ON SUBMANIFOLDS WITH PLANAR NORMAL SECTIONS

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1. Introduction. Let M be a submanifold of dimension n in a Euclidean m-space E^m. For any point p in M and any unit vector t at p tangent to M, the vector t and the normal space N_pM of M at p determine an $(m-n+1)$-dimensional vector subspace $E(p,t)$ of E^m through p. The intersection of M and $E(p,t)$ gives rise to a curve γ in a neighborhood of p which is called the normal section of M at p in the direction t. The submanifold M is said to have planar normal sections if normal sections of M are planar curves. In this case, for any normal section γ, we have $\gamma' \wedge \gamma'' \wedge \gamma''' = 0$. A submanifold M is said to have pointwise planar normal sections if, for each p in M, every normal section γ at p satisfies $\gamma' \wedge \gamma'' \wedge \gamma''' = 0$ at p. Submanifolds with (pointwise) planar normal sections were investigated in [1, 2, 3, 6]. B. Y. Chen [3] classified surfaces in E^m with planar normal sections, and he proved the following theorem:

THEOREM A. *Let M be a surface in E^m with planar normal sections. If, locally, M does not lie in a 3-dimensional hyperplane of E^m, then M is an open subset of a Veronese surface in a 5-dimensional hyperplane of E^m.*

In the following, by a Veronese submanifold V^n we mean a real projective n-space isometrically imbedded in $E^{n+n(n+1)/2}$ by its first standard imbedding (cf. [4, pp. 141–148]).

In this paper, we generalize Theorem A to higher dimensions. We shall prove the following theorems.

THEOREM B. *Let M be an n-dimensional submanifold in E^m with planar normal sections. If, locally, M does not lie in an $(n(n+1)/2)$-dimensional affine subspace of E^m, then M is an open portion of a Veronese submanifold V^n in an $(n+n(n+1)/2)$-subspace of E^m.*

THEOREM C. *Let M be a 3-dimensional submanifold in E^m with planar normal sections. If, locally, M does not lie in a 5-space E^5 of E^m, then M is an open portion of a Veronese submanifold V^3 in E^9 or is the Riemannian direct product of the real line* **R** *with the Veronese surface.*

2. Proof of Theorem B. Let M be a submanifold in E^m, ∇ and $\tilde{\nabla}$ be the covariant derivatives of M and E^m, respectively. For any two vector fields X, Y tangent to M, the second fundamental form h is given by $h(X,Y) = \tilde{\nabla}_X Y - \nabla_X Y$. For any vector field ξ normal to M, we have $\tilde{\nabla}_X \xi = -A_\xi X + \nabla_X^\perp \xi$, where A_ξ is the Weingarten map associated with ξ and $\nabla^\perp$ is the normal connection of the normal bundle $N(M)$. Define the covariant derivative of h by

$$(2.1) \qquad (Dh)(X,Y,Z) = \nabla_X^\perp(h(Y,Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z),$$

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for vector fields X, Y, Z tangent to M, Dh is an $N(M)$-valued tensor of type $(0,3)$.

Assume that M has planar normal sections. Let t be a unit vector tangent to M at a point $p \in M$. Let $\gamma(s)$ be the normal section of M at p in the direction t with s as its arclength and $\gamma(0)=p$. We put $T(s)=\gamma'(s)$. Similar to [3], we can show that

$$h(T, \nabla_T T) \wedge h(T, T) = 0 \quad \text{along } \gamma. \tag{2.2}$$

First, assume that $\gamma(s)$ is a geodesic arc on a small neighborhood of $p=\gamma(0)$, then $\nabla_T T = 0$. Similar to [3], we have:

LEMMA 1. *Let M be a submanifold in E^m with planar normal sections. If a normal section $\gamma(s)$ is a geodesic on a neighborhood of $p=\gamma(0)$, then for every unit vector z orthogonal to $t=\gamma'(0)$, we have*

$$\langle h(t,t), h(t,z)\rangle = 0. \tag{2.3}$$

If γ is not a geodesic in any sufficiently small neighborhood of $p=\gamma(0)$, then there is a sequence $s_n \to 0$, such that $\nabla_{T_n} T \neq 0$, where $T_n = T(s_n)$. Let $u_n = (\nabla_{T_n} T)/|\nabla_{T_n} T|$, by (2.2), $h(T_n, u_n) \wedge h(T_n, T_n) = 0$. By choosing a local coordinate chart, we see that there is a subsequence of $\{u_n\}$ converging to a unit vector $u \in T_pM$, since $u_n \perp T_n$, we have $u \perp t$, and

$$h(t,u) \wedge h(t,t) = 0, \tag{2.4}$$

or equivalently, there is a unit vector $t^* \in T_pM$, such that

$$h(t, t^*) = 0. \tag{2.5}$$

Thus, we have the following.

LEMMA 2. *Let M be a submanifold in E^m with planar normal sections. If a normal section $\gamma(s)$ at p is not a geodesic arc on a sufficiently small neighborhood of p, then there is a unit vector $t^* \in T_pM$, such that $h(t, t^*) = 0$, where $t = \gamma'(0)$.*

Let p be any point on M such that there are $t \in U_pM$, $z \in U_pM$, $t \perp z$ and $\langle h(t,t), h(t,z)\rangle \neq 0$. Thus, there is a neighborhood U of t in U_pM, such that for any $t_1 \in U$, there is a vector $z_1 \perp t_1$, $z_1 \in U_pM$, and $\langle h(t_1,t_1), h(t_1,z_1)\rangle \neq 0$. Thus the normal section on the direction of t_1 is also not a geodesic in any neighborhood of p, so by Lemma 2, there is a unit vector $t_1^* \in U_pM$, such that $h(t_1, t_1^*) = 0$.

We shall show, in this case, $\dim(\operatorname{Im} h) \leq n(n-1)/2$ at p. In fact, since $\dim(\operatorname{Im} h) \leq n(n+1)/2$, we only need to show that there are at least n linearly independent relations between the vectors $h(e_i, e_j)$, $1 \leq i \leq j \leq n$, where $e_1, \dots, e_n$ is a basis for T_pM. May assume $e_1 \in U$. All vectors v satisfying $h(e_1, v) = 0$ form a subspace $V \subset T_pM$. If $V = T_pM$, we have n independent relations already. Otherwise, choose a basis $v_1, \dots, v_{k(1)}$ in V; then we have $k(1)$ linearly independent relations:

$$h(e_1, v_1) = 0, \dots, h(e_1, v_{k(1)}) = 0.$$

By changing basis in T_pM, we may assume

$$v_q = \sum_{i=1}^{q+1} b_{iq} e_i, \quad 1 \le q \le k(1).$$

Suppose we have found vectors $u_1, \dots, u_m \in U$, such that we have linearly independent relations

$$\begin{aligned} &h(u_1, v_1) = 0, \dots, h(u_1, v_{k(1)}) = 0, \\ &h(u_2, v_{k(1)+1}) = 0, \dots, h(u_2, v_{k(2)}) = 0, \\ &\qquad \dots \\ &h(u_m, v_{k(m-1)+1}) = 0, \dots, h(u_m, v_{k(m)}) = 0. \end{aligned} \tag{2.6}$$

So that $v_{k(m-1)+1}, \dots, v_{k(m)}$ span the space $V(u_p)$, where

$$V(u) = \{v \mid v \in T_pM, h(u, v) = 0\}, \quad u \in U,$$

and the dimension of $\mathrm{sp}\{u_1, \dots, u_j, v_1, \dots, v_{k(j)}\}$ is no more than $k(j)+1$, $1 \le j \le m$. Thus, we may assume that

$$u_j = \sum_{i=1}^{r} a_{ij} e_i, \quad \text{where } r = k(j-1)+1, \ a_{rj} \ne 0, \ 2 \le j \le m,$$

$$v_q = \sum_{i=1}^{q+1} b_{iq} e_i, \quad \text{for } 1 \le q \le k(m).$$

If $k(m) < n$, choose a unit vector $u_{m+1} \in U$, $u_{m+1} \notin V(u_m)$, $u_{m+1} = \sum_{i=1}^{s} a_{i,m+1} e_i$, where $s = k(m)+1$, $a_{s,m+1} \ne 0$. Choose any $v \ne 0$, $v \in V(u_{m+1})$. Then the relation $h(u_{m+1}, v) = 0$ is independent of all relations in (2.6).

