THE NONEXISTENCE OF STABLE SUBMANIFOLDS, VARIFOLDS, AND HARMONIC MAPS IN SUFFICIENTLY PINCHED SIMPLY CONNECTED RIEMANNIAN MANIFOLDS

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1. Introduction. A Riemannian manifold M is strictly δ-pinched $0 \leq \delta < 1$ if and only if all sectional curvatures of M are in the half-open interval $(\delta K, K]$ for some $K > 0$. A compact submanifold of M is stable if it is a local minimum for the volume functional. In their paper [8] on stable currents, Lawson and Simons make the following:

CONJECTURE (Lawson–Simons). *Let M^n be a compact simply connected strictly $\frac{1}{4}$-pinched Riemannian manifold. Then there are no stable submanifolds (or, more generally, stable integral currents or stable varifolds) in M.*

This conjecture has been verified for several classes of manifolds. In particular, Lawson and Simons [8] show it holds if M^n can be isometrically immersed in a standard sphere with sufficiently small second fundamental form. The author and S. W. Wei [6] have shown it holds for all metrics in some C^2 neighborhood of the standard metric on the sphere S^n and for all compact hypersurfaces M^n $(n \geq 3)$ in the Euclidean space $\mathbf{R}^{n+1}$ which are pointwise $\frac{1}{4}+3/(n^2+4)$ pinched. In this paper we prove

THEOREM 1. *There is a constant $\delta(n,p) > \frac{1}{4}$ so that if M^n is a compact simply connected strictly $\delta(n,p)$-pinched Riemannian manifold of dimension n, then there are no stable p-dimensional submanifolds, stable p-dimensional integral currents, or stable p-dimensional varifolds in M.*

The number $\delta(n,p)$ is exhibited as a root of a transcendental equation and is in theory computable. Unfortunately $\lim \delta(n,p) = 1$ as $n \to \infty$ and this limit is uniform in p. Computer calculations show that for small values of n, p, the values of $\delta(n,p)$ given by our estimates are shown in Table 1. We remark that a theorem of Fleming and Federer [4] implies that any nonzero homology class in $H_p(M, \mathbf{Z})$ ($\mathbf{Z}$ is the ring of integers) contains a stable integral current. Therefore the above theorem implies that for a simply connected strictly $\delta(n,p)$-pinched Riemannian manifold, the homology group $H_p(M, \mathbf{Z})$ vanishes. Of course, by the classical sphere theorem, when M is strictly $\frac{1}{4}$-pinched then it is homeomorphic to a sphere and therefore Theorem 1 does not imply any new topological result. However we feel that proving a simply connected strictly $\frac{1}{4}$-pinched Riemannian manifold is a homology sphere by verifying the conjecture above not only would be of interest as a natural variational problem, but provides a good test case for the theory.

Received March 12, 1984. Final revision received February 4, 1985.
Michigan Math. J. 32 (1985).

n	p	$\delta(n,p)$	n	p	$\delta(n,p)$
2	1	.90612	8	1	.81186
				6	.99161
3	1	.85408	9	1	.81806
	2	.95637		8	.99315
4	1	.83110	10	1	.82057
	3	.97305		9	.99430
5	1	.82068	50	1	.90144
	4	.98142		49	.99969
6	1	.81640	100	1	.93476
	5	.98632		99	.99988
7	1	.81537			
	6	.98447			

Table 1

We now give an outline of the proof, without the technical details, in the special case of submanifolds N^p of M^n. This will make what follows clearer. The proof involves (as does the proofs of the other results quoted) averaging the second variation formula over a collection of several different deformations of the submanifold and showing the result is negative. This violates the second derivative test for stability. It differs from the others in that an integral average over a continuous family of deformations is used instead of a finite sum. Assume N^p is an imbedded minimal submanifold of M^n. Then, following Lawson and Simons [8], we rewrite the second variation formula as

$$\text{(1-1)} \qquad \frac{d^2}{dt^2}\bigg|_{t=0} \operatorname{vol}(\varphi_t^V N) = \int_N \mathfrak{M}(V, TN_y)\,\Omega_N(y),$$

where V is any smooth vector field on M and φ_t^V its flow (or the one parameter group of diffeomorphisms it generates), and where $\mathfrak{M}(V, TN_y)$ only depends on V and the tangent space TN_y of N at y. For each $x \in M$ let $\rho_x(y) =$ geodesic distance of y from x. For any smooth function $f: \mathbf{R} \to \mathbf{R}$ ($\mathbf{R}$ the real numbers) let $V_x(f)$ be the gradient of $f \circ \rho_x$. It is then shown, for M simply connected and sufficiently pinched and for the proper choice of f, that

$$\text{(1-2)} \qquad \int_M \mathfrak{M}(V_x(f), W)\,\Omega_M(x) < -c \leq 0$$

for every p-dimensional subspace W tangent to M. Therefore, by Fubini's Theorem, for any compact minimal submanifold N^p of M^n

$$\int_M \frac{d^2}{dt^2}\bigg|_{t=0} \operatorname{vol}(\varphi_t^{V_x(f)} N)\,\Omega_M(x) = \int_N \int_M \mathfrak{M}(V_x(f), TN_y)\,\Omega_M(x)\,\Omega_N(y)$$
$$< -c \operatorname{vol}(N) \leq 0.$$

Thus $(d^2/dt^2)\,|_{t=0} \operatorname{vol}(\varphi_t^{V_x(f)} N) < 0$ for some $x \in M$, and therefore N cannot be stable. The main step is proving an inequality of the type (1-2). The estimates

n	$\delta(n)$	n	$\delta(n)$
3	.82842	8	.76166
4	.78058	9	.76549
5	.76411	10	.76997
6	.75907	50	.88265
7	.75910	100	.92397

Table 2

needed to prove (1-2) follow from the Hessian comparison theorem of Greene and Wu [5] and the volume comparison theorem of Bishop and Crittenden [1].

