SECONDARY INVARIANTS OF TRANSVERSELY HOMOGENEOUS FOLIATIONS

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1. Introduction. Let N be a smooth manifold and G a group of diffeomorphisms of N. A transverse (G,N) foliation F on a smooth manifold M consists of an open cover $\{U_\alpha\}_{\alpha\in A}$ of M together with submersions $\varphi_\alpha: U_\alpha \to N$ such that for each pair U_α, U_β with $U_\alpha \cap U_\beta \neq \varnothing$, there is an element $g_{\alpha\beta}$ of G with $g_{\alpha\beta}\circ\varphi_\beta = \varphi_\alpha$. If the dimension of N is q, then F is a smooth codimension q foliation on M and there is a well-defined map $\alpha_F: H^*(WO_q) \to H^*(M)$ giving the characteristic classes of the foliation. Let SL_{q+1} act on S^q as the space of directed lines in $\mathbf{R}^{q+1}$. In this note we prove the following.

THEOREM. *Let F be a transverse (SL_{q+1}, S^q) foliation on M and suppose that $\pi_1(M)$ is finitely generated. Let γ be an element of $H^*(WO_q)$. Then the set of values that $\alpha_F(\gamma)$ can take on is a finite subset of $H^*(M)$.*

This theorem is a generalization of the same result for transverse (PSL_2, S^1) foliations and the Godbillon–Vey invariant $h_1c_1 \in H^*(WO_1)$ due to Brooks–Goldman. The proof appears in [3] which was the inspiration for this paper.

2. Characteristic classes for foliations. The material in this section is fairly standard so we shall recall it only briefly. For more details see [1], [2], and [11].

The differential graded algebra WO_q is

$$WO_q = R_q[c_1,\dots,c_q]\otimes\Lambda(h_1,h_3,\dots,h_{2s+1})$$

where the degree $c_i = 2i$, degree $h_i = 2i-1$, and $2s+1 = q$ or $q-1$. $R_q[c_1,\dots,c_q]$ is the polynomial algebra truncated above degree $2q$ and $\Lambda(h_1,\dots,h_{2s+1})$ is the exterior algebra on the h_i's. The DGA W_q is

$$W_q = R_q[c_1,\dots,c_q]\otimes\Lambda(h_1,\dots,h_q).$$

In both cases the differential is given by setting $d(c_i)=0$, $d(h_i)=c_i$ and extending as a derivation.

Let F be a codimension q foliation on M and let $\tau\subset TM$ be its tangent bundle, where TM is the tangent bundle of M. The normal bundle of F is $\nu = TM/\tau$ and we write $\rho: TM\to\nu$ for the projection. A basic connection θ^b on ν is one whose covariant derivative ∇ satisfies $\nabla_X\rho(Y) = \rho([X,Y])$ for all $X\in\tau$.

Denote the space of differential forms on M by $A^*(M)$. Let θ^r be a Riemannian connection on ν. The map $\alpha_F: WO_q\to A^*(M)$ is given as follows: Let Ω^b be the curvature of θ^b and Ω^t the curvature of the connection $\theta^t = t\theta^b + (1-t)\theta^r$ interpolating between θ^b and θ^r. Then

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$$\alpha_F(c_i) = c_i(\Omega^b),$$

$$\alpha_F(h_i) = \int_0^1 i(\partial/\partial t)c_i(\Omega^t)\,dt.$$

The c_i's on the right are the Chern monomials on the Lie algebra $gl_q\mathbf{R}$. If the normal bundle ν is trivial we define $\alpha_F: W_q \to A^*(M)$ as above but we now take $\Omega^t = t\theta^b + (1-t)\theta^f$, where θ^f is a flat connection on ν. The induced maps in cohomology are independent of all choices.