If fact, if this were not the case, assume $v = \sum_{i=1}^{s} b_i e_i$; then

$$h(u_{m+1}, v) = \sum_{i=1}^{s} \sum_{j=1}^{s+1} a_{i,m+1} b_j h(e_i, e_j) = 0.$$

Since every relation in (2.6) fails to contain $h(e_i, e_{s+1})$ and $h(e_s, e_s)$, we have $b_{s+1} = 0$, $b_s = 0$. And every relation in (2.6) fails to contain $h(e_i, e_s)$ except the last one, hence we have $v \wedge u_m = 0$, thus $u_{m+1} \in V_m$; this is a contradiction.

Thus, if we choose a basis $v_{k(m)+1}, \dots, v_{k(m+1)}$ for $V(u_{m+1})$, then

$$h(u_{m+1}, v_{k(m)+1}) = 0, \dots, h(u_{m+1}, v_{k(m+1)}) = 0,$$

together with (2.6) are independent relations. By induction, there are n independent linear relations between $h(e_i, e_j)$ and we have $\dim(\mathrm{Im}\, h) \le n(n-1)/2$ at p.

Now let

$$M_1 = \{p \in M \mid \dim(\mathrm{Im}\, h) \le n(n-1)/2\}.$$

Then M_1 is a closed subset of M, if $M = M_1$, then by applying Theorem 1 of [2], M is locally contained in an $n(n+1)/2$ dimensional affine subspace of E^m. If $M \ne M_1$, then $M - M_1$ is an open subset of M, at every point $p \in M - M_1$, we have

$$\langle h(t,t), h(t,u)\rangle = 0 \quad \text{for any } t, u \in U_pM,\ t \perp u.$$

This implies that M is isotropic at p [7]. Let G be a component of $M - M_1$, then G is isotropic. By the following lemma, we know that G has planar geodesics.

LEMMA 3. *Let M be a submanifold of E^m with pointwise planar normal sections. If M is isotropic, then M has planar geodesics.*

Proof. Since M is isotropic, there is a differentiable function k on M, such that

$$\langle h(X,X), h(X,Y)\rangle = k^2\langle X,Y\rangle \quad \text{for } X, Y \in U(M).$$

Thus, $A_{h(X,X)}X = k^2X$.

If $k=0$, then $h(X^2)=0$ for any $X \in T(M)$; this means M is a linear subspace of E^m, so M has planar geodesics. Suppose $k>0$ at $p \in M$. Then in some neighborhood U of p, $k>0$, so for any $X \in T(U)$, $h(X,X) \neq 0$. If $\dim(\operatorname{Im} h)=1$ in U, then U is totally umbilical, thus M is a sphere, having planar geodesics. If $\dim(\operatorname{Im} h) \geq 2$ in an open subset $U_1 \subset U$, by Theorem 1 of [6], we have

$$(Dh)(X^3) = b\langle X_0, X\rangle h(X^2),$$

where b is a differentiable function on U_1, and X_0 is a differentiable vector field on U_1. By induction, it is easy to see that for any $n \geq 1$, $(D^n h)(X^{n+2}) \wedge h(X^2) = 0$. Thus, $A_{(D^n h)(X^{n+2})}X \wedge X = 0$. By Theorem 2.3 of [5], U_1 has geodesic normal sections. Thus k is a constant, so by continuity, $U_1 = U = M$ and $(Dh)(X^3) = 0$ for all $X \in T(M)$, hence M has planar geodesics.

