SETS IN E^3 THAT LOCALLY LIE ON FLAT SPHERES

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1. Introduction. The ϵ-boundary, $\partial(\epsilon, X)$, of a subset X of Euclidean 3-space E^3 is the set $\{p \mid d((p, X) = \epsilon\}$, where d is the usual metric for E^3 and ϵ is a positive number. Following earlier discoveries by Brown [1] and Gariepy and Pepe [6], Ferry [5] proved that ϵ-boundaries of sets in E^3 are 2-manifolds for almost all ϵ. As a corollary to our main result in [4], Daverman and I showed that $\partial(\epsilon, X)$ was locally flat at each point where it was known to be a 2-manifold, thus answering a question raised by Weill [8, p. 248]. This paper addresses the local flatness of $\partial(\epsilon, X)$ where it is of dimension one.

Let A be a 1-dimensional subset of E^3 such that $A = \partial(\epsilon, X)$ for some $\epsilon > 0$ and some subset X of E^3. It follows that $E^3 = N(X, \epsilon) \cup A$, where $N(X, \epsilon) = \{p \in E^3 \mid d(X, p) < \epsilon\}$, because otherwise $\partial(\epsilon, X)$ would separate E^3, contradicting the 1-dimensionality of A. In this way one sees, as a corollary to the main theorem, that 1-manifolds, and 1-dimensional sets in general, locally lie on flat 2-spheres when they are realized as the ϵ-boundary of some set.

A 2-sphere Σ in E^3 is said to be *flat,* or flatly embedded, in E^3 if there exists a homeomorphism of E^3 onto itself that takes Σ onto the unit 2-sphere. A set A is said to *locally lie on a flat 2-sphere* in E^3 if, for each $p \in A$, there exist a flat 2-sphere Σ and a neighborhood U of p in A such that $U \subset \Sigma$.

THEOREM. *If A is the ϵ-boundary of a subset X of E^3 for some $\epsilon > 0$, and $E^3 = A \cup N(X, \epsilon)$, then A must locally lie on a flat 2-sphere in E^3.*

Some remarks and examples seem appropriate before presenting the proof of this theorem. Since one can realize a knot as the ϵ-boundary of a set in E^3 it is clear that the conclusion cannot be improved by removing "locally", even when A is a compact 1-manifold. The hypothesis that $E^3 = A \cup N(X, \epsilon)$ is also essential even for the weaker conclusion that A locally lie on a 2-sphere. For an example, let A be obtained by rotating $W \cup Y \cup Z$ about the x-axis where W is the segment $[(-1,0),(0,1)]$ in E^2, $Y = \{(x,y) \mid (x-1)^2 + (y-1)^2 = 1$ and $x, y \in [0,1]\}$, and $Z = [(1,0),(2,0)]$. Let $\epsilon = 1$ and define $X = \{p \in E^3 \mid d(A, p) \geq 1\}$. Then $A = \partial(1, X)$, yet no 2-sphere can contain a neighborhood of $(1,0,0)$ in A.

In this example, and whenever $A = \partial(\epsilon, X)$, it is easily seen that for each $q \in A$ there exists a point $x \in \mathrm{cl}(X)$ and a ball B of radius ϵ, centered at x, such that $A \cap \mathrm{Int}\, B = \varnothing$ and $q \in \mathrm{Bd}\, B$. Such a ball is said to be *tangent to A at q.* Daverman and I [4] proved that a 2-manifold topologically embedded in E^3 so as to have such uniform-sized, tangent balls on one side at each of its points would have to be flatly embedded in E^3. As a corollary we proved that a 2-manifold which is realized as an ϵ-boundary like the rotation of $W \cup Y$ above, must locally lie on a

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flat 2-sphere. An obvious corollary to the theorem in this paper is the analogous result for 1-dimensional sets.

COROLLARY. *If a* 1*-dimensional set A in E^3 is the ϵ-boundary of a set X, for some $\epsilon > 0$, then A locally lies on a flat 2-sphere.*

Perhaps the results here can be obtained from existing theorems or their proofs. A subset A of E^3 is known to locally lie on a flat 2-sphere when A has uniform double tangent balls (see Theorem 3.1 of [7]). A set A is said to have *uniform double tangent balls* if there is a positive number ϵ such that, for each $q \in A$, there exist two balls B_1 and B_2 each of radius ϵ such that $B_1 \cap B_2 = \{q\}$ and $A \cap (\mathrm{Int}(B_1 \cup B_2)) = \varnothing$. I was not able to show that these ϵ-boundary results are consequences of existing theorems on double tangent ball embeddings, although the proofs here often run parallel to those in [7].

A good general reference to embeddings of 2-spheres in E^3 is the survey article [2] by Burgess and Cannon. The references [1], [4], [5], and [6] relate to ϵ-boundaries, while [3], [4], and [7] give more background on embeddings of sets with various types of tangent balls. However, this paper is generally self-contained relative to its proofs and definitions.