Bases for $H^*(WO_q)$ and $H^*(W_q)$, due to J. Vey [5], are given as follows.

$$\begin{array}{lll}
H^*(WO_q): & c_{2j_1}\cdots c_{2j_l} & 2j_1+\cdots+2j_l \le q \\
 & h_{i_1}\cdots h_{i_k}c_{j_1}\cdots c_{j_l} & i_1 < i_2 < \cdots \\
 & & j_1 \le j_2 \le \cdots \\
 & & i_1 \le \text{all odd } j \\
 & & i_1 + |J| = i_1 + j_1 + \cdots + j_l \ge q+1; \\
H^*(W_q): & h_{i_1}\cdots h_{i_k}c_{j_1}\cdots c_{j_l} & i_1 < i_2 < \cdots \\
 & & j_1 \le j_2 \le \cdots \\
 & & i_1 \le j_1 \\
 & & i_1 + |J| \ge q+1.
\end{array}$$

We shall write h_Ic_J for $h_{i_1}\cdots h_{i_k}c_{j_1}\cdots c_{j_l}$ and $h_Ic_J(F)$ for $\alpha_F(h_Ic_J)$. The classes $\alpha_F(c_{2j_1}\cdots c_{2j_l})$ are the Pontrjagin classes $p_{j_1}\cdots p_{j_l}(\nu)$ of ν. If $f: M_1 \to M$ is a smooth map transverse to F, then it induces a codimension q foliation f^*F on M_1 and the following diagram commutes.

$$\begin{tikzcd}
 & H^*(M) \arrow[d, "f^*"] \\
H^*(WO_q) \arrow[ur, "\alpha_F"] \arrow[r, "\alpha_{f^*F}"'] & H^*(M_1).
\end{tikzcd}$$

Finally note that the injection $i_q: WO_q \to W_q$ induces a map

$$i_q^*: H^*(WO_q) \to H^*(W_q).$$

In general, i_q^* is *not* injective. Indeed, it is not difficult to show that the kernel of i_q^* is generated by the $c_{2j_1}\cdots c_{2j_l}$ together with the classes h_Ic_J, where, if j_0 is the least even j in J, then $i_1 > j_0$ and $i_1 + |J| - j_0 > q$ (e.g., $h_3c_2^2 \in H^*(WO_4)$). If ν is a trivial bundle then the diagram

$$\begin{tikzcd}
H^*(W_q) \arrow[dr, "\alpha_F"] & \\
 & H^*(M) \\
H^*(WO_q) \arrow[uu, "i_q^*"] \arrow[ur, "\alpha_F"'] &
\end{tikzcd}$$

commutes. Thus the classes in ker i_q^* are obstructions to ν being a trivial bundle. As far as the author knows, there are no examples known where a class $\alpha_F(h_I c_J)$, $h_I c_J \in \ker i_q^*$, is non-zero.

3. Reduction to the case of flat SL_{q+1} bundles. Let F be a transverse (SL_{q+1}, S^q) foliation on M with associated cover $\{U_\alpha\}$, submersions $\{\varphi_\alpha\}$, and diffeomorphisms $\{g_{\alpha\beta}\}$. Consider the disjoint union $\mathfrak{U} = \dot{\bigcup} U_\alpha \times S^q$. We identify two points $(u_1, x_1) \in U_\alpha \times S^q$ and $(u_2, x_2) \in U_\beta \times S^q$, provided $u_1 = u_2$ and $x_1 = g_{\alpha\beta}(x_2)$. It is easy to check that with these identifications, $\mathfrak{U}$ becomes a flat SL_{q+1} bundle over M with fiber S^q. We denote this bundle by SM. Let F_s be the natural flat foliation on SM. Note that on $U_\alpha \times S^q$, F_s is just the foliation obtained from the point foliation on S^q by the projection.