Slight changes in the proof also gives a result about nonexistence of stable harmonic maps. If N^p and M^n are compact Riemannian manifolds and $\psi: N^p \to M$ is a smooth map, then the energy of ψ is given by

$$E(\psi) = \frac{1}{2}\int_N \|d\psi\|^2 \Omega_N.$$

The function $\psi: N \to M$ is a stable harmonic map if it is a local minimum for the energy integral viewed as a functional on the set of smooth maps from N to M.

THEOREM 2. *For each $n \geq 3$ there is a $\delta(n)$ with $\frac{1}{4} \leq \delta(n) < 1$ such that if M is a simply connected compact strictly $\delta(n)$-pinched Riemannian manifold of dimension n, then there are no stable harmonic maps $\psi: N \to M$ for any compact Riemannian manifold N.*

As before, $\delta(n)$ is exhibited as the root of a transcendental equation and $\lim_{n\to\infty} \delta(n) = 1$. For some small values of n the values of $\delta(n)$ given by our proof are shown in Table 2.

2. Variational formulas. Let M^n be a complete Riemannian manifold with metric $\langle\ ,\ \rangle$ and Riemannian connection ∇. We now recall the definition of a varifold on M. Let $\pi: G_p(M) \to M$ be the bundle of all unoriented p-planes tangent to M. We represent elements of $G_p(M)$ as unit length decomposable p-vectors $\xi = e_1 \wedge \cdots \wedge e_p$ with the understanding that ξ and $-\xi$ represent the same p-plane. A p-dimensional varifold $\mathcal{S}$ on M is a Radon measure on the Borel sets of the total space $G_p(M)$. *It will be assumed all varifolds have compact support,* that is, there is a compact set $K \subseteq G_p(M)$ with $\mathcal{S}(G_p(M)\backslash K) = 0$. The set of all varifolds on M will be denoted by $\mathcal{V}_p(M)$. Given any varifold $\mathcal{S} \in \mathcal{V}_p(M)$ there is associated to $\mathcal{S}$ a radon measure $\|\mathcal{S}\|$ on M by $\|\mathcal{S}\|(B) = \mathcal{S}(\pi^{-1}B)$ for all Borel subsets B of M. The mass $\underset{\sim}{M}(\mathcal{S})$, or p-dimensional area, of a varifold $\mathcal{S} \in \mathcal{V}_p(M)$ is

$$\underset{\sim}{M}(\mathcal{S}) = \mathcal{S}(G_p(M)) = \int_{G_p(M)} 1\, d\mathcal{S}(\xi). \tag{2-1}$$

This is related to the geometry of submanifolds of M as follows. Let N be a compact p-dimensional submanifold of M and define a varifold $|N| \in \mathcal{V}_p(M)$ by requiring

$$\int_{G_p(M)} f(\xi)\, d|N|(\xi) = \int_N f(TN_x)\Omega_N(x) \tag{2-2}$$

for all continuous functions $f: G_p(M) \to \mathbf{R}$. (Here Ω_N is the volume density on N.) It is clear that the mass of $|N|$ in the sense of (2-1) is the same as the volume of N in the usual sense. We also remark that if N is a p-dimensional integral current in M (see [3] for the definition), then N has an "approximate tangent space" TN_x at "almost all" points of its support. Therefore N also defines a varifold $|N|$ by the formula (2-2). It follows that if M has no stable p-dimensional varifolds then it will not have any stable p-dimensional submanifolds or integral currents.

We now give the formulation of the first and second variation formula for varifolds due to Lawson and Simons [8]. First, if $\varphi: M \to M$ is a diffeomorphism then φ induces a map $\varphi_\#: \mathcal{V}_p(M) \to \mathcal{V}_p(M)$. If $\mathcal{S} \in \mathcal{V}_p(M)$ then $\varphi_\# \mathcal{S}$ is the measure on $G_p(M)$, so that

$$\int_{G_p(M)} f(\xi)\, d(\varphi_\# \mathcal{S})(\xi) = \int_{G_p(M)} f(\varphi_* \xi) \|\varphi_* \xi\|\, d\mathcal{S}(\xi) \tag{2-3}$$

for all continuous $f: G_p(M) \to \mathbf{R}$. This clearly implies

$$\underset{\sim}{M}(\varphi_\# \mathcal{S}) = \int_{G_p(M)} \|\varphi_* \xi\|\, d\mathcal{S}(\xi). \tag{2-4}$$

Let V be a smooth vector field on M and let φ_t^V be the flow or one parameter group of diffeomorphisms generated by V. The first and second variations of $\mathcal{S} \in \mathcal{V}_p(M)$ are

$$\frac{d}{dt}\bigg|_{t=0} \underset{\sim}{M}(\varphi^V_{t\#}\mathcal{S}) = \int_{G_p(M)} \frac{d}{dt}\bigg|_{t=0} \|\varphi^V_{t*}\xi\|\, d\mathcal{S}(\xi), \tag{2-5}$$

$$\frac{dt}{dt^2}\bigg|_{t=0} \underset{\sim}{M}(\varphi^V_{t\#}\mathcal{S}) = \int_{G_p(M)} \frac{d^2}{dt^2}\bigg|_{t=0} \|\varphi^V_{t*}\xi\|\, d\mathcal{S}(\xi). \tag{2-6}$$

A varifold $\mathcal{S}$ is *minimal* or *stationary* if and only if the first variation (2-5) vanishes for all smooth vector fields V on M. A varifold $\mathcal{S}$ is *stable* if and only if $\mathcal{S}$ is minimal and also the second variation (2-6) is non-negative for all smooth vector fields V on M. Thus $\mathcal{S}$ is stable if, to second order, it is a local minimum for the mass function on the space $\mathcal{V}_p(M)$ of all varifolds on M.