2. Proof of the theorem. Let $\epsilon > 0$, let A be the ϵ-boundary of a subset X of E^3 such that $E^3 = A \cup N(X, \epsilon)$, and let p be an arbitrary point of A. The objective, to show the existence of a flat 2-sphere Σ in E^3 and a neighborhood U of p in A such that $U \subset \Sigma$, is easily achieved if p is an isolated point of A. In the sequel it will be assumed that p is a limit point of A and that, since $\partial(\epsilon, X) = \partial(\epsilon, \mathrm{cl}(X))$, X is closed. It follows that A is also a closed subset of E^3. For each $q \in A$ let S_q be the sphere of radius ϵ centered at q, let $X \cap S_q$ be denoted by N_q, and let $\mathcal{B}_q$ be the set of all balls of radius ϵ that have their centers in N_q. Lemma 1 restricts the centers of tangent balls to A at p to the intersection of certain hemispheres of S_p. Its proof is not difficult.

LEMMA 1. *If $\{p_i\}$ is a sequence of points of A converging to p such that the sequence $\{R_i\}$ of rays from p through p_i converges to a ray R, then N_p lies in the closed hemisphere of S_p opposite R.*

Since p is a limit point of A one can deduce from Lemma 1 that some hemisphere of S_p contains N_p. By Lemma 3 below there can be no hemisphere of S_p whose interior contains N_p.

The ray from a point q through another point x is denoted by $R(q,x)$ or by $\overrightarrow{qx}$. If two rays R_1 and R_2 have the same initial point, $\theta(R_1, R_2)$ is the degree measure of the smaller angle between R_1 and R_2. Similarly, if B_1 and B_2 are balls in $\mathcal{B}_q$, $\theta(B_1, B_2)$ is defined to be $\theta(R_1, R_2)$, where R_i is the ray from q to the center of B_i. Used frequently in the sequel, Lemma 2 appears here primarily because its proof is similar to that of Lemma 3.

LEMMA 2. *If the hypothesis of Lemma* 1 *is satisfied and H is the hemisphere of S_p opposite R, then* $\limsup N_{p_i} \subset \mathrm{Bd}\, H$.

Proof. Let $x \in \limsup N_{p_i}$. By choosing a subsequence if necessary I may assume $\{x_i\}$ is a sequence of points converging to x such that $x_i \in N_{p_i}$ for each i. For each i, let f_i be the point of R_i such that $\theta(R_i, R(f_i, x_i)) = 90°$; that is, f_i is the foot of x_i on R_i. Because $x \in N_p$ it follows from Lemma 1 that $x \in H$. This makes it clear that $\{f_i\}$ converges to p. Since $\{R_i\}$ converges to R, $\{R(f_i, x_i)\}$ converges to $R(p, x)$, and R_i is perpendicular to $R(f_i, x_i)$ for each i, it follows that $\theta(R, R(p, x)) = 90°$. This means $x \in \mathrm{Bd}\, H$. □

LEMMA 3. *Every closed hemisphere of S_p intersects N_p.*

Proof. Let H be a closed hemisphere of S_p, and let T be the ray beginning at p such that H is symmetric with respect to T and H intersects T. Since A contains no open subset of E^3 there is a sequence $\{t_i\}$ of points of $E^3 - A$ converging to p such that the sequence $\{\overrightarrow{pt_i}\}$ of rays T_i converges to T. Because $E^3 - A \subset N(X, \epsilon)$, each t_i lies in a ball B_i of radius ϵ whose center x_i lies in X and whose interior misses A. By choosing subsequences if necessary I may assume $\{x_i\}$ converges to a point x, and it is clear that $x \in N_p$. For each i let f_i be the foot of x_i on the line through T_i, and note that, since $t_i \in B_i$ and $p \notin \mathrm{Int}\, B_i$, f_i belongs to T_i. Because $\{T_i\}$ converges to T and $\{x_i\}$ converges to x, $\theta(T, R(p, x)) \le 90°$, and it follows that $x \in H$. □

LEMMA 4. *For each $q \in A$ there exist balls B and B' in B_q such that $\theta(B, B') \ge 90°$.*

Proof. Since $q \in \partial(\epsilon, X)$ there exists a ball $B \in \mathcal{B}_q$. Let R be the ray from q through the center of B, and let H be the hemisphere of S_q opposite R. By Lemma 3 there is a point x of N_q in H, so let B' be the ball of $\mathcal{B}_q$ centered at x. □

To help motivate the two cases now examined to complete the proof, recall that N_p cannot lie in the interior of the hemisphere of S_p (Lemma 3) but that N_p must lie in a hemisphere of S_p (Lemma 1). Case 1 includes the situation where there are two hemispheres H_1 and H_2 of S_p such that $N_p \subset H_1 \cap H_2$ and $\mathrm{Bd}\, H_1 \neq \mathrm{Bd}\, H_2$. In this situation $\mathrm{Bd}\, H_2 \cap \mathrm{Bd}\, H_1$ would consist of two points c_1 and c_2 that are antipodal on S_p, and, since the closed set N_p does not lie in the interior of a hemisphere of S_p, c_1 and c_2 would lie in N_p. Furthermore, each hemisphere H of S_p that contains N_p would also contain a closed semicircle $\widehat{c_1 c_2}$ with endpoints c_1 and c_2 such that $N_p \cap \mathrm{Bd}\, H \subset \widehat{c_1 c_2}$. Although Case 1 includes the situation just described, it is not limited to just this. In each of the two mutually exclusive cases that follow there can exist antipodal points of S_p that belong to N_p.

