IMMERSIONS EQUIVARIANT FOR A GIVEN KILLING VECTOR II

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0. Introduction. In [1], we showed that any complete Riemannian manifold with a 1-parameter subgroup of isometries and sectional curvatures bounded above by $-c<0$ cannot be immersed isometrically and equivariantly into any Euclidean space.

On the other hand, we have a negatively curved complete revolution surface whose order of the Gauss curvature at infinity is (distance from a fixed point)$^{-2-e}$, where e is positive.

In this paper, we know that the above estimate is best. That is, we have the following.

THEOREM A. *Let M be a complete Riemannian manifold of negative curvature and ρ a 1-parameter subgroup of isometries acting nontrivially on M. If there exists a point $x\in M$ such that the maximum of sectional curvatures on the geodesic ball of radius s with center x is bounded above by $-As^{-2}$, $A>0$ for large s, then M does not admit any ρ-equivariant isometric immersion into Euclidean spaces.*

Furthermore, we give analogous results to [1] in the case that the ambient space is a hyperbolic space. That is, we obtain the following.

THEOREM B. *Let M be a complete Riemannian manifold, and let $\rho(\theta)$ $(\theta\in\mathbf{R})$ be a 1-parameter subgroup of isometries acting nontrivially on M. If the sectional curvatures of $M\le -c<-1$, then M has no ρ-equivariant isometric immersion into any hyperbolic space with sectional curvature -1.*

THEOREM C. *Let M be an n-dimensional non-compact type symmetric space with Ricci curvature $-(n-1)c$, $c>1$, and let ρ be a 1-parameter subgroup of isometries acting nontrivially on M. Then M has no ρ-equivariant isometric immersion into any hyperbolic space with sectional curvature -1.*

The author is grateful to Professor K. Ogiue for his useful criticism and Professor H. Nagai for his useful suggestions.

1. Revolution surfaces with negative curvatures in $\mathbf{R}^3$. In this section, we study "the order of the Gauss curvature at infinity" of some complete revolution surfaces with negative curvature in $\mathbf{R}^3$.

Let (r,θ) be the polar coordinate of $\mathbf{R}^2$ and (t,r,θ) the coordinate of $\mathbf{R}^3$. We give a revolution surface S by

$$\mathbf{R}\times S^1 \ni (t,\theta)\to (t,\ \tau(t)\cos\theta,\ \tau(t)\sin\theta)\in\mathbf{R}^3,$$

Received April 10, 1984. Final revision received July 11, 1985.
Michigan Math. J. 33 (1986).

where τ is a positive function on $\mathbf{R}$ and S^1 is a unit circle. Therefore the induced metric on $\mathbf{R}\times S^1$ is given by

$$(1+(\tau')^2)\,dt^2+\tau^2\,d\theta^2.$$

Then we obtain the Gauss curvature K of $\mathbf{R}\times S^1$:

$$K=-\frac{\tau''}{\tau(1+(\tau')^2)^2}.$$

Let us assume that $\tau(t)=t^l$ for $t\geq 1$ and $l>1$. Then the Gauss curvature K_l for $t\geq 1$ is given by

$$K_l=-\frac{l(l-1)}{t^2(1+l^2t^{2(l-1)})^2}.$$

Choosing an appropriate $\tau(t)$ for $-1\leq t\leq 1$ such that $\tau(t)$ for $t\leq 1$ is the tractrix, we have a complete revolution surface with negative Gauss curvature. Let K_s be the maximum of the Gauss curvature on the geodesic ball of radius s with center $(1,0)$. Then it is easy to see that

$$K_s=-\frac{l(l-1)}{t^2(1+l^2t^{2(l-1)})^2}$$

for

$$s=\int_1^t\sqrt{1+l^2t^{2(l-1)}}\,dt.$$

Thus there are positive constants α and β such that

$$-\frac{\alpha}{s^{(4l-2)/l}}\leq K_s\leq -\frac{\beta}{s^{(4l-2)/l}}$$

for large s. We find that, for any small $e>0$, there exists a complete revolution surface of negative curvature such that "the order of the Gauss curvature at infinity from a fixed point x" is (distance from $x)^{-2-e}$. In the next section, we show that the above estimate is best; that is, *there is no complete revolution surface of negative curvature such that "the order of the Gauss curvature at infinity from a fixed point x" is (distance from $x)^{-2}$.*

