r$; hence in this case (4) follows from the Liouville bound. So we may suppose $t<\vartheta<(r/2)t<t^{-1}$. Now since $c(x) \leqq \log 3$ $(0<x<1)$ the inequality (3) above implies (4) [6, p. 179], and also (4) implies the inequality preceding this. By Lemma 1 [6, p. 183] the triple (A_1, A_2, τ) is admissible provided α' has degree r over k. So all the hypotheses of [6] are satisfied. Now the alternatives (5A) and (5B) or the inequality just preceding them clearly imply (5). Note that the exponent in (5A) should read $-2\vartheta/(t-\tau)$. This completes the proof, at least in the case in which $k(\alpha')=k(\alpha)$.

If $k(\alpha') \subset k(\alpha)$ the result depends on the fact that the triple (A_1, A_2, τ) remains admissible if $t<\vartheta<(r/2)t$; the proof of this requires a reworking of Dyson's lemma and will not be given here, since the needed result will appear in a forthcoming paper by C. Viola.

3. Wirsing's theorems. In [15], Wirsing investigated the problem of approximating a given real number α by algebraic numbers of fixed degree. This problem appears rather naturally if one considers Koksma's classification of transcendental numbers.

Let α be a real number and let us assume that α is not algebraic of degree $\leqq k$. We are interested in the approximation of α by algebraic numbers β of degree $\leqq k$ and we seek results of the type $|\alpha-\beta| \ll H(\beta)^{-w-1}$ for infinitely many β. Let $w_k^*(\alpha)$ be the best bound of such w's and let $w_k^* = \inf_\alpha w_k^*(\alpha)$.

The following is known:

$$(\mathrm{A}_1) \qquad w_1^* = 1;$$

$$(\mathrm{A}_2) \qquad w_2^* = 2;$$

$$(\mathrm{A}_k) \qquad w_k^* \geqq \frac{k+2}{4} + \tfrac{1}{4}\sqrt{k^2+4k-4}\;;$$

(B) if α is algebraic of degree $\geqq k+1$, then $w_k^*(\alpha) = k$;

(C) if $w_k(\alpha) = \sup w$, where $|P(\alpha)| \ll H(P)^{-w}$ for infinitely many polynomials P of degree $\leqq k$ with integral coefficients, then

$$w_k^*(\alpha) \geqq w_k(\alpha) - k + 1,$$

$$w_k^*(\alpha) \geqq \tfrac{1}{2}(w_k(\alpha)+1),$$

$$w_k^*(\alpha) \geqq \frac{w_k(\alpha)}{w_k(\alpha)-k+1}.$$

Of course, $w_k(\alpha) \geqq k$ is an easy consequence of Minkowski's theorem in the geometry of numbers.

Of these, (A_1) is due to Dirichlet; (A_2) is due to Davenport and Schmidt [8]; and (A_k) is in Wirsing [15]. Also, (B) for $\deg \alpha = k+1$ and (C) are in Wirsing [15] and (B) is a consequence of the last statement in (C) and the fact that if $k < \deg \alpha$ then Schmidt's subspace theorem implies that $w_k(\alpha) \leqq k$, hence $w_k(\alpha) = k$; since $w_k^*(\alpha) \leqq w_k(\alpha)$ for every α, we get (B).

Schmidt's result is ineffective and this leads to the following curious situation: if $\epsilon > 0$ is given and $k < \deg \alpha$ then we obtain that there are infinitely many algebraic numbers β of degree k such that $|\alpha-\beta| \leqq H(\beta)^{-k-1+\epsilon}$, but we cannot give an upper bound for first solution of this inequality. On the other hand one could at least theoretically test the inequality for solutions, and we would produce one by persevering long enough in our tries; the ineffectivity lies in the fact that we cannot tell a priori when our tries will come to a satisfactory conclusion. The special case in which $k = \deg \alpha - 1$ does not depend on Schmidt's deep result and can be treated elementarily in an effective and satisfactory manner, as Wirsing's proof of

$$w_k^*(\alpha) \geqq \frac{w_k(\alpha)}{w_k(\alpha)-k+1}$$

shows. In what follows we shall give a new proof of the fact that if α is real algebraic and $k = \deg \alpha - 1$ then $w_k^*(\alpha) = k$. Our argument could also be used to prove lower bounds for $w_k^*(\alpha)$ in terms of $w_k(\alpha)$; although the results one obtains in this way appear to be slightly inferior to Wirsing's, we feel that our treatment is sufficiently different to merit independent consideration.