Case 1. This is the case where there exist antipodal points c_1 and c_2 of S_p such that, for every hemisphere H of S_p containing N_p, $N_p \cap \mathrm{Bd}\, H$ lies in a closed semicircle $\widehat{c_1 c_2}$ of $\mathrm{Bd}\, H$. As mentioned above, it follows from Lemmas 1 and 3 that $\{c_1, c_2\} \subset N_p$. Let L be the line through $\{c_1, c_2, p\}$, let B_1 be a ball in $\mathcal{B}_p$ with its center at c_1, and choose a point x on L between p and c_1 such that a ball B in $\mathcal{B}_p$ must contain x in its interior whenever $\theta(B, B_1) \le 89°$. The object is to show the existence of an open subset U of A containing p such that, for every $q \in U$, there is a ball in $\mathcal{B}_q$ with x in its interior. Once this is proved, B^* is identified as the union of all balls in $\bigcup \{\mathcal{B}_q \mid q \in U\}$, so that Bd B^* is star-like from x. Then Bd B^* is the desired flat 2-sphere that contains U.

Suppose no such open set exists. Then there must be a sequence $\{q_i\}$ of points of A converging to p such that, for each i, $\mathcal{B}_{q_i}$ contains no ball with x in its interior. For each i, let R_i be the ray from p through q_i, and, for convenience (choose an appropriate subsequence of $\{q_i\}$ if necessary), assume $\{R_i\}$ converges to a ray R. By Lemma 1, N_p must lie in the hemisphere H of S_p opposite R. Let $\widehat{c_1c_2}$ denote a closed semicircle of Bd H that contains $N_p \cap \operatorname{Bd} H$. By Lemma 2 $\limsup N_{q_i} \subset \widehat{c_1c_2}$. Once it is proved that, for sufficiently large n, $\mathcal{B}_{q_n}$ contains two balls B_n and B_n' such that $\theta(B_n, B_n') > 91°$, a contradiction to the existence of $\{q_i\}$ can be obtained. This is because the centers of B_n and B_n' can be made so close to $\widehat{c_1c_2}$ that one of B_n and B_n', say B_n, would have the property that $\theta(B_n, B_1) \leq 89°$ and hence would have x in its interior.

Fix n and let V and W be two balls in $\mathcal{B}_{q_n}$ such that $\theta(V, W) \geq 90°$ (see Lemma 4). If $\theta(V, W) > 91°$, the objective is achieved, so suppose $\theta(V, W) \leq 91°$. Let C_n be the circle centered at q_n that contains the centers v and w of V and W (respectively), let t and t' be the two points of C_n equidistant from v and w with t' on the minor arc of $\widehat{vw}$, and let T be the ray from q_n through t. By Lemma 3 there must exist a ball B_n in $\mathcal{B}_{q_n}$ such that $\theta(B_n, t)$, the measure of the angle at q_n subtended by t and the center b_n of B_n, is no larger than $90°$. If b_n were known to lie on or near C_n, it would follow that either $\theta(V, B_n)$ or $\theta(W, B_n)$ would be larger than $180 - \theta(V, W)/2$, which would exceed $134.5°$. Then one of V or W could be chosen as B_n' so that $\theta(B_n, B_n') > 91°$.

However, $\{C_n\}$ converges to Bd H because $\limsup N_{q_n} \subset \widehat{c_1c_2}$. This means that for sufficiently large n, b_n lies near enough to C_n that the required balls B_n and B_n' can be obtained with x in the interior of one of them.

Case 2. In this case there do not exist antipodal points of S_p as in Case 1; that is, for every pair of antipodal points c_1 and c_2 of S_p there must exist a corresponding hemisphere H of S_p such that $N_p \subset H$ and $N_p \cap \operatorname{Bd} H$ fails to lie in a semicircle $\widehat{c_1c_2}$. Let H be a hemisphere of S_p containing N_p (Lemma 1), and impose a coordinate system on E^3 with p the origin, Bd H in the xy-plane, and such that the positive z-axis fails to intersect H. The immediate objective is to show the existence of a positive number δ and a solid Z, obtained by revolving the disk $\{(0, y, z) \mid (y-\delta)^2 + z^2 \leq \delta^2\}$ about the z-axis, such that $Z \cap A = \emptyset$. To accomplish this the hemisphere H may need changing.