2. Complete Riemannian manifolds with negative curvature. Let M be an n-dimensional complete Riemannian manifold with negative curvature and X a Killing vector field on M. Since X is a Jacobi field on τ, we have

$$\nabla_{\tau_*}\nabla_{\tau_*}X=-R_{\tau_*X}\tau_*,$$

where τ is a geodesic with arc length parameter, ∇ is the covariant differentiation for the metric $\langle\ ,\ \rangle$ on M, and R is the curvature tensor for ∇. Let K_s be the maximum of the sectional curvatures of the geodesic ball of radius s with center $x\in M$. Then we assume that $K_s\leq -A/s^2$ for large s, where A is a positive constant. It implies that, for any point in place of x, there is a positive constant B such that $K_s\leq -B/s^2$ for large s. Thus, for a point x such that $X(x)\neq 0$, we may consider that there exists a constant A such that $K_s < -A/s^2$ for large s. Let τ be a geodesic with arc length parameter s such that $\tau(0)=x$ and $e_1\ (=\tau_*), \dots, e_n$ a parallel field of orthonormal frames along τ. Then, expressing X by $X=\sum_{i=1}^n f^i e_i$, we have the Jacobi equation:

$$f^1 = \text{constant } (=\beta)$$

$$f^{i''} = -\sum_{j=2}^n f^j \langle R_{\tau_* e_j}\tau_*, e_i\rangle \quad \text{for } i>1.$$

Setting $\sum_{i=2}^n (f^i)^2 = g$, we obtain

$$\begin{aligned} g'' &= \sum_{i=2}^n 2((f^i)'^2 + f^i(f^i)'') \\ &= 2\sum_{i=2}^n (f^i)'^2 - 2\sum_{i,j=2}^n f^i f^j \langle R_{\tau_* e_j}\tau_*, e_i\rangle. \end{aligned}$$

By the assumption on the sectional curvatures of M, we get

$$(2.1) \qquad \begin{aligned} & g'' \geq 0 \ \text{ on } (-s_0, s_0), \\ & g'' \geq 2\sum_{i=2}^n (f')^2 + \frac{1}{As_0^2} g \ \text{ on } s \leq -s_0 \text{ and } s \geq s_0. \end{aligned}$$

Choosing τ such that $X(x)//e_2$, we have $f^2(x) \neq 0$, $\beta = 0$ and hence obtain $g(0)>0$. If necessary, we set $s \to -s$ and obtain $g'(0) \geq 0$. Then $g'' \geq 0$ implies that $g(s_0)>0$ and $g'(s_0) \geq 0$. Therefore we find that $g>0$, $g' \geq 0$, and

$$g'' \geq \frac{(g')2}{2g} + 2\frac{1}{As^2} g \quad \text{for } s \geq s_0.$$

Setting $G = g'/g$, by (2.1) we obtain

$$G' \geq -\frac{1}{2}G^2 + \frac{2}{As^2}.$$

Thus, for any $B>A$,

$$G' > -\frac{1}{2}G^2 + \frac{2}{Bs^2}$$

holds. Now we consider the following ordinary differential equation of Riccati type:

$$(2.2) \qquad F' = -\frac{1}{2}(F)^2 + \frac{2}{Bs^2} \quad \text{for } s \geq s_0.$$

It is easy to see that c/s, where $c = 1+\sqrt{1+4/B}$, is a solution of (2.2). Using the theory of the ordinary differential equation of Riccati type, we obtain the general solution F:

$$(2.3) \qquad F = \frac{c}{s} + \frac{(c-1)s_0^c s^{-c}}{s_0^c(s_0^{1-c} - s^{1-c})/2 + (c-1)\alpha},$$

where α is a real number. We can choose α such that $G(s_0)=F(s_0)$. That is, when $G(s_0)=c/s_0$ we set $F=c/s$, while in another case we put $\alpha=1/(G(s_0)-c/s_0)$. By properties of G and F, we have

$$(G-F)'>-\tfrac{1}{2}(G-F)(G+F). \tag{2.4}$$

Thus $(G-F)'(s_0)>0$ holds and hence there exists s_1 such that $G-F>0$ on (s_0, s_1). If there exists a positive number s_2 such that $(G-F)(s_2)=0$ and $G-F>0$ on (s_0, s_2), then $(G-F)'(s_2)\leq 0$. This is a contradiction for (2.4). We have $G(s)>F(s)$ for $s>s_0$. If $F\neq c/s$, then F is given by (2.3). It is easy to show that $s_0/2+(c-1)\alpha\neq 0$. If it is zero, then $G(s_0)=F(s_0)\geq 0$ implies $c/s_0+1/\alpha\geq 0$, which contradicts

$$\frac{c}{s_0}+\frac{1}{\alpha}=\frac{c}{s_0}-\frac{2(c-1)}{s_0}=\frac{2-c}{s_0}<0.$$