On U_α define a cross section $\sigma_\alpha : U_\alpha \to SM$ by $\sigma_\alpha(u) = (u, \varphi_\alpha(u))$; as $\varphi_\alpha(u) = g_{\alpha\beta} \circ \varphi_\beta(u)$, these local cross sections are compatible and so define a global cross section σ of SM which is transverse to F_s. In particular $\sigma^*(F_s) = F$, so we have the commutative diagram

$$\begin{tikzcd} & H^*(SM) \arrow[d, "\sigma^*"] \\ H^*(WO_q) \arrow[ur, "\alpha_{F_s}"] \arrow[r, "\alpha_F"'] & H^*(M). \end{tikzcd}$$

The foliation F_s depends only on the flat structure on SM which in turn depends only on the holonomy homomorphism $h: \pi_1(M) \to \mathrm{SL}_{q+1}$ associated to the flat bundle SM. Thus the classes $\alpha_F(\gamma) = \sigma^* \alpha_{F_s}(\gamma)$ depend only on h and the cross section σ.

4. The Pontrjagin classes. Denote by $\tilde{M}$ the simply connected covering space of M. Then $SM = \tilde{M} \times_h S^q$. The normal bundle ν_s of F_s may be identified with the tangent bundle along the fiber of SM. Consider now the bundle $RM = \tilde{M} \times_h (\mathbf{R}^{q+1} \setminus \{0\})$ where SL_{q+1} acts on $\mathbf{R}^{q+1} \setminus \{0\}$ naturally.

The natural projection $\Pi: RM \to SM$ (along radial lines) induces an isomorphism $\Pi^*: H^*(SM) \to H^*(RM)$. The tangent bundle along the fiber of RM is a flat SL_{q+1} bundle so all of its Pontrjagin classes are zero. It is easy to see that this bundle is equivalent to the bundle $\Pi^* \nu_s \oplus 1$ where 1 is a trivial $\mathbf{R}$ bundle over RM (1 = tangent bundle along the fiber of Π). We have immediately that all the Pontrjagin classes of ν_s are zero. As $\nu = \sigma^* \nu_s$, we have that all the Pontrjagin classes of ν are zero, that is, $\alpha_F(c_{2j_1} \dots c_{2j_l}) = 0$ for all $c_{2j_1} \dots c_{2j_l} \in H^*(WO_q)$.

5. α_F is independent of σ. We now need only consider the classes $\alpha_F(h_I c_J) = h_I c_J(F)$. Denote integration over the fiber of SM by $⨍$.

LEMMA. *For all $h_I c_J \in H^*(WO_q)$, $⨍ h_I c_J(F_s) = 0$.*

Proof. Theorem 4.2 of [8] combined with Theorem 2.3 of [9] states that $⨍ h_I c_J(F_s)$ is a multiple of the Euler class of SM. As SM admits a section, the Euler class of SM is zero.

The Euler class and all the Pontrjagin classes of the bundle SM are zero. By the Serre spectral sequence, the map $\Pi^*: H^*(M) \to H^*(SM)$ induced by the projection $\Pi: SM \to M$ is injective, and $H^*(SM)$ is isomorphic to $H^*(M) \otimes H^*(S^q)$. An element $\eta \in H^*(SM)$ is in the image of Π^* if and only if $\fint \eta = 0$. Thus $h_I c_J(F_s)$ is in the image of Π^*. But

$$\begin{aligned}\sigma^*(h_I c_J(F_s)) &= h_I c_J(F) \\ &= \sigma^* \Pi^*(h_I c_J(F))\end{aligned}$$

and σ^* is one-to-one on $\Pi^*(H^*(M))$. Thus

$$h_I c_J(F_s) = \Pi^*(h_I c_J(F)),$$

and for any section σ_0 of SM,

$$h_I c_J(F) = \sigma_0^*(h_I c_J(F_s)).$$

Therefore $h_I c_J(F)$ depends only on the holonomy h of SM and not on the particular section σ.

6. End of the proof. We now show that $h_I c_J(F)$ depends only on the homotopy class of $h: \Pi_1 M \to \mathrm{SL}_{q+1}$. This completes the proof of the Theorem since Sullivan [13] has observed that there are only finitely many homotopy classes of homomorphisms of a finitely generated group to SL_{q+1}.