Now, we know that G has planar geodesics, thus G is a helical immersion of a compact rank one symmetric space or its open subset [5, 10]. Thus we can choose an orthonormal basis $\{e_1, \dots, e_n\}$ of $T(G)$, such that the components R_{ijkr} of the curvature tensor R are independent of the points. Then we have [6]

$$\langle h(e_i,e_j), h(e_k,e_r)\rangle = (1/3)[R_{ikjr} + R_{irjk} + k_1^2(\delta_{ij}\delta_{kr} + \delta_{ik}\delta_{jr} + \delta_{ir}\delta_{jk})], \tag{2.12}$$

where $k_1 = |h(e_1,e_1)|$ is constant. By continuity, (2.12) is also true on boundary points of G, thus on the boundary, $\dim(\operatorname{Im} h)$ has the same value as in the interior. This implies G is closed in M. Thus $G=M$. Therefore by applying a result of Little [7] and Sakamoto [9], M is either a linear subspace or an open portion of a compact rank one symmetric space imbedded in E^m by its first standard imbedding. If M is an open portion of an n-sphere, a complex projective space, a quarternion projective space, or a Cayley plane imbedded in E^m by its first standard imbedding, then M lies in an $n(n+1)/2$ dimensional affine subspace of E^m (cf. [4, pp. 144, 155]). Thus we conclude that if locally M does not lie in an $n(n+1)/2$ dimensional affine space, then M is an open portion of a Veronese submanifold. □

3. The proof of Theorem C. Let M^3 be a 3-dimensional submanifold in E^m with planar normal sections, but at some point p, the normal section on some direction $t \in T_pM$ is not a geodesic. Then by Theorem B, M^3 is contained in a

6-dimensional space E^6. We shall show that in this case, either $\dim(\operatorname{Im} h) \leq 2$ at p or we can find a unit vector $u \in U_pM$, such that $h(u,v)=0$ for all $v \in T_pM$. We need the following lemma.

LEMMA 4. *Let A, B, C be linearly independent symmetric linear transformations on a 3-dimensional Euclidean space V. If for every vector $v \in V$, Av, Bv, Cv are linearly dependent, then there is a unit vector $u \in V$, such that $Au = Bu = Cu = 0$.*

Proof. Choose orthonormal eigenvectors e_1, e_2, e_3 of A such that $Ae_1 \neq 0$. Since Ae_1, Be_1, Ce_1 are linearly dependent by replacing one of B, C by a linear combination of A, B, C, we may assume $Ce_1 = 0$. Put

$$a_i = \langle Ae_i, e_i \rangle, \quad b_{ij} = \langle Be_i, e_j \rangle, \quad c_{ij} = \langle Ce_i, e_j \rangle, \quad i, j = 1, 2, 3.$$

Thus for any vector $v = x_1e_1 + x_2e_2 + x_3e_3$, the condition that Av, Bv, Cv are linearly dependent can be written as

$$\begin{vmatrix} a_1x_1 & b_{11}x_1 + b_{12}x_2 + b_{13}x_3 & 0 \\ a_2x_2 & b_{21}x_1 + b_{22}x_2 + b_{23}x_3 & c_{22}x_2 + c_{23}x_3 \\ a_3x_3 & b_{31}x_1 + b_{32}x_2 + b_{33}x_3 & c_{32}x_2 + c_{33}x_3 \end{vmatrix} = 0. \tag{3.1}$$

By linear combination again, we may assume $b_{11} = 0$. If $b_{21} = b_{31} = 0$, (3.1) would become

$$\begin{vmatrix} b_{22}x_2 + b_{23}x_3 & c_{22}x_2 + c_{23}x_3 \\ b_{32}x_2 + b_{33}x_3 & c_{32}x_2 + c_{33}x_3 \end{vmatrix} = 0,$$

this would imply that B, C were linearly dependent. Thus, at least one of b_{21}, b_{31} is not 0. Now in (3.1), the terms containing x_1 are

$$a_1x_1 \begin{vmatrix} b_{21}x_1 + b_{22}x_2 + b_{23}x_3 & c_{22}x_2 + c_{23}x_3 \\ b_{31}x_1 + b_{32}x_2 + b_{33}x_3 & c_{32}x_2 + c_{33}x_3 \end{vmatrix} = 0.$$

It is easy to see that $(b_{21}, b_{22}, b_{23}, c_{22}, c_{23})$ and $(b_{31}, b_{32}, b_{33}, c_{32}, c_{33})$ are proportional. If $a_2 = a_3 = 0$, we can find a unit vector $u = x_2e_2 + x_3e_3$, such that

$$b_{22}x_2 + b_{23}x_3 = 0, \quad c_{22}x_2 + c_{23}x_3 = 0;$$

u is the vector desired.