Hence $s_0/2+(c-1)\alpha\neq 0$. For large s, we find that there is a positive constant K such that

$$\left|\frac{(c-1)s_0^c}{s_0/2+(c-1)\alpha-s_0^c s^{1-c}/2}\right|<K.$$

Therefore we conclude

$$F(s)\geq\frac{c}{s}-\frac{K}{s^c}=\frac{c}{s}\left(1-\frac{K}{cs^{c-1}}\right)\quad\text{for large } s>0.$$

Therefore for any small ϵ we may consider that $F(s)\geq c(1-\epsilon)/s$ for large s. In both cases, $F(s)\geq c(1-\epsilon)/s$ for large s holds. Since $g'/g=G$, we obtain $g\geq Ls^{c(1-\epsilon)}$ for large s. For large s, we note $c(1-\epsilon)>2$.

Using the same argument as in [1] or as in §3 on the length of a Killing vector field of Euclidean space, we have Theorem A.

3. Killing vector fields of an n-dimensional hyperbolic space $H^n(-c)$ with constant sectional curvature $-c$, $c>0$. Let τ be a geodesic with arc length parameter s in $H^n(-c)$ and X a Killing vector field on $H^n(-c)$. Then X is a Jacobi field along τ, which satisfies the Jacobi equation:

$$\nabla_{\tau_*}\nabla_{\tau_*}X=cX-c\langle X,\tau_*\rangle\tau_*,$$

where $\langle\ ,\ \rangle$ is the metric on $H^n(-c)$ and ∇ is the covariant differentiation with respect to $\langle\ ,\ \rangle$. Let $e_1\ (=\tau_*), e_2, \ldots, e_n$ be a parallel field of orthonormal frames along τ. Then, setting $X=\sum_{i=1}^n f^i e_i$, we obtain the Jacobi equation

$$f''=0 \text{ and } f^{i\prime\prime}=cf^i \text{ for } i>1.$$

Thus there are constants A^i and B^i for $i>1$ such that

$$f^i=A^i e^{\sqrt{c}s}+B^i e^{-\sqrt{c}s}\quad\text{for } i>1.$$

Furthermore, since X is a Killing vector field, f^1 must be constant $(=\beta)$. Consequently we have

$$X = \alpha\tau_* + \sum_{i=2}^{n} (A^i e^{\sqrt{c}s} + B^i e^{-\sqrt{c}s}) e_i, \tag{3.1}$$

$$\nabla_{\tau_*} X = \sum_{i=2}^{n} \sqrt{c}(A^i e^{\sqrt{c}s} - B^i e^{-\sqrt{c}s}) e_i, \tag{3.2}$$

which imply

$$\|X\|^2(\tau(0)) = \alpha^2 + \sum_{i=1}^{n} (A^i + B^i)^2,$$

$$\|\nabla_{\tau_*} X\|^2(\tau(0)) = \sum_{i=2}^{n} c(A^i - B^i)^2.$$

Then we have the following.

LEMMA 3.1. *Let α_τ, A^i_τ and B^i_τ be the constants determined by a geodesic with arc length parameter s such that $\tau(0) = x \in H^n(-c)$. Then, independent of τ, $|\alpha_\tau|$, $|A^i_\tau|$ and $|B^i_\alpha|$ are bounded. Thus there is a constant ϵ such that ϵ is independent of τ ($\tau(0) = x$) and the length of X at $\tau(s)$ for large s is bounded above by $\epsilon e^{\sqrt{c}s}$.*