Denote by P the flat SL_{q+1} bundle $\Pi_0: \tilde{M} \times_h \mathrm{SL}_{q+1} \to M$. There is a natural map $\Pi_1: P \to SM$ induced from the map $\mathrm{SL}_{q+1} \to S^q$ given by sending g to $g(1, 0, \dots, 0)$. Denote by F_p the foliation on P induced from F_s by Π_1. The normal bundle ν_p of F_p is trivial (see below). Consider the following commutative diagram.

$$\begin{tikzcd}
H^*(W_q) \arrow[rr, "\alpha_{F_p}"] & & H^*(P) \\
 & & H^*(SM) \arrow[u, "\Pi_1^*"'] \\
H^*(WO_q) \arrow[uu, "i_q^*"] \arrow[uurr, "\alpha_{F_p}"] \arrow[urr, "\alpha_{F_s}"'] \arrow[rr, "\alpha_F"'] & & H^*(M) \arrow[u, "\Pi^*"'] \arrow[uu, bend right=60, "\Pi_0^*"']
\end{tikzcd}$$

By the Serre spectral sequence, Π_0^* is injective. Also,

$$\begin{aligned}\Pi_0^* h_I c_J(F) &= \Pi_1^* \Pi^* h_I c_J(F) \\ &= \Pi_1^* h_I c_J(F_s) \\ &= h_I c_J(F_p).\end{aligned}$$

Thus to show that $h_I c_J(F)$ depends only on the homotopy class of h, we need only show this for $h_I c_J(F_p)$.

The foliation F_0 on $\tilde{M} \times \mathrm{SL}_{q+1}$ induced from F_p may be described as follows. Let $\mathrm{SL}(1, q)$ be the subgroup of SL_{q+1} of matrices fixing $(1, 0, \dots, 0)$, that is, of the form

$$\begin{bmatrix} * & * & \cdots & * \\ 0 & * & & * \\ \vdots & \vdots & & \vdots \\ 0 & * & \cdots & * \end{bmatrix}.$$

F_0 is induced from the foliation of SL_{q+1} by the left cosets of $\mathrm{SL}(1,q)$. Thus F_0 has trivial normal bundle. Let $A_1,\ldots,A_q$ be vector fields on $\tilde{M}\times\mathrm{SL}_{q+1}$ tangent to SL_{q+1} and which come from the following left invariant vector fields on SL_{q+1}. A_l corresponds to the matrix $A_l\in sl_{q+1}$ whose only non-zero entry is $(A_l)_{l+1,1}=1$. The $A_1,\ldots,A_q$ provide a framing of the normal bundle ν_0 of F_0 and this framing descends to a framing of ν_P, which is thus trivial. To compute the map $\alpha_{F_p}\colon W_q\to A^*(P)$, we may work with F_0 on $\tilde{M}\times\mathrm{SL}_{q+1}$ provided we use objects which descend to P.

Let θ^f be the flat connection on ν_0 defined by requiring $A_1,\ldots,A_q$ to be a flat framing. Let θ^b be the basic connection on ν_0 whose covariant derivative satisfies

$$\nabla_X A_i=0 \qquad X\in T\tilde{M}$$

$$\nabla_X A_i=\rho[X,A_i] \qquad X\in T\,\mathrm{SL}_{q+1}$$

(recall $\rho\colon T(\tilde{M}\times\mathrm{SL}_{q+1})\to\nu_0$ is the projection). It is easily seen that both θ^f and θ^b descend to well-defined connections on ν_P, θ^f to a flat connection and θ^b to a basic connection. The connection matrix of θ^b computed with respect to the framing $A_1,\ldots,A_q$ consists entirely of one-forms on $\tilde{M}\times\mathrm{SL}_{q+1}$ which are the pullbacks of left invariant one-forms on SL_{q+1}. In particular it is a straightforward computation to show that the i,j entry of this matrix is $\omega_{i+1,j+1}-\delta^i_j\omega_{1,1}$ where the $\omega_{i,j}$ are the Maurier Cartan forms on SL_{q+1}. The connection matrix of θ^f with respect to the framing $A_1,\ldots,A_q$ is the zero matrix.

