ON QUASICONVEX FUNCTIONS

W. K. Hayman

Dedicated to Professor George Piranian on his seventieth birthday

1. Introduction. Let $f(z)$ be a convex univalent function not assuming the value d and let $F \equiv f$ or

$$F(z)=\frac{af(z)+b}{f(z)-d}=\sum_0^\infty c_n z^n \tag{1.1}$$

be a Möbius transformation of f. We shall call such functions $F(z)$ quasiconvex and denote the class of all such functions by Q. The class Q was considered by R. R. Hall [2] who proved the following.

THEOREM A. *If $F\in Q$ then for $|z|=\rho$*

$$|F(z)-c_0|<\frac{\pi^2|c_1|}{2}\frac{\rho}{(1-\rho)},$$

and hence $|c_n|\le A_0|c_1|$, where A_0 is an absolute constant.

Further results were obtained by Barnard and Schober [1], by variational techniques. They denoted by $\hat{K}$ the subclass of Q for which $c_0=0$ and $c_1=1$ and by K the class of normalized convex functions and proved the following.

THEOREM B. *If $0<r<1$, $F\in\hat{K}$, and*

$$m(r,F)=\inf_{|z|=r}|F(z)|,\qquad M(r,F)=\sup_{|z|=r}|F(z)|,$$

then the extreme values of $m(r,F)$ and $M(r,F)$ for given r and $F\in\hat{K}$ occur when $f(z)$ maps the unit disk Δ onto a vertical strip. In particular $\max|c_2|=1.3270$.

2. Statement of new results. Although Theorem B gives sharp results for the class $\hat{K}$, the individual bounds usually seem to be solutions of rather complicated transcendental equations. This is certainly the case for $|c_2|$. If we confine ourselves to the subclass of Q consisting of functions omitting a fixed value (e.g., zero), the bounds become more manageable and only elementary considerations such as subordination are necessary. The extremals turn out once again to be the functions considered by Barnard and Schober [1]. We have the following results.

THEOREM 1. *If $F\in Q$, and $F(z)\neq 0$ in Δ, then we have the sharp inequality*

$$|c_1|\le\frac{8}{\pi}|c_0|. \tag{2.1}$$

Further we have for $|z|=\rho$ the sharp inequalities

Received August 15, 1984.
Michigan Math. J. 32 (1985).

(2.2) $$\left(\frac{\pi}{\cos^{-1}\rho}-1\right)^{-2} \leq \left|\frac{F(z)}{c_0}\right| \leq \left(\frac{\pi}{\cos^{-1}\rho}-1\right)^{2}.$$

It seems likely to be hard to obtain the right value for the coefficient bound A_0 in Hall's Theorem A. However it is not difficult to get sharp asymptotic bounds. It turns out that if $F \in Q$ then there is always a constant $c = c(F)$, such that

(2.3) $$\phi(z) = (F(z)+c)^2$$

is univalent in Δ. From this and classical results [3, Chapter V] we deduce the following.

THEOREM 2. *If $F \in Q$, there exists $\alpha = \alpha(F)$ such that $0 \leq \alpha < \infty$ and*

(2.4) $$\lim_{\rho \to 1} (1-\rho) M(\rho, F) = \alpha;$$

(2.5) $$\lim_{n \to \infty} |c_n| = \alpha.$$

We can also obtain sharp bounds for α in terms of $|c_1|$, or the nearest omitted value. It is curious that the upper bounds cannot be attained. It does not look as if the two bounds can be obtained from each other.

THEOREM 3. *If $F \in \hat{K}$, then*

(2.6) $$M(\rho, F) < \frac{\pi}{2\cos^{-1}\rho}\left(\frac{\pi}{\cos^{-1}\rho}-2\right)$$

and hence we have

(2.7) $$\alpha(F) < \frac{\pi^2}{4}.$$

If $F \in Q$, and $F(z) \neq 0$ in Δ, then

(2.8) $$\alpha(F) < \frac{\pi^2}{2}|c_0|.$$

The inequalities (2.7) *and* (2.8) *are sharp, but* (2.6) *is not.*

Theorem 3 shows that Hall's constant A_0 cannot be less than $\pi^2/4$. Also (2.6) shows that the bound for $M(\rho, F)$ obtained by Hall [2] is about twice the correct bound.