To be more explicit about the form of the integrands in (2-5) and (2-6) some notation is needed. For any smooth vector field V on M define a smooth field of linear endomorphisms of the tangent spaces to M by

$$\mathfrak{A}^V(X) = \nabla'_X V. \tag{2-7}$$

Extend $\mathfrak{A}^V$ to the full tensor algebra over M as a derivation. Then, on decomposable p-vectors, $\mathfrak{A}^V$ is given by

$$\mathfrak{A}^V(X_1 \wedge \cdots \wedge X_p) = \sum_{i=1}^{p} X_1 \wedge \cdots \wedge \mathfrak{A}^V X_i \wedge \cdots \wedge X_p. \tag{2-8}$$

Note that on $\Lambda^p(TM)$ the square of the extension of $\mathcal{A}^V$ (denoted by $\mathcal{A}^V\mathcal{A}^V$) differs from the extension of the square of $\mathcal{A}^V$ (denoted by $(\mathcal{A}^V)^2$). The first variation formula can then be written as [8, p. 435]

$$\text{(2-9)} \qquad \left.\frac{d}{dt}\right|_{t=0} \underset{\sim}{M}(\varphi^V_{t\#}\mathcal{S}) = \int_{G_p(M)} \langle \mathcal{A}^V\xi, \xi\rangle \, d\mathcal{S}(\xi).$$

For the second variation formula one more piece of notation is needed. We choose the sign on the curvature tensor so that

$$R(X, Y)Z = (\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})Z.$$

Then for any vector $V \in TM_x$ the Ricci map $R_V: TM_x \to TM_x$ is given by

$$\text{(2-10)} \qquad R_V(X) = R(X, V)V.$$

The symmetries of the curvature tensor imply that R_V is self-adjoint. Let $\mathcal{S} \in \mathcal{V}_p(M)$ be a stationary varifold. Then the first variation formula implies

$$\text{(2-11)} \qquad \int_{G_p(M)} \langle \mathcal{A}^{\mathcal{A}^V(V)}\xi, \xi\rangle \, d\mathcal{S}(\xi) = 0.$$

Now assume that $\mathcal{A}^V$ is a self-adjoint map on tangent spaces (this will be the case in our applications). Then for any p-vector ξ, $\langle \mathcal{A}^V\mathcal{A}^V\xi, \xi\rangle = \langle \mathcal{A}^V\xi, \mathcal{A}^V\xi\rangle = \|\mathcal{A}^V\xi\|^2$. In this case (i.e., $\mathcal{S}$ stationary and $\mathcal{A}^V$ self-adjoint) the second variation formula [8, eq. (2.8′), p. 435] can be rewritten as

$$\text{(2-12)} \qquad \left.\frac{d^2}{dt^2}\right|_{t=0} \underset{\sim}{M}(\varphi^V_{t\#}\mathcal{S}) = \int_{G_p(M)} \mathfrak{M}(\mathcal{A}^V, \xi) \, d\mathcal{S}(\xi),$$

where

$$\text{(2-13)} \qquad \mathfrak{M}(\mathcal{A}^V, \xi) = 2\langle \mathcal{A}^V\mathcal{A}^V\xi, \xi\rangle - \langle \mathcal{A}^V\xi, \xi\rangle^2 - \langle (\mathcal{A}^V)^2\xi, \xi\rangle - \langle R_V\xi, \xi\rangle.$$

3. Comparison theorems. We now show that if V is the gradient of a smooth function then the Hessian comparison theorem of Greene and Wu can be used to give bounds on the size of the tensor $\mathcal{A}^V$ appearing in the first and second variation formulas. If f is a C^1 function on M then the gradient ∇f of f is the vector field dual to the differential df, that is $\langle \nabla f, X\rangle = df(X)$ for all vectors X. The Hessian D^2f of a C^2 function is the symmetric bilinear form defined on $TM_x \times TM_x$ by $D^2f(X, Y) = (XY - \nabla_X Y)f$. For gradient vector fields a straightforward calculation shows

$$\text{(3-1)} \qquad V = \nabla f \quad \text{implies} \quad \langle \mathcal{A}^V X, Y\rangle = D^2 f(X, Y).$$

Thus when $V = \nabla f$ the symmetry of $D^2 f$ implies that $\mathcal{A}^V$ is self-adjoint. (Therefore when V is a gradient field we are justified in using the form of the second variation formula given by (2-12) and (2-13).)

We now introduce some notation. For any unit vector u tangent to M let $P_1(t, u)$ and $P_2(t, u)$ be the fields of linear maps along the geodesic $\gamma(t) = \exp(tu)$ given by

$$
\begin{aligned}
&P_1(t,u) = \text{orthogonal projection onto span of } \gamma'(t), \\
&P_2(t,u) = (\mathrm{id}) - P_1(t,u) = \text{orthogonal projection onto } \gamma'(t)^{\perp}.
\end{aligned} \tag{3-2}
$$

Because γ is a geodesic both $P_1(t,u)$ and $P_2(t,u)$ are parallel along γ. Also, for each real number λ define $c_\lambda, s_\lambda\colon \mathbf{R}\to\mathbf{R}$ by the initial value problems

$$
\begin{aligned}
&c_\lambda''(t)+\lambda c_\lambda(t)=0 \quad c_\lambda(0)=1, \quad c_\lambda'(0)=0 \\
&s_\lambda''(t)+\lambda s_\lambda(t)=0 \quad s_\lambda(0)=0, \quad s_\lambda'(0)=1.
\end{aligned} \tag{3-3}
$$

Thus, when $\lambda>0$, $c_\lambda(t)=\cos(\sqrt{\lambda}t)$ and $s_\lambda(t)=\sin(\sqrt{\lambda}t)/\sqrt{\lambda}$.