4. A Killing vector field on a complete Riemannian manifold. Let M be an n-dimensional complete Riemannian manifold with sectional curvature $\leq -c < 0$. Let τ be a geodesic with arc length parameter s and X a Killing vector field on M. Since X is a Jacobi field on τ, we have

$$\nabla_{\tau_*} \nabla_{\tau_*} X = -R_{\tau_* X} \tau_*,$$

where ∇ is the covariant differentiation for the metric $\langle\ ,\ \rangle$ on M and R is the curvature tensor for ∇. Let e_1 $(= \tau_*), \ldots, e_n$ be a parallel field of orthonormal frames along τ. Then, expressing X by $X = \sum_{i=1}^n f^i e_i$ and using the same argument as in §2, we have, for $\sum_{i=2}^n (f^i)^2 = g$,

$$g'' \geq 2 \sum_{i=2}^{n} (f^i)'^2 + 2cg. \tag{4.1}$$

In particular, $g'' \geq 2cg$ holds. We assume without loss of generality that $g(\tau(0)) > 0$ and $g'(\tau(0)) > 0$. $g'' \geq 2cg$ implies that g and g' are positive on $[0, \infty)$. On the other hand, by Schwarz inequality we have

$$g' \leq 2\sqrt{g} \sqrt{\sum_{i=2}^{n} (f^i)'^2},$$

which, together with the positiveness of g and g' on $[0, \infty)$, implies

$$g'^2 \leq 4g\left(\sum_{i=2}^{n} (f^i)'^2\right).$$

Thus we obtain

$$g'' \geq \frac{g'^2}{2g} + 2cg.$$

It follows from Schwarz inequality that $g'' \geq 2\sqrt{c}g'$, which implies

$$g \geq g(0) + \frac{1}{2\sqrt{c}} e^{2\sqrt{c}s + \log g'(0)}.$$

THEOREM 4.1. *Let M be a complete Riemannian manifold with sectional curvatures $\leq -c$ $(c > 0)$ and X a Killing vector field on M. Then there is a geodesic τ with arc length parameter s and a positive constant ϵ such that the length of $X(\tau(s))$ for large s is bounded below by $\epsilon e^{\sqrt{c}s}$.*

5. Proof of Theorem B. Let τ be the same geodesic as in Theorem 4.1. Let χ be a ρ-equivariant isometric immersion of M into $H^{n+p}(-1)$. That is, there is a 1-parameter subgroup $\tilde{\rho}$ of isometries of M such that

$$\chi(\rho(\theta)x) = \tilde{\rho}(\theta)\chi(x) \quad \text{for all } x \in M.$$

Let X_ρ and $X_{\tilde{\rho}}$ be Killing vector fields on M and $H^{n+p}(-1)$ generated by ρ and $\tilde{\rho}$, respectively. Then we have $\chi_*(X_\rho) = X_{\tilde{\rho}}$. We denote by $\tilde{\tau}_s$ the geodesic segment which joins $\chi(\tau(0))$ to $\chi(\tau(s))$. Furthermore, with respect to the arc length parameter $\tilde{s}$ of $\tilde{\tau}_s$ with $\tilde{\tau}_s(0) = \chi(\tau(0))$, we denote by $\tilde{s}(s)$ the positive number such that $\tilde{\tau}_s(\tilde{s}(s)) = \chi(\tau(s))$. It is clear that $\tilde{s}(s) \leq s$. Since $X_{\tilde{\rho}}$ is a Killing vector field, there are real numbers $\tilde{\alpha}_s$, $\tilde{A}_s^i$ and $\tilde{B}_s^i$ such that

$$X_{\tilde{\rho}}(\tilde{s}) = \tilde{\alpha}_s \tilde{\tau}_{s*}(\tilde{s}) + \sum_{i=2}^{n+p} (\tilde{A}_s^i e^{\tilde{s}} + \tilde{B}_s^i e^{-\tilde{s}}) \tilde{e}_{si},$$

where $\tilde{e}_{s1}$ $(= \tau_{s*}), \ldots, \tilde{e}_{s(n+p)}$ is a parallel field of orthonormal frames along $\tilde{\tau}_s$. By Lemma 3.1, $|\tilde{\alpha}_s|$, $|\tilde{A}_s^i|$ and $|\tilde{B}_s^i|$ for all $s > 0$ are bounded above by some positive number L. Thus we have