To illustrate the choice of δ first consider the case where $(\operatorname{Bd} H) \cap N_p$ contains antipodal points c_1 and c_2. Let A_{12} and A_{12}' be the two open semicircles of Bd H each having endpoints c_1 and c_2. If there are points c_3 and c_4 of $A_{12} \cap N_p$ and $A_{12}' \cap N_p$, respectively, then the union of the four balls of N_p with centers c_1, c_2, c_3, and c_4 would contain the desired solid Z. On the other hand if one of A_{12} and A_{12}', say A_{12}, fails to intersect N_p, then, by the stipulations of this case, there would be a hemisphere H' of S_p such that $N_p \subset H'$ and both open semicircles of Bd H' with endpoints c_1 and c_2 would intersect N_p. In this situation H' could be renamed H and δ and Z chosen as before. Finally, consider the case where there are no antipodal pairs of points of $N_p \cap \operatorname{Bd} H$. Then every open semicircle of $\operatorname{Bd} H$ must intersect N_p as I now show. Suppose α is an open semicircle of $\operatorname{Bd} H$ such that $\alpha \cap N_p = \varnothing$, and let c_1 and c_2 be its endpoints with c_2 not belonging to the closed set N_p. Rotate α around $\operatorname{Bd} H$ slightly if necessary to obtain a closed semicircle α' and a positive number $*$ such that the regular neighborhood $N(\alpha', *)$ fails to intersect N_p. Now rotate H slightly about the line through the endpoints of α', moving the interior of α' only within $N(\alpha', *)$, to obtain a hemisphere H' of S_p with N_p in its interior. This contradicts Lemma 3 and proves that each open semicircle of $\operatorname{Bd} H$ must intersect N_p. Then to choose δ and Z, let $c_1 \in N_p \cap \operatorname{Bd} H$, and let α_1 be the open semicircle of $\operatorname{Bd} H$ opposite c_1. There must exist a point c_2 in $\alpha_1 \cap N_p$ and c_2 is not the antipode of c_1. Let α_2 be the open semicircle of $\operatorname{Bd} H$ opposite the midpoint of the minor arc $\widehat{c_1 c_2}$, and let c_3 be a point of $\alpha_2 \cap N_p$. By construction, the union of the three balls of $\mathcal{B}_p$ centered at the three points c_1, c_2, and c_3 must contain a solid Z as desired.

For convenience in writing assume $\delta < \epsilon/2$, and, for each t, let P_t denote the horizontal plane defined by $z = t$ and let M denote the union of all balls of radius $\delta/2$ whose centers lie on the circle $\{(x, y, z) \mid x^2 + y^2 = (\delta/2)^2 \text{ and } z = 0\}$. Then $M \subset Z \subset \bigcup \mathcal{B}_p$, so $M \cap A = \varnothing$. For $t \in [-\delta/2, \delta/2]$ and $t \neq 0$, let G_t denote the open circular disk in P_t whose center is $(0, 0, t)$ such that $G_t \cap M = \operatorname{Bd} G_t$. Let $G = \bigcup \{G_t \mid t \neq 0 \text{ and } -\delta/2 < t < \delta/2\}$; that is, G is the union of two open 3-cells each shaped like a trumpet. The remainder of the proof in Case 2 is based on a sequence of nine lemmas whose hypotheses include unstated conditions previously established.

LEMMA 2.1. *There is a positive number u such that if $0 < |t| < u$, $q \in G_t \cap A$, and $B \in \mathcal{B}_q$, then the disk $P_t \cap B$ has radius greater than δ.*

Lemma 2.1 follows easily from Lemma 2. However, to see that Lemma 2 applies to H note that there cannot be two hemispheres of S_p containing N_p unless they share the same boundary (see the remarks preceding Case 1). Thus $\operatorname{Bd} H$ is unique with respect to being the boundary of every hemisphere of S_p containing N_p.

A disk D is said to be a *normal disk* at a point q of $A \cap P_t$ if D lies in P_t, D has radius δ, $q \in \operatorname{Bd} D$, and there is a ball B in $\mathcal{B}_q$ such that $D \subset B \cap P_t$. A ray R is called a *normal ray* at a point q of $A \cap P_t$ if q is the endpoint of R and R contains the center of a normal disk at q. A line is said to be *normal* at q if it contains a normal ray at q. If D_1 and D_2 are two normal disks at q, $\theta(D_1, D_2)$ is the degree measure of the smaller angle subtended at q by the centers of D_1 and D_2. Similarly $\theta(B_1 \cap P_t, B_2 \cap P_t)$ measures the smaller angle subtended at q by the centers of the disks $B_1 \cap P_t$ and $B_2 \cap P_t$, where $B_1, B_2 \in \mathcal{B}_q$.