If $x_0\in M$ let $\rho_{x_0}(x)$ = geodesic distance at x from x_0 and let $\mathrm{cut}(x_0)$ be the cut locus of x_0 in M. Then ρ_{x_0} is a smooth function on the open set $M\setminus(\{x_0\}\cup\mathrm{cut}(x_0))$. Finally if A and B are self-adjoint linear maps on an inner product space write $A\le B$ to mean that $B-A$ is positive semidefinite. Using the relation between $\mathfrak{A}^V$ (when $V=\nabla f$) and $D^2 f$ given by (3-1), the Hessian comparison theorem in [5, p. 19] yields the

3.1. COMPARISON THEOREM (Greene–Wu). *Let M be a Riemannian manifold and assume*

(1) *all sectional curvatures of M are in the interval* $[\alpha,\beta]$,

(2) $f\colon(0,\infty)\to\mathbf{R}$ *is smooth and* $f'\ge 0$,

(3) $V=\nabla(f\circ\rho_{x_0})$ *for some* $x_0\in M$.

Then on the open set $M\setminus(\{x_0\}\cup\mathrm{cut}(x_0))$ *the inequalities*

$$
\begin{aligned}
&f''(\rho)P_1(\rho,u)+f'(\rho)\frac{c_\beta(\rho)}{s_\beta(\rho)}P_2(\rho,u) \\
&\le \mathfrak{A}^V \le f''(\rho)P_1(\rho,u)+f'(\rho)\frac{c_\alpha(\rho)}{s_\alpha(\rho)}P_2(\rho,u)
\end{aligned} \tag{3-4}
$$

hold where u varies over the unit sphere of TM_{x_0} and $\rho=\rho_{x_0}$. The lower bound only holds up to the first positive root of $s_\beta(\rho)=0$.

We give two elementary corollaries to this which are well known in the case $\mathcal{S}$ is a submanifold.

3.2. COROLLARY. *Let M^n be a complete Riemannian manifold with all sectional curvatures nonpositive. Then for any compactly supported p-dimensional minimal varifold $\mathcal{S}$ the support,* $\mathrm{spt}(\mathcal{S})$, *of $\mathcal{S}$ (i.e., the support of the measure $\|\mathcal{S}\|$) intersects the cut locus* $\mathrm{cut}(x_0)$ *of every point $x_0\in M$. Thus if M is simply connected (so that by the Cartan–Hadamard theorem every cut locus* $\mathrm{cut}(x_0)$ *is empty) then M contains no compactly supported minimal varifolds.*

Proof. Let $\mathcal{S}$ be any varifold so that $\mathrm{spt}(\mathcal{S})\cap\mathrm{cut}(x_0)$ is empty for some x_0 in M. In the comparison theorem use $\beta=0$ and $f(t)=\frac{1}{2}t^2$ (then $f\circ\rho_{x_0}$ is smooth on $M\setminus\mathrm{cut}(x_0)$) and $V=\nabla f\circ\rho_{x_0}$ to get $\mathrm{id}=P_1(\rho,u)+P_2(\rho,u)\le\mathfrak{A}^V$ on $M\setminus\mathrm{cut}(x_0)$. But for a decomposable p-vector ξ and the extension of id to $\Lambda^p TM$ we have $\mathrm{id}(\xi)=p\xi$. Thus, by the first variation formula (2-9),

$$\frac{d}{dt}\bigg|_{t=0} \underset{\sim}{M}(\varphi_{t\#}^{V} \mathcal{S}) \geq p\underset{\sim}{M}(\mathcal{S}).$$

Therefore $\mathcal{S}$ is clearly not stationary. □

3.3. COROLLARY. *Let M^n be a complete Riemannian manifold with all sectional curvatures $\leq K$ (where $K>0$) and let $r<\pi/2\sqrt{K}$. For $x_0 \in M$ let $B_r(x_0)$ be the closed geodesic ball of radius r centered at x_0. If $B_r(x_0) \cap \operatorname{cut}(x_0)$ is empty then there are no minimal varifolds $\mathcal{S} \in \mathcal{V}_p(M)$ with $\operatorname{spt}(\mathcal{S}) \subseteq B_r(x_0)$.*

Proof. This time use $\beta=K$ and $f=-\cos(t\sqrt{K})=-c_K(t)$ in the comparison theorem to get for $V=\nabla f \circ \rho_{x_0}$ that

$$\mathcal{C}^V \geq K\cos(\sqrt{K}\,\rho)(P_1(\rho,u)+P_2(\rho,u))=K\cos(\sqrt{K}\,\rho)\,\mathrm{id}$$

on $B_r(x_0)$. Therefore if $\mathcal{S}$ is a p-dimensional varifold with $\operatorname{spt}(\mathcal{S}) \subseteq B_r(x_0)$, then

$$\frac{d}{dt}\bigg|_{t=0} \underset{\sim}{M}(\varphi_{t\#}^{V} \mathcal{S}) \geq p\int_{G_p(M)} \cos(\rho(\pi\xi))\, d\mathcal{S}(\xi)>0,$$

where $\pi: G_p(M) \to M$ is a projection. This implies $\mathcal{S}$ is not minimal.

The other comparison theorem needed is due to Bishop and Crittenden. If $x_0 \in M^n$ then let UM_{x_0} be the unit sphere in TM_{x_0}. Then, letting $\rho=\rho_{x_0}$ and $u \in UM_{x_0}$, we can view (ρ, u) as polar coordinates on M near x_0 in the obvious way. Let Ω_M be the volume density on M and $\Omega_{UM_{x_0}}$ the volume density on UM_{x_0}. Assume that all sectional curvatures of M are in the interval $[\alpha, \beta]$. Then [1, chap. 11] on the open set $M \backslash \operatorname{cut}(x_0)$

$$s_\beta(\rho)^{n-1}\, d\rho\, \Omega_{UM_{x_0}}(u) \leq \Omega_M \leq s_\alpha(\rho)^{n-1}\, d\rho\, \Omega_{UM_{x_0}}(u), \tag{3-4}$$

where the lower bound only holds up to the first positive zero of $s_\beta(\rho)$. □

4. The main estimate. For the rest of this paper we will be assuming M is a compact simply connected Riemannian manifold with all sectional curvatures in the half-open interval $(\delta K, K]$ for some $K, \delta>0$ and $\delta<1$. We start by multiplying the metric by a positive constant to normalize so that all sectional curvatures are in the open interval $(\delta, 1)$. We also assume $\frac{1}{4}<\delta<1$. By a well known theorem due to Klingenberg [2, p. 100] this implies the injectivity radius of M is greater than π.