Denote the algebra of left invariant forms on SL_{q+1} by $A^*(sl_{q+1})$. The projection $\varphi\colon\tilde{M}\times\mathrm{SL}_{q+1}\to\mathrm{SL}_{q+1}$ induces $\varphi^*\colon H^*(sl_{q+1})\to H^*(\tilde{M}\times\mathrm{SL}_{q+1})$ and so also $\varphi_P^*\colon H^*(sl_{q+1})\to H^*(P)$. If we use θ^f and θ^b above to construct α_{F_0} and α_{F_P} we see immediately that $\alpha_{F_P}(H^*(W_q))\subset\varphi_P^*(H^*(sl_{q+1}))$.

Note that the constructions above are *purely* formal. Thus given $h_Ic_J\in H^*(W_q)$ there is a $y_{IJ}\in H^*(sl_{q+1})$ such that for any bundle P, $\alpha_{F_P}(h_Ic_J)=\varphi_P^*(y_{IJ})$.

As this is true universally, we may determine y_{IJ} by computing φ_P^* and α_{F_P} for a flat bundle P where φ_P^* is injective. This is done in [12] where it is shown that $\alpha_{F_P}(h_Ic_J)=0$ if $i_1+|J|>q+1$ and

$$\alpha_{F_p}(h_Ic_J)=(-1)^{|I|}\cdot\frac{c_{i_1}c_J(\mathrm{Id})}{c_{q+1}(\mathrm{Id})}\cdot\varphi_P^*(y_{I'}y_{q+1})$$

if $i_1+|J|=q+1$. Here $h_Ic_J=h_{i_1}\ldots h_{i_k}c_{j_1}\ldots c_{j_l}$, $|I|=k$, $c_{i_1}c_J(\mathrm{Id})$ and $c_{q+1}(\mathrm{Id})$ are the Chern monomials applied to the $q+1$-by-$q+1$ identity matrix, $H^*(sl_{q+1})=\Lambda(y_2,\ldots,y_{q+1})$, and $y_{I'}=y_{i_2}\ldots y_{i_k}$.

Define $\alpha\colon H^*(W_q)\to H^*(sl_{q+1})$ by

$$\alpha(h_Ic_J)=0 \qquad \text{if } i_1+|J|>q+1$$

$$\alpha(h_Ic_J)=(-1)^{|I|}\frac{c_{i_1}c_J(\mathrm{Id})}{c_{q+1}(\mathrm{Id})}y_{I'}y_{q+1} \qquad \text{if } i_1+|J|=q+1.$$

Then for any flat SL_{q+1} principal P we have

$$\alpha_{F_P}=\varphi_P^*\circ\alpha.$$

Suppose that F_0 and F_1 are two transverse (SL_{q+1}, S^q) foliations on M whose associated holonomy maps, h_0 and h_1, are homotopic. Let $h_t : \Pi_1(M) \to \mathrm{SL}_{q+1}$, $t \in [0,1]$ be a homotopy of h_0 to h_1. For each t, the flat principal bundle $\beta_t = \tilde{M} \times_{h_t} \mathrm{SL}_{q+1}$ is bundle isomorphic to P_0 by an isomorphism ψ_t which may be chosen to be continuous in t. Consider the diagram

$$\begin{tikzcd} H^*(sl_{q+1}) \arrow[rr, "\varphi_{P_t}"] & & H^*(P_t) \arrow[r, "\psi_t^*"] & H^*(P_0) \\ H^*(W_q) \arrow[u, "\alpha"] & H^*(WO_q) \arrow[l, "i_q^*"'] \arrow[rr, shift left, "\alpha_{F_1}"] \arrow[rr, shift right, "\alpha_{F_0}"'] & & H^*(M). \arrow[ul, "\Pi_t^*"'] \arrow[u, "\Pi_0^*"'] \end{tikzcd}$$