For $0 < |t| < u$ and $q \in A \cap P_t$, let C_q be the circle in P_t of radius δ whose center is q, and let M_q be the set of centers of all normal disks at q. Then $M_q \subset C_q$.

LEMMA 2.2. *If $0 < |t| < u$ and $q \in A \cap P_t$, then every semicircle of C_q intersects M_q.*

Proof. Let S be a semicircle of C_q, and let R be the ray in P_t opposite S. By Lemma 3 there is a ball B in B_q whose center lies in the hemisphere of S_q opposite R. Since B is tangent at q to the line through q and the endpoints of S, it follows from Lemma 2.1 that $B \cap P_t$ contains a normal disk whose center lies in S. □

LEMMA 2.3. *If* $0 < |t| < u$, $q \in A \cap P_t$, *and* $\gamma = 2\delta \sin 0.5°$, *then either*
(i) *there exist two balls* B_1 *and* B_2 *in* $\mathcal{B}_q$ *such that* $\theta(B_1 \cap P_t, B_2 \cap P_t) \geq 179°$; *or*
(ii) *there exist three balls* B_1, B_2, *and* B_3 *in* $\mathcal{B}_q$ *whose union contains a circular disk in* P_t *with radius* γ *and center* q.

Proof. Suppose condition (i) fails for some appropriate choices of t and q, and let B_1 and B_2 be balls of $\mathcal{B}_q$ such that $\theta(D_1, D_2)$ is as large as possible, where D_1 and D_2 are normal disks at q lying in $B_1 \cap P_t$ and $B_2 \cap P_t$, respectively. By the supposition, $\theta(D_1, D_2) < 179°$, and by Lemma 2.2, $D_1 \neq D_2$. Let c_1 and c_2 be the centers of D_1 and D_2, respectively, and let S be the closed semicircle of C_q opposite the bisecting ray of $\measuredangle c_1 q c_2$. Using Lemma 2.2 and incorporating the maximality of $\theta(D_1, D_2)$, one can deduce that $120° \leq \theta(D_1, D_2)$.

By Lemma 2.2 there is a normal disk D_3 whose center lies in S, and, since $120° \leq \theta(D_1, D_2) < 179°$ and $\theta(D_1, D_2)$ is maximal, it is clear that the union of D_1, D_2 and D_3 contains a circular disk centered at q. To detect its radius notice that $D_1 \cap D_2$ contains a segment with q as one endpoint whose length h is $2\delta \cos(\theta(D_1, D_2)/2) = 2\delta \sin K/2$, where K is the angle subtended at c_2 by the segment. The length h decreases as $\theta(D_1, D_2)$ approaches $179°$, so γ is a lower bound for $2\delta \sin K/2$. Since $\theta(D_3, D_i) \leq \theta(D_1, D_2)$ for $i = 1, 2$, both $D_1 \cap D_3$ and $D_2 \cap D_3$ contain segments of length no less than γ, and condition (ii) follows. □

A line L in a plane P_t is called a *projective line* if no line in P_t parallel to L, including L itself, meets $A \cap G_t$ in two points. Not every normal line at q is a projective line, but Lemma 2.5 states that any line close to what is called a double normal at q must be a projective line. A *double normal* W_q at q is the union of two normal rays R and R' at q such that $\theta(R, R') \geq 179°$. For two coplanar lines L and M, $\theta(L, M)$ is the degree measure of the smaller of the two angles made by $L \cup M$ or $0°$ if $L \cap M = \emptyset$. If W_x is a double normal $R \cup R'$ at x in P_t and K is a line in P_t, define $\theta(W_x, K) = \max(\theta(L, K), (\theta(M, K))$ where L and M are the two lines containing R and R'. Finally, if W_y is a double normal at y in P_t and W_x is as above, define $\theta(W_x, W_y) = \max(\theta(W_y, L), \theta(W_y, M))$ where L and M are as before.

The immediate goal is to identify a number u_2 and a corresponding continuous family $\{L_t \mid 0 < |t| < u_2\}$ of projective lines. This is completed with Lemma 2.8. The proof of Lemma 2.9 shows how to rotate the segments $\{L_t \cap G_t \mid 0 < |t| < u_2\}$ using a space homeomorphism h so that the orthogonal projection of $h(\{A \cap G_t \mid 0 < |t| < u_2\})$ into the yz-plane is one-to-one.

Choose a positive number u_1 such that $u_1 < u$ and, for $0 < |t| < u_1$, G_t has diameter less than $\gamma/2$. If $|t| < u_1$ and $q \in A \cap P_t$, then it follows from Lemma 2.3 that either $G_t \cap A = \{q\}$ or there exists a double normal at q.

LEMMA 2.4. *There exists* $u_2 \in (0, u_1)$ *such that if* $0 < |t| < u_2$ *and* P_t *contains double normals* W_x *and* W_y *at two distinct points* x *and* y (*respectively*) *of* $A \cap G_t$, *then* $\theta(W_x, W_y) < 5°$.