Define a function $f: \mathbf{R} \to \mathbf{R}$ by

$$f(t)=\begin{cases} -\cos(t)=-c_1(t) & |t| \leq \pi, \\ 1 & |t| \geq \pi. \end{cases} \tag{4-1}$$

Then f and f' are continuous. For any $x \in M$ let $\rho_x(y)=$ geodesic distance of y from x and let $V_x(f)$ be the vector field defined by

$$V_x(f)=\nabla(f \circ \rho_x)=f'(\rho_x)\nabla\rho_x. \tag{4-2}$$

Because the injectivity radius of M is greater than π and f is an even function (so that $f \circ \rho_x$ is smooth at x), the vector field $V_x(f)$ is continuous and smooth off of the locus defined by $\rho_x = \pi$, and $V_x(f) \equiv 0$ on the set defined by $\rho_x \geq \pi$.

Fix a point $x_0 \in M$ and a unit decomposable p-vector $\xi = e_1 \wedge \cdots \wedge e_p \in \Lambda^p TM_{x_0}$. Our object is to give bounds on $\langle \mathfrak{A}^{V_x(f)}\xi, \xi\rangle$ and related expressions and their integrals in terms of the distance of x from x_0. Let UM_{x_0} be the unit sphere of TM_{x_0} and $\rho = \rho_{x_0}$. Then denote by (ρ, u) polar coordinates on M centered at x_0. For $u \in UM_{x_0}$ let $P_1(u) = P_1(0, u)$ and $P_2(u) = P_2(0, u)$ (thus $P_1(u)X = \langle X, u\rangle u$ and $P_2(u) = \mathrm{id} - P_1(u)$). Then $\mathfrak{A}^{V_x(f)} = 0$ when $\rho(x) \geq \pi$ and the comparison Theorem 3.1 implies (letting $I = \mathrm{id}_{TM_{x_0}}$)

$$
\begin{aligned}
&c_1(\rho(x))I \leq (\mathfrak{A}^{V_x(f)})x_0 \\
&\leq c_1(\rho(x))P_1(u) + \frac{s_1(\rho(x))c_\delta(\rho(x))}{s_\delta(\rho(x))}P_2(u)
\end{aligned}
\tag{4-3}
$$

for $\rho(x) < \pi$. (Here we have used that $\rho_x(x_0) = \rho_{x_0}(x) = \rho(x)$ so that $c_1(\rho_x(x_0)) = c_1(\rho(x))$ etc.) This implies the eigenvalues $\kappa_1 \leq \kappa_2 \leq \cdots \leq \kappa_n$ of $(\mathfrak{A}^{V_x(f)})x_0$ (which are real as $\mathfrak{A}^{V_x(f)}$ is self-adjoint) satisfy

$$
c_1(\rho(x)) \leq \kappa_i \leq \frac{s_1(\rho(x))c_\delta(\rho(x))}{s_\delta(\rho(x))}, \quad 1 \leq i \leq n.
\tag{4-4}
$$

When $(\mathfrak{A}^{V_x(f)})x_0$ is extended to $\Lambda^p TM_{x_0}$ as a derivation its eigenvalues are $\kappa_{i_1} + \cdots + \kappa_{i_p}$ $(1 \leq i_1 < \cdots < i_p \leq n)$. Likewise the eigenvalues of $\mathfrak{A}^{V_x(f)}\mathfrak{A}^{V_x(f)}$ are $(\kappa_{i_1} + \cdots + \kappa_{i_p})^2$ and those of the extension of $(\mathfrak{A}^{V_x(f)})^2$ are $\kappa_{i_1}^2 + \cdots + \kappa_{i_p}^2$. Set

$$
\begin{aligned}
&\tilde{g}_1(t, \delta) = \text{middle value of } \left\{c_1(t), 0, \frac{s_1(t)c_\delta(t)}{s_\delta(t)}\right\}, \\
&g_1(t, \delta) = (\tilde{g}_1(t, \delta))^2, \quad 0 \leq t \leq \pi, \\
&g_2(t, \delta) = \max\left\{c_1(t)^2, \left(\frac{s_1(t)c_\delta(t)}{s_\delta(t)}\right)^2\right\}, \quad 0 \leq t \leq \pi, \\
&g_1(t, \delta) = g_2(t, \delta) = 0, \quad \pi < t.
\end{aligned}
\tag{4-5}
$$

If $a \leq x \leq b$ and $y =$ middle value of $\{a, 0, b\}$ then $y^2 \leq x^2$. Also if A is a self-adjoint linear map on any inner product space with largest eigenvalue λ_2 and smallest eigenvalue λ_1, then for any unit vector u, $\lambda_1 \leq \langle Au, u\rangle \leq \lambda_2$. Therefore (4-4) and the estimates on Ω_M given by (3-4) imply

$$
\begin{aligned}
&p^2 g_1(\rho(x), \delta)(s_1(\rho(x)))^{n-1}\, d\rho\, \Omega_{UM_{x_0}} \\
&\leq \langle \mathfrak{A}^{V_x(f)}\xi, \xi\rangle^2 \Omega_M,\ \langle \mathfrak{A}^{V_x(f)}\mathfrak{A}^{V_x(f)}\xi, \xi\rangle \Omega_M,\ p\langle (\mathfrak{A}^{V_x(f)})^2\xi, \xi\rangle \Omega_M \\
&\leq p^2 g_2(\rho(x), \delta)(s_\delta(\rho(x)))^{n-1}\, d\rho\, \Omega_{UM_{x_0}}.
\end{aligned}
\tag{4-6}
$$

whence, if $\mathfrak{I}$ is any one of the integrals

$$
\begin{aligned}
&\int_M \langle \mathfrak{A}^{V_x(f)}\xi, \xi\rangle^2 \Omega_M(x), \quad \int_M \langle \mathfrak{A}^{V_x(f)}\mathfrak{A}^{V_x(f)}\xi, \xi\rangle \Omega_M(x), \\
&p\int_M \langle (\mathfrak{A}^{V_x(f)})^2\xi, \xi\rangle \Omega_M(x),
\end{aligned}
\tag{4-7}
$$

then integration in polar coordinates yields

$$p^2 \operatorname{vol}(S^{n-1}) \int_0^\pi g_1(\rho,\delta) s_1(\rho)^{n-1}\, d\rho \leq \mathcal{G} \leq p^2 \operatorname{vol}(S^{n-1}) \int_0^\pi g_2(\rho,\delta)(s_\delta(\rho))^{n-1}\, d\rho. \tag{4-8}$$