3. Characterization of quasiconvex domains. Let Δ be the unit disk and let D be the image of Δ by $F(z) \in Q$. We shall call such a domain D quasiconvex. We have the following simple characterization of quasiconvex domains.

THEOREM 4. *A domain D is quasiconvex if and only if there is a finite or infinite point ξ_0 outside D, such that if ξ is any other point outside D then there is a circle or straight line $C(\xi)$ through ξ_0 and ξ lying outside D.*

Proof. If $\xi_0 = \infty$, $C(\xi)$ is a straight line and we obtain the usual characterization of convex domains. This corresponds to the limiting case $F \equiv f$ in (1.1). If D

satisfies the conditions of Theorem 4 and $W=F(z)$ maps Δ onto D, write

$$w=\frac{1}{W-\xi_0}=f(z).$$

Then $f(z)$ maps Δ onto a domain D_0 in the w plane, such that if w' is any point outside D_0 there is a straight line $c(w')$ lying outside D_0. Thus D_0 is convex, f is a convex function and $f\neq 0$. Thus $F(z)=\xi_0+1/f\in Q$. Conversely, if f is convex, D_0 is the image of Δ by f, and $F(z)$ is given by (1.1), then the image of Δ by $F(z)$ is the image of D_0 by

$$W=\frac{aw+b}{w-d}. \tag{3.1}$$

If ξ lies outside D then

$$\xi=\frac{a\xi'+b}{\xi'-d},$$

where ξ' lies outside D_0. Thus there is a line $L(\xi')$ lying outside D_0, since D_0 is convex. So the image of $L(\xi')$ by (3.1) is a circle or line $C(\xi)$, containing ξ and $L(\infty)=a=\xi_0$, and lying outside D. This completes the proof of Theorem 4. □

We deduce the following.

THEOREM 5. *Let D_F ($F(z)\in Q$) be the image of Δ by $w=F(z)$, and suppose that w_0 is a point outside D_F. There exists a circle C, containing the point w_0 and a line L touching C at $w_1\neq w_0$, such that*

$$D_F\subset C_E\cap H_L, \tag{3.1}$$

where C_E is the exterior of C and H_L is that half plane complementary to L, which contains w_0.

Proof. It follows from Theorem 4 that there exists a point w_1 and a line L through w_1 and a circle or line C through w_0 and w_1, both lying outside $D=D_F$. If L contains w_0, let H be the half plane complementary to L containing D_F. We then draw a circle C' touching L at w_0 and lying otherwise in the other complementary half plane H' of L. We also draw a line L' in H', parallel to L and touching C'. With C' and L' instead of C and L we clearly have (3.1). If C is a straight line we argue similarly. Thus we may assume that C is a circle which contains w_0, but that line L does not. If C touches L, we have (3.1). Otherwise C meets L in two points and so there exists a circle C', containing w_0 but lying otherwise inside C and touching L. Replacing C by C' we again have (3.1), and Theorem 5 is proved. □

By using Theorem 5 and subordination we can confine our study of Q largely to the study of functions mapping Δ onto the type of domains occurring on the right-hand side of (3.1). In the next section we accordingly study the hyperbolic metric in these domains.

4. Hyperbolic metric and hyperbolic distance. We recall [4, p. 48] the hyperbolic metric $\sigma(w)$ in a simply connected domain D. If $w=f(z)$ maps Δ onto D, then

$$\sigma_D(w)\,|dw|=\frac{|dz|}{1-|z|^2}. \tag{4.1}$$

The hyperbolic distance between two points w_1, w_2 in D is defined to be

$$d(w_1,w_2,D)=\inf\int_J \sigma(w)\,|dw|, \tag{4.2}$$

where the infimum is taken over all curves J joining w_1 to w_2 in D. The infimum is attained if J is the image of a segment of the real axis by a conformal map of Δ onto D. We need the following.