Proof. Choose $u_2 \in (0, u_1)$ such that if c is any point of P_t, $0 < |t| < u_2$, at a distance δ or more from some point of G_t, then G_t subtends an angle of measure less than $3°$ at c. Let t be given such that $|t| < u_2$, $G_t \cap A$ contains two distinct points x and y, and P_t contains double normals W_x and W_y at x and y, respectively. Suppose $\theta(W_x, W_y) \geq 5°$. Then $W_x \cap W_y$ contains at least one point b, for otherwise $\theta(W_x, W_y)$ could be at most $2°$. Suppose $d(b, y) \geq \delta$. It follows from the definition of u_2 that the measure of $\sphericalangle ybx$ is less than $3°$; therefore, if R and S are the rays of W_x and W_y (respectively) that contain b, then $\theta(R, S) < 3°$. But then $\theta(W_x, W_y) < 5°$, contradicting the first supposition. Therefore $d(b, y) < \delta$, and, by similar argument, $d(b, x) < \delta$. For convenience assume $d(b, x) \leq d(b, y)$, so that x lies in the disk Y of radius $d(b, y)$ centered at b. Let D_1 and D_2 be normal disks at y corresponding to the two normal rays at y whose union is W_y. Since $d(b, y) < \delta$ and b is on a ray of W_y, it follows that Y must be a subset of D_1 or D_2, say D_1. From Lemma 2.1 there is a ball B_1 in $\mathcal{B}_y$ such that $D_1 \subset P_t \cap B_1$ and $P_t \cap B_1$ has radius larger than δ. This means $D_1 \cap A = \{y\}$, contradicting the facts that x lies in D_1 and $x \neq y$. □

LEMMA 2.5. *If* $0 < |t| < u_2$, $q \in G_t \cap A$, W_q *is double normal at* q *in* P_t, *and* L *is a line in* P_t *such that* $\theta(W_q, L) \leq 60°$, *then* L *is a projective line.*

Proof. If q is the only point of $G_t \cap A$, then every line in P_t is projective. In the other case it follows from Lemma 2.3 and the choice of u_1 that there is a double normal W_x at each point x of $A \cap G_t$. Let L' be a line in P_t parallel to L such that L' contains a point x of $G_t \cap (A - \{q\})$, and let W_x be a double normal at x. By Lemma 2.4, $\theta(W_x, W_q) < 5°$, and, from the hypothesis, $\theta(W_x, L') < 5° + 60° = 65°$. The normals rays at x in W_x contain the centers c and c' of two normal disks D and D' at x. If N is the normal ray at x through c, then, by definition, $\theta(L', N) \leq \theta(W_x, L') = 65°$. Therefore the chord $L' \cap D$ subtends an angle at c of measure larger than $180° - 2(65°) > 5°$. Then by the definition of u_2 in the first line of the proof of Lemma 2.4 and a similar analysis relative to $L' \cap D'$, $L' \cap G_t \subset D \cup D'$. Therefore $L' \cap (A \cap G_t) = \{x\}$, and L' is projective. □

LEMMA 2.6. *If* $0 < |t| < u_2$, $q \in A \cap G_t$, *and* W *and* W' *are double normals at* q *such that* $\theta(W, W') \geq 5°$, *then the disk in* P_t *centered at* q *and having radius* $2\delta \sin 2°$ *intersects* A *only at* q.

Proof. Let R, T, R', and T' be rays such that $W = R \cup T$ and $W' = R' \cup T'$, and for convenience assume $\theta(W, W') = \theta(R, R')$. Then $5° \leq \theta(R, R') \leq 90°$. Let D and E be normal disks at q centered on R and T, respectively, and let D' and E' be normal disks at q centered on R' and T', respectively. Since $\theta(D, E') \leq 176°$, $\theta(D', E) \leq 176°$, and D' and E' are centered on opposite sides of the line through R, trigonometry reveals the existence of a disk in $D \cup D' \cup E \cup E'$ centered at q whose radius can be chosen as large as $2\delta \cos(176°/2) = 2\delta \sin 2°$. □

Let $J = \{q \in A \mid \text{there exist double normals } W \text{ and } W' \text{ at } q \text{ such that } \theta(W, W') \geq 5°\}$. If $q \in J$ and $|t| < u_2$ it follows from Lemma 2.6 and the choice of u_1 that $G_t \cap A = \{q\}$. Let $T = \{t \mid t \in [-u_2, u_2] \text{ and } G_t \cap J \neq \varnothing\}$, so that $G_t \cap A$ is a singleton set whenever $t \in T$. It follows that if $t \notin T$, $|t| < u_2$, and W and W'

are double normals at a point q of $A \cap G_t$, then $\theta(W, W') < 5°$. Let DN be $[-u_2, u_2] \cap \{t \mid \text{there exists a double normal at some point of } G_t \cap A\}$. Then $T \subset DN$.