Next, a lower bound on the integral of $\langle R_{V_x(f)}\xi, \xi\rangle$ is needed. Let $x \in M$ have polar coordinates (ρ, u) with $0 < \rho < \pi$. Then $V_x(f) = f'(\rho_x)\nabla(\rho_x)$. By the Gauss lemma the restriction of $\nabla(\rho_x)$ to the geodesic $\gamma(t) = \exp(tu)$, $0 \leq t < \rho(x)$, is $-\gamma'(t)$. Therefore $\nabla(\rho_x)_{x_0} = -u$. So, for $X \in TM_{x_0}$,

$$\begin{aligned} R_{V_x(f)}(X) &= R(X, f'(\rho_x(x_0))\nabla(\rho_x)_{x_0} f'(\rho_x \mid x_0))(\nabla\rho_x)_{x_0} \\ &= (f'(\rho_{x_0}(x))^2 R(x, -u)(-u) \\ &= (s_1(\rho(x))^2 R_u(x). \end{aligned} \tag{4-9}$$

But $R_u(u) = 0$, and if X is a unit vector orthogonal to u then $\langle R_u X, X\rangle$ is the sectional curvature of the plane spanned by u and X, and thus $\delta < \langle R_u X, X\rangle < 1$. This gives that

$$\delta s_1(\rho(x))^2 P_2(u) < R_{V_x(f)} < s_1(\rho(x))^2 P_2(u) \tag{4-10}$$

when $0 < \rho(x) < \pi$. This in turn can be combined with (3-4) to give

$$\langle R_{V_x(f)}\xi, \xi\rangle \Omega_M \geq \delta\langle P_2(u)\xi, \xi\rangle s_1(\rho(x))^{n+1}\, d\rho\, \Omega_{UM_{x_0}}. \tag{4-11}$$

To integrate this inequality we need

LEMMA (A). *Let $Q\colon \mathbf{R}^n \to \mathbf{R}$ be a quadratic form; then*

$$\int_{S^{n-1}} Q(u)\Omega_{S^{n-1}}(u) = \frac{1}{n}\operatorname{vol}(S^{n-1})\operatorname{trace}(Q). \tag{4-12}$$

LEMMA (B).

$$\int_{UM_{x_0}} \langle P_2(u)\xi, \xi\rangle \Omega_{UM_{x_0}}(u) = \frac{(n-1)p}{n}\operatorname{vol}(S^{n-1}). \tag{4-13}$$

Proof. Let $x^1, \dots, x^n$ be the standard coordinates on $\mathbf{R}^n$ and let

$$c = \int_{S^{n-1}} (x^i)^2 \Omega_{S^{n-1}}(x).$$

Then c is independent of i. Also, if $i \neq j$ then $\int_{S^{n-1}} x^i x^j \Omega_{S^{n-1}}(x) = 0$. Let $Q(x) = \sum_{i,j} Q_{ij} x^i x^j$. Then

$$\begin{aligned} \int_{S^{n-1}} Q(x)\Omega_{S^{n-1}}(x) &= \sum_{i,j} \int_{S^{n-1}} x^i x^j \Omega_{S^{n-1}}(x) Q_{ij} \\ &= c\sum_i Q_{ii} = c\operatorname{trace}(Q). \end{aligned}$$

The value of c is the same for all quadratic forms and may be found by letting $Q(x) = \|x\|^2$. Then

$$\operatorname{vol}(S^{n-1}) = \int_{S^{n-1}} Q(x)\Omega_{S^{n-1}}(x) = c\operatorname{trace}(Q) = nc.$$

To prove part (B) note that $P_1(u)X = \langle X, u\rangle u$ so the function $u \to \langle P_1(u)\xi, \xi\rangle$ is a quadratic form on TM_{x_0}. Choose an orthonormal basis $e_1, \dots, e_n$ of TM_{x_0} with $e_1 \wedge \cdots \wedge e_p = \xi$. Then

$$P_1(e_i)\xi = \sum_{j=1}^{p} e_1 \wedge \cdots \wedge P_1(e_i)e_j \wedge \cdots \wedge e_p = \begin{cases} \xi & 1 \le i \le p \\ 0 & p+1 \le i \le n \end{cases}$$

and so $\operatorname{trace}(u \to \langle P_1(u)\xi, \xi\rangle) = p$. Thus by part (A)

$$\int_{UM_{x_0}} \langle P_1(u)\xi, \xi\rangle \Omega_{UM_{x_0}}(u) = \frac{p}{n}\operatorname{vol}(S^{n-1}).$$

But $P_2(u) = I - P_1(u)$, so

$$\int_{UM_{x_0}} \langle P_2(u)\xi, \xi\rangle \Omega_{UM_{x_0}}(u) = \int_{UM_{x_0}} \langle I\xi, \xi\rangle \Omega_{UM_{x_0}}(u) - \int_{UM_{x_0}} \langle P_1(u)\xi, \xi\rangle \Omega_{UM_{x_0}}(u)$$

$$= p\operatorname{vol}(S^{n-1}) - \frac{p}{n}\operatorname{vol}(S^{n-1}) = \frac{(n-1)p}{n}\operatorname{vol}(S^{n-1}).$$