LEMMA 1. *If D is given by the right-hand side of* (3.1), *where C is the circle $|w-a|=r$ and $w_1=a-re^{i\theta_1}$, then for $w\in D$ and $|w-a|=R>r$,*

$$\sigma_D(w)\geq\frac{\pi r}{(r+R)^2\sin\{2\pi r/(r+R)\}}. \tag{4.3}$$

Equality holds if and only if $w=a+Re^{i\theta_1}$.

Proof. This lemma will be proved entirely elementarily by calculating the mapping function.

Since $\sigma_D(w)$ is conformally invariant we may rotate and translate D without altering (4.3). So we assume without loss of generality that $\theta_1=0$ and $a=-r$, so that $D=D_r$ where, for $0<r<\infty$,

$$D_r=\{w\mid |w+r|>r,\ \operatorname{Re} w>-2r\}. \tag{4.4}$$

We set

$$W=\frac{\pi r}{w+2r}. \tag{4.5}$$

Then D_r corresponds in the W plane to the vertical strip

$$S=\{W\mid 0<\operatorname{Re} W<\pi/2\}.$$

The function

$$W=\tan^{-1}z+\frac{\pi}{4}=\frac{1}{2i}\log\frac{1+iz}{1-iz}+\frac{\pi}{4}$$

maps Δ onto S. Thus we have at least for real z and W

$$\sigma_S(W)=\sigma_\Delta(z)\frac{dz}{dW}=\frac{1+z^2}{1-z^2}=\frac{1}{\sin 2W}.$$

Evidently $\sigma_S(W+ic)=\sigma_S(W)$, when c is real, so that for $W=U+iV$

$$\sigma_S(W)=\frac{1}{\sin(2U)}. \tag{4.6}$$

Suppose now that $w=-r+Re^{i\theta}$ is a point in D_r, so that $R>r$ and W and w are related by (4.5). Then, writing $D=D_r$, we have

$$\sigma_D(w)=\sigma_S(W)\left|\frac{dW}{dw}\right|=\frac{\pi r}{|r+Re^{i\theta}|^2\sin(2U)},$$

where

$$U=\operatorname{Re}\frac{\pi r}{(r+Re^{i\theta})}=\frac{\pi r(r+R\cos\theta)}{r^2+2rR\cos\theta+R^2}=\frac{\pi}{2}-\frac{\pi(R^2-r^2)}{2|r+Re^{i\theta}|^2}.$$

Thus writing $|r+Re^{i\theta}|^2=x$, we have

$$\frac{1}{\sigma_D(w)}=\frac{x}{\pi r}\sin\{\pi(R^2-r^2)/x\}\leq\frac{(r+R)^2}{\pi r}\sin\{\pi(R^2-r^2)/(r+R)^2\},$$

since $t^{-1}\sin t$ decreases with increasing t. Equality holds if and only if $\theta=0$, and this proves Lemma 1. □

Lemma 1 enables us to confine our estimates for the hyperbolic metric and hyperbolic distance to those for the domains D_r on the positive w axis. To prove (2.1) we need the following.

LEMMA 2. *If D_r is defined by* (4.4) *and w is a point of D_r such that*

$$|w+r|=r+d,$$

then

$$\sigma_{D_r}(w)\geq\frac{\pi}{8d}.$$

Equality holds if and only if $w=d$ and $r=\frac{1}{2}d$.

Proof. We may write $w=-r+Re^{i\theta}$, where $R=r+d$, and deduce from Lemma 1 that

$$\sigma_D(r)\geq\frac{\pi r}{(r+R)^2\sin\{2\pi r/(r+R)\}}\tag{4.7}$$

with equality if and only if $\theta=0$. We now calculate the minimum value of the right-hand side of (4.7) when $R=r+d$, and r varies. To do this we let

$$\frac{2r}{r+R}=\frac{1}{2}+t,\qquad 2r=\left(\frac{1}{2}+t\right)(2r+d),\qquad r=\frac{d}{2}\frac{1+2t}{1-2t},$$

$$\frac{r}{(r+R)^2}=\frac{1}{r}\left(\frac{r}{r+R}\right)^2=\frac{2}{d}\frac{1-2t}{1+2t}\left(\frac{1+2t}{4}\right)^2=\frac{1-4t^2}{8d}.$$

Thus

$$\frac{\pi r}{(r+R)^2\sin\{2\pi r/(r+R)\}}=\frac{\pi(1-4t^2)}{8d\sin\{\pi/2+\pi t\}}=\frac{\pi(1-4t^2)}{8d\cos\pi t}\geq\frac{\pi}{8d},$$

with equality only when $t=0$, as we see from the cosine product. In this case $R=3r$ and $d=2r$, and Lemma 2 is proved. □

We finish this section by proving a result involving hyperbolic distances in D_r.