A family $\{L_w \mid w \in (r,s)\}$ of lines L_w in P_w is said to be continuous provided $\{L_{t_i}\}$ converges to L_t whenever $\{t_i\}$ converges to t in the interval (r,s).

LEMMA 2.7. *If $|t| < u_2$ and $t \in (DN - T)$, then there exist an open interval (r,s) containing t, a continuous family $\{L_w \mid w \in (r,s)\}$ of projective lines, and a family $\{W_w \mid w \in (r,s) \cap DN\}$ of double normals such that, whenever $w \in (r,s) \cap DN$, $L_w \cup W_w \subset P_w$ and $\theta(L_w, W_w) < 15°$.*

Proof. Let q be a point of $A \cap G_t$ such that W_t is a double normal at q, and suppose there exists a sequence $\{t_i\}$ converging to t such that each P_{t_i} contains a double normal W_i such that $\theta(W_t, \rho(W_i)) \geq 15°$, where ρ denotes the vertical projection of E^3 onto P_t. Then the limiting set of $\{W_i\}$ in P_t contains a double normal W where $\theta(W, W_t) \geq 15°$. If W is not a double normal at q, then Lemma 2.4 is contradicted. On the other hand if W were a double normal at q, then q would belong to J and t to T—another contradiction. Since no such sequence $\{t_i\}$ exists there must be an interval (r,s) containing t and a family $\{W_w \mid w \in (r,s) \cap DN\}$ of double normals such that $W_w \subset P_w$ and $\theta(W_t, \rho(W_w)) < 15°$ for each w in $(r,s) \cap DN$. Let N_t be a line in P_t containing one of the two normal rays whose union is W_t, and define $L_w = P_w \cap \rho^{-1}(N_t)$ for each $w \in (r,s)$. From Lemma 2.5, L_w is a projective line for each $w \in (r,s) \cap DN$, and L_w is projective for $w \notin DN$ since then every horizontal line is projective (see Lemma 2.3). □

Let T' be the union of all points of $[-u_2, u_2]$ that are either endpoints of T or in point-components of T, and notice that T' is closed. It is convenient to also include $-u_2$ and u_2 in T', so that each component of $[-u_2, u_2] - T'$ is an open interval.

LEMMA 2.8. *For each component V of $[-u_2, u_2] - T'$, there exists a continuous family $\{L_w \mid w \in V\}$ of projective lines.*

Proof. If $V \subset T$, define L_w to be the intersection of P_w with the xz-plane P for all $w \in V$. Then L_w is projective because $G_w \cap A$ is a singleton for each $w \in V$. Otherwise, where $V \cap T = \varnothing$, define, for each $t \in V$, an open interval (r_t, s_t) and a continuous family $\{L_w \mid w \in (r_t, s_t)\}$ of projective lines such that $t \in (r_t, s_t) \subset V$ as follows. If $t \notin DN$, then either $G_t \cap A = \varnothing$ or $G_t \cap A$ is a singleton (Lemma 2.3). Since both A and DN are closed, there must exist an open interval (r_t, s_t) in $V - DN$ such that $G_w \cap A$ contains at most one point for each $w \in (r_t, s_t)$. In this case define L_w as $P_w \cap P$. If $t \in DN$, then $t \in DN - T$ and Lemma 2.7 applies to identify (r_t, s_t) and corresponding families $\{L_w\}$ of projective lines and $\{W_w\}$ of double normals, where $\theta(L_w, W_w) < 15°$ for $w \in (r_t, s_t) \cap DN$.

Since $Q = \{(r_t, s_t) \mid t \in V\}$ is an open cover for V, one can mark an increasing sequence $\{s_i\}$ converging to s such that $s_i \in V$ for each i and each of the intervals $[s_{i-1}, s_i]$ lies in an open interval Q_i of Q. The objective, to fit two families of projective lines together at a point s_i where $[s_{i-1}, s_i]$ and $[s_i, s_{i+1}]$ overlap, is easily accomplished when $s_i \notin DN$. This is because in one of Q_i and Q_{i+1}, say Q_i, all horizontal directions are projective so that, near P_{s_i}, the family $\{L_w \mid w \in Q_i\}$ can be continuously rotated to match $\{L_w \mid w \in Q_{i+1}\}$ at s_i. In the more difficult case where $s_i \in DN$, both Q_i and Q_{i+1} were obtained through the use of Lemma 2.7. This means there are projective lines L and L' and double normals W and W', all in P_{s_i}, such that $\theta(L, W) < 15°$ and $\theta(L', W') < 15°$. Since $s_i \notin T$, then $\theta(W, W') < 5°$ if W and W' are double normals at the same point of $G_{s_i} \cap A$. Otherwise, where W and W' are double normals at different points of $G_{s_i} \cap A$, it follows from Lemma 2.4 that $\theta(W, W') < 5°$. Then $\theta(L, L') < 15° + 5° + 15° = 35°$. It follows from Lemma 2.5 that any line M within $35°$ of L must also be a projective line, because M would then lie within $60°$ of the double normal W. ($\theta(M, W) \leq \theta(M, L) + \theta(L, W) < 35° + 15° < 60°$.) Thus one can rotate L' through projective lines to bring it parallel to L. To obtain a continuous family $\{L_w \mid w \in [s_{i-1}, s_{i+1}]\}$ one does this rotation, together with a translation if necessary, using a small vertical interval containing s_i. A continuous family $\{L_w \mid w \in (r, s)\}$ of projective lines is obtained using this procedure at each s_i, then by repeating it relative to the lower endpoint r of V. □