Theorem 2 of [3] implies that if $\beta_t : A^*(sl_{q+1}) \to A^*(P_0)$ is a family of differential graded algebra maps, then the induced maps $\beta_t^* : H^*(sl_{q+1}) \to H^*(P_0)$ are independent of t. Thus the maps $\psi_t^* \circ \varphi_{P_t}$ are independent of t and we have

$$\begin{aligned} \Pi_0^* \circ \alpha_{F_1} &= \psi_1^* \circ \Pi_1^* \circ \alpha_{F_1} \\ &= \psi_1^* \circ \varphi_{P_1} \circ \alpha \circ i_q^* \\ &= \psi_0^* \circ \varphi_{P_0} \circ \alpha \circ i_q^* \\ &= \Pi_0^* \circ \alpha_{F_0}. \end{aligned}$$

As Π_0^* is an injection, $\alpha_{F_0} = \alpha_{F_1}$ and the Theorem is proven. □

It has been pointed out by Haefliger [7] that Theorem 2 of [3] is a special case of an older and more general theorem, to wit:

THEOREM. *Let Γ be the pseudogroup generated by a semisimple Lie group G acting on a homogeneous space G/H. Then the continuous cohomology of $B\Gamma$ is rigid.*

Haefliger's proof goes as follows. Let K be a maximal compact subgroup of the closed subgroup H. Let F be a foliation on a manifold M transversely homogeneous of type G/H where $\dim G/H = n$. More generally let F be a Γ structure where Γ is the pseudogroup generated by G acting on G/H, that is, $\Gamma = G/H \times G^\delta$ ($G^\delta = G$ with the discrete topology). Then there is a commutative diagram

$$\begin{tikzcd} H^*(M; \mathbf{R}) & H^*(\mathfrak{g}, K) \arrow[l] \\ H^*(WO_n) \arrow[u] \arrow[r, phantom, "="] & H^*(\mathfrak{a}_n, O_n), \arrow[u] \end{tikzcd}$$

which can be interpreted as coming from the diagram

$$\begin{tikzcd} M \arrow[rr] \arrow[dr] & & B\Gamma \arrow[dl] \\ & B\Gamma_n. & \end{tikzcd}$$

Now $B\Gamma = EG^{\delta} \times_{G^{\delta}} G/H$ and has continuous cohomology $H^*(\mathfrak{g}, K)$ by the theorem of Van Est. Suppose F_t is a smooth family of Γ structures, and denote by $C^{\infty}(I, H^*(M; \mathbf{R}))$ the smooth families of cohomology classes on M parametrized by $I = [0,1]$. The family F_t induces the commutative diagram

$$\begin{tikzcd}
C^{\infty}(I, H^*(M;\mathbf{R})) \arrow[rrr, "d/dt\,|_0"] \arrow[dr] & & & H^*(M;\mathbf{R}) \\
& H^*(\mathfrak{g}, K; \mathbf{R}) \arrow[r, "\mathrm{var}"] & H^*(\mathfrak{g}, K; \mathfrak{g}') \arrow[ur] \arrow[dr] & \\
H^*(\mathfrak{a}_n, O_n; \mathbf{R}) \arrow[uu] \arrow[ur] \arrow[rrr, "\mathrm{var}"] & & & H^*(\mathfrak{a}_n, O_n; \mathfrak{a}_n'), \arrow[uu]
\end{tikzcd}$$

where the top horizontal arrow is the value of the derivative d/dt at 0, and the maps "var" are as defined in [4]. The $\mathfrak{g}$ module $\mathfrak{g}'$ is the dual of $\mathfrak{g}$ with the adjoint representation.