If $a_2 \neq 0$, $a_3 = 0$, then (3.1) becomes

$$a_2x_2(b_{12}x_2 + b_{13}x_3)(c_{32}x_2 + c_{33}x_3) = 0,$$

thus $c_{32} = c_{33} = 0$. But $C \neq 0$, so $c_{22} \neq 0$; this means $b_{31} = b_{32} = b_{33} = 0$. Thus, e_3 is the vector desired. If a_2 and a_3 are both not zero, the same argument shows that $C = 0$, which is impossible. □

Now, assume $\dim(\operatorname{Im} h) = 3$ at p, and denote an orthonormal basis for N_pM by ξ_4, ξ_5, ξ_6. By

$$\langle A_x t, u \rangle = \langle h(t,u), \xi_x \rangle \quad t, u \in T_pM, \ x = 4, 5, 6,$$

we define symmetric linear transformations A_4, A_5, A_6 on T_pM. Since $\dim(\operatorname{Im} h) = 3$, they are linearly independent. By Lemma 2, there is a neighborhood U of t in T_pM, such that if $t_1 \in U$, we can find a unit vector $t_1^* \in U_pM$, $h(t_1, t_1^*) = 0$. This means $\langle A_x t_1, t_1^* \rangle = 0$, $x = 4, 5, 6$. So $A_4 t_1, A_5 t_1, A_6 t_1$ are linearly dependent. Although this condition is satisfied in an open set U, but (3.1) is a polynomial equation on (x_1, x_2, x_3); if it is satisfied in some open set of the (x_1, x_2, x_3) space, it is satisfied for all x_1, x_2, x_3. Thus, by Lemma 4, there is a unit vector x, $h(t, x) = 0$ for all $t \in T_pM$.

Now, in some neighborhood G of p, $\dim(\operatorname{Im} h) = 3$. It is easy to see that x is a differentiable vector field in G. Thus, we may assume $e_1 = x$, so

$$h(e_1, e_1) = h(e_1, e_2) = h(e_1, e_3) = 0. \tag{3.2}$$

It is also easy to see that

$$\begin{aligned}
&(Dh)(e_1^3) = (Dh)(e_1^2, e_2) = (Dh)(e_1^2, e_3) = 0,\\
&(Dh)(e_1^2, e_2) = \nabla_{e_1}^{\perp} h(e_1, e_2) - h(\nabla_{e_1} e_1, e_2) - h(e_1, \nabla_{e_1} e_2)\\
&\qquad = -w_1^2(e_1) h(e_2, e_2) - w_1^3(e_1) h(e_2, e_3),
\end{aligned}$$

where w_1^2, w_2^3, w_1^3 are connection forms on M. Since $h(e_2, e_2)$ and $h(e_2, e_3)$ are linearly independent, we have

$$w_1^2(e_1) = w_1^3(e_1) = 0. \tag{3.3}$$

Also, by the equation of Codazzi and

$$\begin{aligned}
(Dh)(e_2, e_1, e_3) &= -h(\nabla_{e_2} e_1, e_3)\\
&= -w_1^2(e_2) h(e_2, e_3) - w_1^3(e_2) h(e_3, e_3),\\
(Dh)(e_3, e_1, e_2) &= -h(\nabla_{e_3} e_1, e_2)\\
&= -w_1^2(e_3) h(e_2, e_2) - w_1^3(e_3) h(e_2, e_3),
\end{aligned}$$

since $h(e_2, e_2), h(e_2, e_3), h(e_3, e_3)$ are linearly independent, we have

$$w_1^2(e_2) = w_1^3(e_3), \quad w_1^3(e_2) = w_1^2(e_3) = 0. \tag{3.4}$$

By (3.2) and (3.3), we have $\tilde{\nabla}_{e_1} e_1 = \nabla_{e_1} e_1 = 0$. Thus, the integral curves of e_1 are straight lines. By (3.4) we have

$$[e_2, e_3] = w_3^2(e_2) e_2 - w_2^3(e_3) e_3,$$

so the 2-dimensional distribution spanned by e_2, e_3 is integrable, thus we see that M^3 is the Riemannian direct product of $\mathbf{R}$ with a 2-dimensional manifold M_1. Since M has planar normal sections, so does M_1. But M_1 does not lie in a 3-space, by Theorem A, M_1 is an open subset of a Veronese surface.