LEMMA 2.9. *There exist a neighborhood U of p in A and a homeomorphism h of E^3 onto itself such that the orthogonal projection of $h(U)$ into the yz-plane is injective. It follows that $h(U)$ lies on a flat 2-sphere in E^3.*

Proof. Let

$$U = \bigcup \{G_t \cap A \mid -u_2 < t < u_2\},$$

let V be a component of $[-u_2, u_2] - T'$, and let $\{L_w \mid w \in V\}$ be a continuous family of projective lines as given by Lemma 2.8. First consider the special case where 0 is the lower endpoint of V. By translation of L_w in P_w if necessary, one may assume that L_w intersects the z-axis for each $w \in V$. The homeomorphism h takes each plane P_w onto itself, is fixed on the z-axis and outside $\bigcup \{G_t \mid t \in V\}$, and rotates the various concentric circular sections of G_t to bring the segments $G_t \cap L_t$ into the xz-plane. More detail follows, but, once h is understood, it is clear that $h(U)$ projects as desired because no line parallel to a segment $h(G_t \cap L_t)$ intersects U at two distinct points.

In the general case let r and s be the two endpoints of V where $r < s$, and define $A(V)$ as $\bigcup \{A \cap G_t \mid r \leq t \leq s\}$. Then $A(V) \cap P_r$ and $A(V) \cap P_s$ are singletons $\{x\}$ and $\{y\}$, respectively. With no loss in generality, assume $x = (0, 0, r)$ and $y = (0, 0, s)$. There exist two 3-cells $C_1(V)$ and $C_2(V)$ such that

$$A(V) \subset C_1(V) \subset \{x, y\} \cup \operatorname{Int} C_2(V), \qquad C_i(V) \cap P_r = \{x\}, \qquad C_i(V) \cap P_s = \{y\},$$

and, for each $t \in V$, $C_i(V) \cap P_t$ is a circular disk in G_t with center $(0, 0, t)$. For each $t \in V$, let L'_t be the open diameter of $C_1(V) \cap P_t$ that is parallel to the projective line L_t. There is a homeomorphism h_V of E^3 onto itself that is fixed on $E^3 - \operatorname{Int} C_2(V)$ and rotates each $C_1(V) \cup P_t$ so that L'_t is carried into the xz-plane.

The desired space homeomorphism h is constructed to agree with h_V on $\bigcup \{P_t \mid t \in V\}$ for each component V of $[-u_3, u_3] - T'$ and to be the identity elsewhere. The construction of $C_1(V)$ insures that a sequence $\{h_{V_i}\}$ of homeomorphisms, where $\{V_i\}$ is a sequence of components of $[-u_3, u_3] - T'$ converging to a point t_0 in T', must converge to the identity on P_{t_0}. As in the special case the orthogonal projection of $h(U)$ into the yz-plane is injective because of the realignment of the projective segments $\{L_t'\}$ parallel to the xz-plane. Thus the proof of Case 2 is completed. □

REFERENCES

1. M. Brown, *Sets of constant distance from a planar set,* Michigan Math. J. 19 (1972), 321–323.
2. C. E. Burgess and J. W. Cannon, *Embeddings of surfaces in E^3,* Rocky Mountain J. Math. 1 (1971), 259–344.
3. C. E. Burgess and L. D. Loveland, *Wild double tangent ball embeddings of spheres in E^3,* Topology Appl. 15 (1983), 231–237.
4. R. J. Daverman and L. D. Loveland, *Any 2-sphere in E^3 with uniform interior tangent balls is flat,* Canad. J. Math. 33 (1981), 150–167.
5. S. Ferry, *When ϵ-boundaries are manifolds,* Fund. Math. 90 (1976), 199–210.
6. R. Gariepy and W. D. Pepe, *On the level sets of a distance function in a Minkowski space,* Proc. Amer. Math. Soc. 31 (1972), 255–259.
7. L. D. Loveland, *Double tangent ball embeddings of curves in E^3,* Pacific J. Math. 104 (1983), 391–399.
8. L. R. Weill, *A new characterization of tame 2-spheres in E^3,* Trans. Amer. Math. Soc. 190 (1974), 243–252.

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