THEOREM 2. *Let α, $|\alpha| \leqq \frac{1}{2}$, be real algebraic of degree r and height $H(\alpha)$. For every $X \geqq 2$ there is β, algebraic of degree at most $r-1$, such that*

$$|\alpha-\beta| \leqq \frac{r!(r-1)}{X^r},$$

$$H(\beta) \leqq 2^r (r(r+1)H(\alpha))^{(r-1)^2/r} X.$$

Moreover, β is real as soon as $X > (r(r+1))^r H(\alpha)^{r-1}$.

The proof of Theorem 2 is an easy consequence of the following two lemmata.

LEMMA 1. *Let $Q(x) = a_k x^k + \cdots + a_0$ be a polynomial with real or complex coefficients, not identically 0. Let α be any complex number with $Q'(\alpha) \neq 0$. Then the closest root β of Q to α satisfies*

$$|\alpha-\beta| \leqq k \frac{|Q(\alpha)|}{|Q'(\alpha)|}.$$

Proof. Clear from the identity

$$\frac{Q'(\alpha)}{Q(\alpha)} = \sum_{\beta} \frac{1}{\alpha-\beta},$$

where β runs over all roots, counted with multiplicity, of $Q(x)$. (For this argument, see Davenport and Schmidt, [8, p. 217].) □

Let α be real and let $S(X)$ be the convex symmetrical body in $\mathbf{R}^{k+1}$ defined by

$$\begin{cases} |x_0 + x_1\alpha + \cdots + x_k\alpha^k| \leqq X^{-k} \\ |x_1| \leqq X \\ \quad \vdots \\ |x_k| \leqq X, \end{cases}$$

and let $\lambda_i = \lambda_i(X)$, $i = 1, \ldots, k+1$ be the successive minima of $S(X)$. Let $x^{(i)}$, $i = 1, \ldots, k+1$ be points at which the minimum λ_i is attained and let $P_i(y)$ denote the polynomial

$$P_i(y) = x_0^{(i)} + x_1^{(i)} y + \cdots + x_k^{(i)} y^k.$$

LEMMA 2. *We have*

$$|P_i'(\alpha)| \geqq \frac{\lambda_i}{(k+1)!} X$$

for at least one suffix i.

Proof. Let

$$Y = \max_i |P_i'(\alpha)| / \lambda_i;$$

by definition of Y, $\lambda_1, \ldots, \lambda_{k+1}$ are still the successive minima of the convex body

$$S(X, Y) = S(X) \cap \{|x_1 + 2\alpha x_2 + \cdots + k\alpha^{k-1} x_k| \leqq Y\}$$

and, by Minkowski's second theorem, we have

$$\frac{2^{k+1}}{(k+1)!} \leqq \lambda_1 \dots \lambda_{k+1} \operatorname{vol}(S(X,Y)).$$

Again by Minkowski's second theorem we have

$$\lambda_1 \dots \lambda_{k+1} \operatorname{vol}(S(X)) \leqq 2^{k+1}.$$

Thus $\lambda_1 \dots \lambda_{k+1} \leqq 1$, and we get

$$\frac{2^{k+1}}{(k+1)!} \leqq \operatorname{vol}(S(X,Y)).$$

Clearly

$$\operatorname{vol}(S(X,Y)) \leqq \operatorname{vol}\begin{cases} |x_0+x_1\alpha+\cdots+x_k\alpha^k| \leqq X^{-k} \\ |x_1+x_2 2\alpha+\cdots+x_k k\alpha^{k-1}| \leqq Y \\ |x_2| \leqq X, \dots, |x_k| \leqq X \end{cases}$$