By the following lemma, we know that the direct product of $\mathbf{R}$ with the Veronese surface has planar normal sections.

LEMMA 5. *Let M^n be a submanifold of E^m with planar normal sections. Then the Riemannian direct product $N = E^k \times M^n$ is a submanifold of E^{m+k} with planar normal sections.*

Proof. Let the equation of M^n in E^m be (locally)

$$x_i = f_i(x_{m-n+1}, \dots, x_m), \quad I \le i \le m-n. \tag{3.5}$$

Assume the origin $o \in M^n$ and the plane $x_1 = \cdots = x_{m-n} = 0$ is the tangent plane of M^n, and the equation of $N = E^k \times M$ in E^{m+k} is the same as (3.5), the coordinate in E^{m+k} is $(x_1, \dots, x_m, x_{m+1}, \dots, x_{m+k})$. Let $X_i = \partial/\partial x_i$, then a basis for $T_o N$ is $X_{m-n+1}, \dots, X_{m+k}$, and a basis for $N_o M$ is $X_1, \dots, X_{m-n}$, which is also a basis for the normal space of N at o.

We only need to show that the normal section at o on the direction $t = X_{m-n+1} \cos\theta + X_{m+k} \sin\theta$ $(0 < \theta < 2)$ is planar. In fact, the normal section on the direction X_{m-n+1} has equation

$$x_j = 0, \quad x_i = f_i(x_{m-n+1}, 0, \dots, 0), \quad 1 \le i \le m-n, \; m-n+1 < j \le m+k.$$

By assumption, it is planar. The normal section on the direction $t = X_{m-n+1} \cos\theta + X_{m+k} \sin\theta$ has equation

$$x_{m-n+1} \sin\theta - x_{m+k} \cos\theta = 0, \quad x_i = f_i(x_{m-n+1}, 0, \dots, 0), \; 1 \le i \le m-n,$$
$$x_j = 0, \quad m-n+1 < j < m+k.$$

It is easy to see that this is also planar.

Now we prove Theorem C. Theorem B shows that either M is the second standard immersion of a 3-sphere or $\dim(\operatorname{Im} h) \le 3$ at each point $p \in M$.

Suppose $\dim(\operatorname{Im} h) \le 3$ at each point $p \in M$. Let $M_1 = \{p \in M \mid \dim(\operatorname{Im} h) = 3\}$. Then by the arguments in the last paragraph, we know that each component U of M_1 is an open subset of the direct product of **R** with the Veronese surface, hence it is also closed in M. Hence either $U = M$ or M_1 is empty. Theorem C is proved. □

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REFERENCES

1. B. Y. Chen, *Submanifolds with planar normal sections,* Soochow J. Math. 7 (1981), 19–24.
2. ———, *Differential geometry of submanifolds with planar normal sections,* Ann. Mat. Pura Appl. 130 (1982), 59–66.
3. ———, *Classification of surfaces with planar normal sections,* J. Geom. 20 (1983), 122–127.
4. ———, *Total mean curvature and submanifolds of finite type,* Ser. in Pure Math., Vol. I, World Scientific, Singapore, 1984.
5. Y. Hong and C. S. Houh, *Helical immersions and normal sections,* Kōdai Math. J., to appear.
6. Y. Hong, C. S. Houh, and G. Q. Wang, *Some surfaces with pointwise planar normal sections,* Bull. Soc. Math. Belg., to appear.

7. J. A. Little, *Manifolds with planar geodesics,* J. Differential Geom. 11 (1976), 265–285.
8. B. O'Neill, *Isotropic and Kähler immersions,* Canad. J. Math. 17 (1965), 907–915.
9. K. Sakamoto, *Planar geodesic immersions,* Tôhoku Math. J. 29 (1977), 25–56.
10. P. Verheyen, *Submanifolds with geodesic normal sections are helical,* to appear.

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