If G is semisimple and V is a $\mathfrak{g}$ module, then $H^*(\mathfrak{g}; V) \simeq H^*(\mathfrak{g}; V^{\mathfrak{g}})$ for V finite dimensional [10]. This implies that the inclusion $H^*(\mathfrak{g}, K; V^{\mathfrak{g}}) \to H^*(\mathfrak{g}, K; V)$ is also an isomorphism. To see this, apply one of the spectral sequence comparison theorems to the Koszul-Hochschild-Serre spectral sequence related to the reductive Lie subalgebra k, using that both $H^*(\mathfrak{g}; V^{\mathfrak{g}}) \to H^*(\mathfrak{g}; V)$ and $H^*(K; V^{\mathfrak{g}}) \to H^*(K; V)$ are isomorphisms. Setting $V = \mathfrak{g}'$ as $V^{\mathfrak{g}} = 0$, we have always $H^*(\mathfrak{g}, K; \mathfrak{g}') = 0$, and the theorem.

7. A remark and a conjecture. It would be very interesting to have explicit bounds for the $h_I c_J(F)$. In one special case we can give such a bound. Suppose $E \to M$ is a flat SL_{q+1} bundle with fiber S^q, and assume dimension $M = q+1$. Denote by F_E the natural flat foliation on E (F_E is then a transverse (SL_{q+1}, S^q) foliation). Denote by $[E]$ and $[M]$ the homology classes determined by E and M. If $h_i c_J \in H^{2q+1}(WO_q)$ we have, by [9].

$$\begin{aligned}
|h_i c_J(F_E)[E]| &= \left| \int_M \textstyle\int c_i c_J(F_E) \right| \\
&= \left| \int_M \frac{c_i c_J(\mathrm{Id})}{c_{q+1}(\mathrm{Id})} \cdot \chi(E) \right| \\
&= \left| \frac{c_i c_J(\mathrm{Id})}{c_{q+1}(\mathrm{Id})} \cdot \chi(E)[M] \right|.
\end{aligned}$$

Thus for $q+1$ odd we have $h_i c_J(F_E)[E] = 0$, so $h_i c_J(F_E) = 0$. For $q+1$ even, the Theorem of Milnor-Sullivan (as improved by Smillie; see [6]) gives

$$|\chi(E)[M]| \leq \frac{1}{2^{q+1}} \|M\|,$$

where $\|M\|$ is Gromov's simplicial volume of M. Thus for the situation described above,

$$|h_i c_J(F_E)[E]| \leq \frac{1}{2^{q+1}} \left| \frac{c_i c_J(\mathrm{Id})}{c_{q+1}(\mathrm{Id})} \right| \cdot \|M\|.$$

Note that if one such $h_i c_J(F_E)=0$ then $\chi(E)=0$ and all such $h_i c_J(F_E)=0$ (and vice versa).

In [3] a bound is given for $h_1 c_1 \in H^3(WO_1)$ for certain transverse (PSL_2, S^1) foliations. This bound suggests the following. Suppose $q+1$ is even and M is a $q+1$ dimensional manifold whose tangent bundle is SL_{q+1} flat. Let F_1 be the natural transverse (SL_{q+1}, S^q) foliation on the unit tangent bundle T^1M of M. Let E and F_E be as above and suppose F is *any* transverse (SL_{q+1}, S^q) foliation on E.

CONJECTURE. For $h_i c_J \in H^{2q+1}(WO_q)$,

$$|h_i c_J(F)[E]|\,|h_i c_J(F_E)[E]| \le |h_i c_J(F_1)[T^1M]|^2.$$

If we set $F=F_E$ and apply the results of [9], we obtain:

COROLLARY TO THE CONJECTURE:

$$|\chi(E)[M]| \le |\chi(TM)[M]| = |\chi(M)|.$$

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