$$= 2^{k+1} Y/X,$$

and we obtain

$$\frac{2^{k+1}}{(k+1)!} \leqq 2^{k+1} Y/X.$$

Thus $Y \geqq (1/(k+1)!)X$ and Lemma 2 follows. □

Proof of Theorem 2. We apply Lemma 1 and Lemma 2 with $k=r-1$. By definition of successive minima,

$$\text{(6)} \qquad \begin{aligned} &|P_i(\alpha)| \leqq \lambda_i X^{-k}, \\ &\max_{1\leqq h\leqq k} |x_h^{(i)}| \leqq \lambda_i X \end{aligned}$$

and thus

$$\begin{aligned} |x_0^{(i)}| &\leqq |P_i(\alpha)| + \sum_{h=1}^{k} |x_h^{(i)}|\,|\alpha|^h \\ &\leqq \lambda_i X^{-k} + \lambda_i X \sum_{h=1}^{k} 2^{-h} \leqq \lambda_i X \end{aligned}$$

because $X \geqq 2$. It follows that

$$\text{(7)} \qquad |H(P_i)| \leqq \lambda_i X.$$

By Lemma 2, there is i such that

$$|P_i'(\alpha)| \geqq \frac{1}{r!}\lambda_i X,$$

and now Lemma 1 and (6) yield

$$\text{(8)} \qquad |\alpha-\beta| \leqq \frac{r!(r-1)}{X^r}.$$

Also, β is a root of P_i, thus

$$H(\beta) \leqq 2^r H(P_i) \leqq 2^r \lambda_i X, \tag{9}$$

by using the inequality $H(PQ) \geqq 2^{-\deg P - \deg Q} H(P)H(Q)$ (see Duncan [9]).

It remains to obtain an upper bound for λ_i and this is done as follows. As remarked in the proof of Lemma 2, we have $\lambda_1 \lambda_2 \dots \lambda_r \leqq 1$, hence

$$\lambda_i \leqq \lambda_r \leqq \lambda_1^{-r+1} \tag{10}$$

so that we need a lower bound for λ_1. By definition of λ_1, the polynomial P_1 satisfies

$$\begin{aligned} &|P_1(\alpha)| \leqq \lambda_1 X^{-k}, \\ &\max_{1 \leqq h \leqq k} |x_h^{(1)}| \leqq \lambda_1 X; \end{aligned} \tag{11}$$

$$H(P_1) \leqq \lambda_1 X. \tag{12}$$

Now (11) and (12) imply

$$|P_1(\alpha)| \leqq \lambda_1^r H(P_1)^{-k}. \tag{13}$$

On the other hand, by taking norms or, even better, by considering absolute heights, if P is any polynomial with integral coefficients of degree at most $k = r-1$ and not identically 0 we have

$$|P(\alpha)| \geqq \frac{1}{(r(r+1)H(\alpha)H(P))^k}. \tag{14}$$

Indeed, let $\xi = P(\alpha)$ and let $F = \mathbf{Q}(\alpha)$. Let w be the real normalized absolute value of F corresponding to the real embedding of F determined by α. We have

$$\begin{aligned} \log|\xi|_w &\geqq -\sum_{v \neq w} \log^+|\xi|_v \\ &\geqq -\sum_{v \neq w} \left(\frac{\epsilon_v}{[F:\mathbf{Q}]} \log(rH(P)) + k \log^+|\alpha|_v \right) \end{aligned}$$

(where $\epsilon_v = 0$ if v is non-Archimedean, $\epsilon_v = 1$ if v is real, $\epsilon_v = 2$ if v is complex), because

$$\log^+|\xi|_v = \log^+|P(\alpha)|_v \leqq \frac{\epsilon_v}{[F:\mathbf{Q}]} \log(rH(P)) + k \log^+|\alpha|_v$$

for every v. Now (14) follows from $|P(\alpha)| = |\xi|_w^r$, from

$$\sum_{v \neq w} \frac{\epsilon_v}{[F:\mathbf{Q}]} = 1 - \frac{1}{r},$$

and

$$\sum_v \log^+|\alpha|_v = \log h(\alpha) \leqq \frac{1}{r} \log((r+1)H(\alpha)).$$

If we combine (13) and (14) we get

$$\lambda_1 \geqq (r(r+1)H(\alpha))^{-(r-1)/r}; \tag{15}$$

the first clause of Theorem 2 follows from (8), (9), (10), and (15).