This completes the proof of the lemma. □

Using (4-13) the inequality (4-11) can be integrated over M and then integrated by parts to get

$$\text{(4-14)} \quad \begin{aligned} \int_M \langle R_{V_x(f)}\xi, \xi\rangle \Omega_M(x) &> \frac{(n-1)p}{n}\operatorname{vol}(S^{n-1}) \int_0^\pi \delta s_1(\rho)^{n+1}\,d\rho \\ &= p(n-1)\operatorname{vol}(S^{n-1}) \int_0^\pi \delta c_1(\rho)^2 s_1(\rho)^{n-1}\,d\rho. \end{aligned}$$

Combining the above gives

The main estimate. If $\mathfrak{M}(\mathcal{A}^V, \xi)$ is given by (2-13) and under the hypothesis and notation of this section then

$$\text{(4-15)} \quad \int_M \mathfrak{M}(\mathcal{A}^{V_x(f)}, \xi)\Omega_M(x) < p\operatorname{vol}(S^{n-1})F_{n,p}(\delta)$$

where $F_{n,p}(\delta)$ is the continuous function of δ given by

$$\text{(4-16)} \quad \begin{aligned} F_{n,p}(\delta) = \int_0^\pi &(2pg_2(t,\delta)(s_\delta(t))^{n-1} \\ &- (p+1)g_1(t,\delta)s_1(t)^{n-1} - (n-1)\delta c_1(t)^2 s_1(t)^{n-1})\,dt \end{aligned}$$

and

$$\text{(4-17)} \quad F_{n,p}(1) = -(n-p)\int_0^\pi \cos^2(t)\sin^{n-1}(t)\,dt < 0.$$

Proof. The inequality (4-15) follows at once from (4-6), (4-7), (4-14) and the definition of $\mathfrak{M}(\mathfrak{A}^V, \xi)$. It is clear that $F_{n,p}(\delta)$ is a continuous function of δ. When $\delta=1$ then $g_1(t,\delta)=g_2(t,\delta)=c_\delta(t)^2=c_1(t)^2=\cos^2(t)$ and $s_\delta(t)=s_1(t)=\sin(t)$. This implies (4-17) and completes the proof. □

5. Proof of the theorem.

5.1. THEOREM. *Let* $F_{n,p}(\delta)$ *be given by* (4-16) *and set*

$$\delta(n,p)=\inf\{\delta: \delta>\tfrac{1}{4} \text{ and } F_{n,p}(\delta)\le 0\}. \tag{5-1}$$

Then no compact simply connected strictly $\delta(n,p)$*-pinched Riemannian manifold of dimension n supports any stable p-dimensional varifolds.*

Proof. First note that $F_{n,p}(1)<0$ and so $\frac{1}{4}\le\delta(n,p)<1$. To simplify notation set $\delta=\delta(n,p)$. By normalizing we may assume all sectional curvatures of M are in the interval $(\delta,1)$. Let f be the function given by (4-1). Then we wish to use the method of proof outlined in the introduction along with the estimate (4-15) to show M has no stable varifolds. The only problem is that the vector field $V_x(f)$ is not smooth so we must first approximate f by smooth functions. The functions f and f' are continuous and f'' is bounded and also continuous everywhere except at the points π and $-\pi$. Choose $\epsilon>0$ so that $\pi+\epsilon$ is less than the injectivity radius of M. Then it is not hard to show the existence of smooth functions f_k $(k=1,2,\dots)$ with $f_k(-t)=f_k(t)$, f_k converging to f uniformly, f_k' converging to f' uniformly, $f_k''(t)$ uniformly bounded with respect to both t and k, f_k'' converging uniformly to f on all compact sets disjoint from $\{\pi,-\pi\}$, and each f_k constant on the interval $[\pi+\epsilon,\infty)$.

Then for each k and each $x\in M$ the vector field $V_x(f_k)=\nabla(f_k\circ\rho_x)$ is smooth on M and converges uniformly to $V_x(f)$ as $k\to\infty$. Because the functions f_k'' are uniformly bounded with respect to k the comparison Theorem 3.1 implies the components of $\mathfrak{A}^{V_x(f_k)}$ can be bounded independently of x and k. Therefore $\mathfrak{M}(\mathfrak{A}^{V_x(f_k)},\xi)$ can be bounded by a constant independent of x, k, ξ. For fixed ξ tangent to M at a point x_0 the function $x\to\mathfrak{M}(\mathfrak{A}^{V_x(f_k)},\xi)$ converges to the function $x\to\mathfrak{M}(\mathfrak{A}^{V_x(f)},\xi)$ except on the set defined by $\rho_{x_0}(x)=\pi$. Thus the convergence is almost everywhere.

Now assume that $\mathcal{S}$ is any minimal varifold in $\mathcal{V}_p(M)$. Then by the second variation formula (2-12), the main estimate (4-15), Fubini's theorem and Lebesgue's dominated convergence theorem

$$\lim_{k\to\infty}\int_M \frac{d^2}{dt^2}\bigg|_{t=0} \underset{\sim}{M}(\varphi_{t\#}^{V_x(f_k)}\mathcal{S})\,\Omega_M(x)=\int_{G_p(M)}\int_M \mathfrak{M}(\mathfrak{A}^{V_x(f)},\xi)\,\Omega_M(x)\,d\mathcal{S}(\xi)$$
$$<p\operatorname{vol}(S^{n-1})\underset{\sim}{M}(\mathcal{S})F_{n,p}(\delta)\le 0,$$

and thus for some $k\ge 1$, $x\in M$

$$\frac{d^2}{dt^2}\bigg|_{t=0}\underset{\sim}{M}(\varphi_{t\#}^{V_x(f_k)}\mathcal{S})<0.$$

Therefore $\mathcal{S}$ is not stable.