$$\|\tilde{X}_{\tilde{\rho}}(\tilde{s}(s))\|^2 \leq L + 4(n+p-1)Le^{2s},$$

which contradicts the definition of τ. □

6. A Killing vector field on a noncompact type symmetric space. Let M be an n-dimensional noncompact type symmetric space with Ricci curvature $-(n-1)c$, $c > 0$. Then there is a symmetric pair (G, K) such that G/K, where G is the connected component of the Lie group of isometries of M and K is an isotropy subgroup of G, which fixes $o \in M$ (see [2], for example). We denote by $\mathcal{G}$ and $\mathcal{K}$ $(\subset \mathcal{G})$ the Lie algebra of G and the Lie algebra of K, respectively. Let $\mathcal{G} = \mathcal{K} + \mathcal{P}$ be the canonical decomposition of $\mathcal{G}$ and B the Killing form of $\mathcal{G}$. Then B is negative definite over $\mathcal{K}$, positive definite over $\mathcal{P}$, and $B(\mathcal{K}, \mathcal{P}) = 0$. Considering T_0M as $\mathcal{P}$, we give the metric by positive scalar multiple of B. Let ρ be a 1-parameter subgroup of isometries acting nontrivially on M. Then there is a vector $\bar{X} \in \mathcal{G}$ such that $\rho(\theta) = \exp \theta \bar{X}$. Since $k \in K$ acts on M as an isometry, we have

$$k_*(X_{\exp \theta \bar{X}}) = X_{\exp \theta \operatorname{Ad}(k) \bar{X}},$$

where Ad is the adjoint representation of G. Thus it is enough to prove Theorem C with respect to an appropriate $X_{\exp\theta \mathrm{Ad}(k)\bar{X}}$. We assume without loss of generality that the $\mathcal{P}$-component $\bar{X}^{\mathcal{P}}$ of $\bar{X}$ is not zero. Then $(\mathrm{Ad}(k)\bar{X})^{\mathcal{P}}$ is not zero.

Let τ be a geodesic with arc length parameter s such that $\tau(0)=o$ and $\tilde{X}$ a Killing vector field of M. Then $\tilde{X}$ along τ satisfies the Jacobi equation:

$$\nabla_{\tau_*}\nabla_{\tau_*}\tilde{X} = -R_{\tau_*\tilde{X}}\tau_*,$$

where ∇ is the covariant differentiation of M and R is the curvature tensor. Let $e_1\ (=\tau_*), \dots, e_n$ be a parallel field of orthonormal frames along τ such that

$$\langle R_{\tau_* e_j}\tau_*, e_k\rangle = a_j\delta_{ij} \text{ at } o.$$

By the fact that M is a symmetric space, we have

$$\langle R_{\tau_* e_j}\tau_*, e_k\rangle = a_j\delta_{jk} \text{ on } \tau.$$

Since

$$\sum_{j=2}^{n} a_j = -(n-1)c \quad \text{and} \quad a_j \le 0,$$

there is i_0 such that $a_{i_0} \le -c$. We assume without loss of generality that $i_0=2$. Furthermore, since the adjoint representation of K on $\mathcal{P}$ is irreducible, there is an element k_0 such that

$$\langle(\mathrm{Ad}(k_0)\bar{X}, e_2\rangle \ne 0.$$

Let $\tilde{X}$ be the Killing vector field generated by $\exp\theta\,\mathrm{Ad}(k_0)\bar{X}$. Then there are functions f_i on τ such that $\tilde{X} = \sum_{i=1}^{n} f^i e_i$, and the Jacobi equation gives $f^{2''} = -a_2 f^2$. If necessary, replacing e_2 by $-e_2$, we obtain $f^2(0)>0$ and $f^{2''} \ge cf^2$ on $[0,\infty)$. Thus we have real numbers $A>0$ and B such that

$$f^2 \ge Ae^{\sqrt{c}s} + Be^{-\sqrt{c}s}$$

by changing s into $-s$ if necessary. Consequently the length of $\tilde{X}(\tau(s))$ for large s is bounded below by (some positive constant)$e^{\sqrt{c}s}$.

7. Proof of Theorem C. By the same argument as in the proof of Theorem B, we have Theorem C. □

REMARK. (1) We note that the result obtained in §6 gives a simple proof of Theorem B in [1].

(2) In general, we obtain the following:

Let M be a noncompact type symmetric space and N a noncompact type symmetric space with Ricci curvature $-(n-1)c$. We denote by τ the minimum value of the sectional curvatures of M. If $c > -\tau$, then N does not admit a ρ-equivariant isometric immersion into M.

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