It remains to prove that β is real provided X is sufficiently large. If β were non-real then we would have

$$|\alpha-\beta| \leqq \frac{r!(r-1)}{X^r}, \qquad |\alpha-\bar{\beta}| \leqq \frac{r!(r-1)}{X^r}.$$

Therefore, if $s=\deg P_i$, we would have

$$\begin{aligned}
|P_i(\alpha)| &\leqq |x_s^{(i)}| \prod_{h=1}^{s} |\alpha-\beta_h| \\
&\leqq \left(\frac{r!(r-1)}{X^r}\right)^2 |x_s^{(i)}| \prod_{h=3}^{s} |\alpha-\beta_h| \\
&\leqq \left(\frac{r!(r-1)}{X^r}\right)^2 2^{s-2} M(P_i) \\
&\leqq \frac{(r!(r-1))^2 2^{s-2} r\lambda_i}{X^{2r-1}},
\end{aligned}$$

where $\beta_1=\beta$, $\beta_2=\bar{\beta}$, $\beta_3,\ldots,\beta_r$ are the roots of P_1. If we compare this upper bound with the lower bound (14) (note that $H(P_i)\leqq \lambda_i X$) we obtain, after some calculation, the upper bound for X in the last clause of Theorem 2. □

4. Proof of Theorem 1 and concluding remarks. In order to prove our Theorem 1 we apply Theorem 2, with X very large. Then we have

$$\log \frac{1}{|\alpha-\beta|} \geqq r\log X + O(1), \tag{16}$$

$$\log h(\beta) = \frac{1}{[k:\mathbf{Q}]}\log X + O(1), \tag{17}$$

and thus

$$\mu_{\text{eff}}(\alpha,k) \leqq 2\frac{1+a^2/r}{(1-a)^2}\left(1+O\left(\frac{1}{a^2\log X}\right)\right),$$

where the constant involved in the $O(\ldots)$ term depends only on α. If we choose a of order $(\log X)^{-1/3}$ we find

$$\mu_{\text{eff}}(\alpha,k) \leqq 2+O((\log X)^{-1/3}),$$

and k is generated by an equation of degree $\leqq r-1$ and height $O(X)$.

We have tacitly assumed here that $[k(\alpha):k]=r$, since we need this in the Thue Principle. This follows immediately for large X from the Liouville bound, which gives

$$\log \frac{1}{|\alpha-\beta|} \leqq [k(\alpha):k]\log X + O(1)$$

as $X\to\infty$.

Theorem 1 also asserts that the field k has exact degree $r-1$ infinitely many times and also that we obtain infinitely many distinct fields this way. If we appeal to Schmidt's theorem, it is easy to prove these assertions, because if we had $[k:\mathbf{Q}]\leqq r-2$ infinitely many times we would deduce $w^*_{r-2}(\alpha)\geqq r-1$, contradicting (B) of §2. Similarly if we did not have infinitely many distinct fields $k=\mathbf{Q}(\beta)$ we would contradict the Roth theorem stated in (1) of §1.

The above argument is not effective in the sense that we cannot determine an X_1 such that if $X>X_1$ and (16) and (17) hold then β has exact degree $r-1$, although the preceding argument shows that such an X_1 exists. However, in the special but non-trivial case in which $r=3$ it is possible to appeal to Baker's effective results in [2] instead of Schmidt's theorem and make our Theorem 1 completely effective.

It is possible to obtain other estimates of this type, such as bounds in the case in which k is further restricted to degree $\leqq s<r$. On the other hand, it may be of interest to look at specific examples. The following treatment of $\alpha=\sqrt[3]{2}$ can be easily extended to general α's and it provides an alternative way of finding β's of degree $r-1$ and exceptionally close to α.