5.2. REMARK. We now indicate the dependence of $\delta(n,p)$ on n. The relevant relations are

$$\delta(p+1,n)>\delta(p,n), \tag{5-2}$$

$$\lim_{n\to\infty}\delta(n,p)=1. \tag{5-3}$$

The limit is uniform in p.

First note

$$F_{n,p+1}(\delta)-F_{n,p}(\delta)=2\int_0^\pi g_2(t,\delta)S_\delta(t)^{n-1}\,dt-\int_0^\pi g_1(t,\delta)s_1(t)^{n-1}\,dt>0 \tag{5-4}$$

as $g_2(t,\delta)\geq g_1(t,\delta)$ and $S_\delta(t)\geq s_1(t)$. This, together with the definition of $\delta(n,p)$, proves (5-2).

To prove (5-3) choose any δ_0 in the open interval $(\frac{1}{4},1)$ and set

$$r(\delta_0)=\frac{1}{2}\left(1+\frac{1}{\sqrt{\delta_0}}\right)>1. \tag{5-5}$$

For $\delta\in[\frac{1}{4},\delta_0]$, $s_\delta(\pi/2\sqrt{\delta})=1/\sqrt{\delta}>r(\delta_0)$. Therefore $s_\delta(t)\geq r(\delta_0)$ on some interval $\mathcal{I}(\delta)$. Moreover the length of $\mathcal{I}(\delta)$ is bounded uniformly from below for δ in $[\frac{1}{4},\delta_0]$. Also, $g_2(t,\delta)$ is bounded below by a positive constant when $\frac{1}{4}\leq\delta\leq\delta_0$, $0\leq t\leq\pi$. Therefore there is a positive constant $A(\delta_0)>0$ such that for all $\delta\in[\frac{1}{4},\delta_0]$

$$2\int_0^\pi g_2(t,\delta)s_\delta(t)^{n-1}\,dt\geq A(\delta_0)\,r(\delta_0)^{n-1}. \tag{5-6}$$

From this, the definition of $F_{n,p}(\delta)$, and (5-4) it follows that there is a $B(\delta_0)>0$ such that

$$F_{n,p}(\delta)\geq F_{n,1}(\delta)\geq A(\delta_0)r(\delta_0)^{n-1}-nB(\delta_0) \tag{5-7}$$

for all $\delta\in[\frac{1}{4},\delta_0]$. But $r(\delta_0)>1$, so the last inequality implies $F_{n,p}(\delta)\geq 0$ with $\frac{1}{4}\leq\delta\leq\delta_0$ can only hold for finitely many n. This proves (5-3).

6. Harmonic maps. Let N^p and M^n be compact Riemannian manifolds and $\psi: N\to M$ a smooth map. Then the energy of ψ is defined to be

$$E(\psi)=\frac{1}{2}\int_N \|d\psi\|^2\Omega_N=\frac{1}{2}\int_N\sum_{i=1}^{p}\langle\psi_*e_i,\psi_*e_i\rangle\Omega_N \tag{6-1}$$

where $e_1,\ldots,e_p$ is an orthonormal basis for the tangent space to N. A map $\psi: N\to M$ is *harmonic* if and only if it is a critical point for the energy integral. If $\psi: N\to M$ is harmonic then it is possible to write the second variation formula for the energy integral as ([7, p. 126])

$$\left.\frac{d^2}{dt^2}\right|_{t=0}E(\varphi_t^V\circ\psi)=\int_N\sum_i \mathcal{H}(\mathfrak{A}^V,\psi_*e_i)\Omega_N \tag{6-2}$$

where V is any smooth vector field on M and

(6-3) $$\mathcal{H}(\mathcal{A}^V, X) = \|\mathcal{A}^V X\|^2 - \langle R_V X, X\rangle.$$

The harmonic map ψ is *stable* if the second variation (6-2) is non-negative for all smooth vector fields V on M. Now assume M is as in Section 4. Let f be given by (4-1) and using the notation and estimates of Section 4 we find, for $X \neq 0$,

(6-4) $$\int_M \mathcal{H}(\mathcal{A}^{V_x(f)}, X)\Omega_M(x) < \mathrm{vol}(S^{n-1})F_n(\delta)\|X\|^2,$$

where

(6-5) $$F_n(\delta) = \int_0^\pi (g_2(t,\delta)s_\delta(t))^{n-1} - (n-1)\delta c_1(t)^2(s_1(t))^{n-1})\,dt.$$

So for any harmonic map $\psi: N^p \to M^n$

(6-6) $$\begin{aligned}\int_M \frac{d^2}{dt^2}\bigg|_{t=0} E(\varphi_t^{V_x(f)}\circ\psi)\Omega_M(x) &= \int_N \sum_{i=1}^p \int_M \mathcal{H}(\mathcal{A}^{V_x(f)}, \psi_* e_i)\Omega_M(x)\Omega_N \\ &\le p\,\mathrm{vol}(S^{n-1})E(\psi)F_n(\delta).\end{aligned}$$

But $E(\psi) \ge 0$ with equality if and only if ψ is constant and

(6-7) $$F_n(1) = -(n-2)\int_0^\pi \cos^2(t)\sin^{n-1}(t)\,dt,$$

which is negative when $n \ge 3$. Therefore the proof of Theorem 5.1 can easily be modified to give

6.1. THEOREM. *Let $n \ge 3$, $F_n(\delta)$ be given by* (6-5), *and set*

(6-8) $$\delta(n) = \inf\{\delta : \delta > \tfrac{1}{4} \text{ and } F_n(\delta) \le 0\}.$$

Then if M is a compact simply connected strictly $\delta(n)$-pinched Riemannian manifold of dimension n there are no nonconstant stable harmonic maps $\psi: N \to M$ for any compact Riemannian manifold N.

6.2. REMARK. As in Remark 5.2, it can be shown that $\delta(n) \to 1$ as $n \to \infty$.

ACKNOWLEDGEMENT. The computer calculations for the tables of values of $\delta(n,p)$ and $\delta(n)$ were done by Valerie Miller.

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