Let $\rho=e^{2\pi i/3}$ and let us define $\eta=\sqrt[3]{2}-1$, $\vartheta=\rho\sqrt[3]{2}-1$, $\bar{\vartheta}=\bar{\rho}\sqrt[3]{2}-1$. η is a unit of the cubic field $\mathbf{Q}(\sqrt[3]{2})$ and its absolute height is $h(\eta)=(\sqrt[3]{2}-1)^{-1/3}$. We have $|\eta|=h(\eta)^{-3}$; in the notation of [6], η^{-1} is a Thue number.

Let us define integers a_n, b_n, c_n by

$$\eta^n=a_n+b_n\sqrt[3]{2}+c_n(\sqrt[3]{2})^2, \tag{18}$$

and let

$$P_n(x)=a_n+b_nx+c_nx^2.$$

Then $P_n(x)$ has a root β such that

$$|\sqrt[3]{2}-\beta|\leqq 2\frac{|P_n(\sqrt[3]{2})|}{|P_n'(\sqrt[3]{2})|}.$$

We have

$$a_n=\frac{1}{3}(\vartheta^n+\bar{\vartheta}^n+\eta^n),$$

$$b_n=\frac{1}{3\sqrt[3]{2}}(\bar{\rho}\vartheta^n+\rho\bar{\vartheta}^n+\eta^n),$$

$$c_n=\frac{1}{3\sqrt[3]{4}}(\rho\vartheta^n+\bar{\rho}\vartheta^n+\eta^n),$$

and we obtain

$$|\sqrt[3]{2}-\beta|\leqq\frac{2\eta^n}{|b_n+2\sqrt[3]{2c_n}\;|}. \tag{19}$$

Since $1/\eta^n=|\vartheta|^{2n}$ and

N	$\mu_{\mathrm{eff}}(\sqrt[3]{2}, \mathbf{Q}(\sqrt{D_N}))$
435	2.99865
436	2.99854
437	2.99908
438	2.99880
439	2.99571
500	2.94154
1000	2.70467
5000	2.37793
10000	2.29271
50000	2.16483

Table 1

$$a_n \sim \frac{1}{3}(\vartheta^n + \bar{\vartheta}^n),$$

$$b_n \sim \frac{1}{3\sqrt[3]{2}}(\bar{\rho}\vartheta^n + \rho\bar{\vartheta}^n),$$

$$c_n \sim \frac{1}{3\sqrt[3]{4}}(\rho\vartheta^n + \overline{\rho\vartheta}^n),$$

we see that the right-hand side of (16) is usually of order $H(\beta)^{-3}$ and then, for large $H(\beta)$, $(\sqrt[3]{2}, \beta)$ is an anchor pair for the field $k_n(\sqrt[3]{2})$, where k_n is the quadratic field $k_n = \mathbf{Q}(\sqrt{D_n})$ with $D_n = b_n^2 - 4a_n c_n$.

We have computed some values for $\mu_{\mathrm{eff}}(\sqrt[3]{2}, k_n)$, using a refined version of the Thue–Siegel principle in [6]; our calculations in Table 1 show that our procedure is still far from obtaining interesting results valid for fields k with a small discriminant.

The results of Table 1 have been obtained using the Thue–Siegel principle in [6, p. 179], together with an improvement of Lemma 1 of [6]. This improvement allows us to use the principle with the values of A_1 given in [6] and with $\tau \leqq \frac{2}{3}\sqrt{2 - rt^2} + O(\delta)$, rather than the bound $\tau \leqq \sqrt{2 - rt^2} + O(\delta)$ of Lemma 1 of [6]. We have also computed D_N for $N < 500$ and, for example, $D_{435} \cong 2.31 \times 10^{254}$.

Finally, we should observe that the major obstacle to finding substantially better results seems to be in the estimation of A_1 (i.e., the height of the auxiliary polynomial). In the special case $\deg \alpha = 3$ one can profitably use the technique of Chudnovsky [7] to obtain good improvements of our results. Thus the bounds in Table 1 should be considered only as providing an example of what can be proved in the general case, rather than being associated with the special number $\sqrt[3]{2}$.

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