LEMMA 3. *Suppose that R_1, R_2 are fixed $0<R_1<R_2$ and that w_1, w_2 are two points of D_r such that*

$$|w_1+r| \le r+R_1 \quad \text{and} \quad |w_2-w_1| \ge R_2-R_1.$$

Then

$$d(w_1, w_2, D_r) \ge \log\left(\frac{1+\tan h}{1-\tan h}\right), \quad \text{where } h=\frac{\pi}{4}\frac{(R_2/R_1)^{1/2}-1}{(R_2/R_1)^{1/2}+1}.$$

Equality holds if and only if $w_1=R_1$, $w_2=R_2$, and $2r=(R_1R_2)^{1/2}$.

Proof. We suppose that J is a geodesic joining w_1, w_2, so that

$$d(w_1, w_2)=\int_J \sigma_D(w)\,|dw|. \tag{4.8}$$

We write $|w-w_1|=t$, and denote points of J on this circle by w_t. Thus it follows from Lemma 1 that, since $|w_t+r| \le r+R_1+t$,

$$\begin{aligned}\sigma_D(w_t) &\ge \frac{\pi r}{(r+|w_t+r|)^2 \sin\{2\pi r/(r+|w_t+r|)\}} \\ &\ge \frac{\pi r}{(2r+R_1+t)^2 \sin\{2\pi r/(2r+R_1+t)\}}.\end{aligned} \tag{4.9}$$

The first inequality follows from Lemma 1. For the second we use the fact that $\theta^{-1}\sin\theta$ and so $\theta^{-2}\sin\theta$ decreases with increasing θ for $0<\theta<\pi$. Thus if $0<x_1<x_2<\pi$,

$$\frac{1}{x_1^2}\sin x_1 > \frac{1}{x_2^2}\sin x_2.$$

We set

$$x_1=\frac{2\pi r}{2r+R_1+t}, \qquad x_2=\frac{2\pi r}{r+|w_t+r|},$$

and recall that, since $w_t \in D$, $|w_t+r|>r$.

Now (4.8) and (4.9) show that

$$\begin{aligned} d(w_1, w_2, D) &\ge \int_J \sigma_D(w_t)\,|dt| \ge \int_0^{R_2-R_1} \frac{\pi r\,dt}{(2r+R_1+t)^2 \sin\{2\pi r/(2r+R_1+t)\}} \\ &= \int_{R_1}^{R_2} \sigma_D(x)\,dx = d(R_1, R_2, D), \end{aligned}$$

since the positive axis is a geodesic for the hyperbolic metric in D. Equality is possible in the above inequalities only if

$$|w_t+r|=r+R_1+t, \quad 0<t<R_2-R_1,$$

i.e., $w_1=R_1$ and $w_2=R_2$. Thus, with the hypotheses of Lemma 3, $d(w_1, w_2, D_r)$ can attain its minimum value only in this case. It remains to calculate the minimum for varying r.

To do this, we return to the transformation (4.5). Then $w=R_j$ correspond to $W=U_j$ where

$$U_j=\frac{\pi r}{2r+R_j}. \tag{4.10}$$

Also by conformal invariance we have

$$d(R_1,R_2,D_r)=d(U_2,U_1,S)=\int_{U_2}^{U_1}\sigma_S(U)\,dU=\int_{U_2}^{U_1}\frac{dU}{\sin(2U)}.$$

We note that by (4.10) we have

$$0<U_2<U_1<\frac{\pi}{2}\quad\text{and}\quad \frac{R_2U_2}{\pi-2U_2}=\frac{R_1U_1}{\pi-2U_1}. \tag{4.11}$$

We proceed to minimize the integral

$$I=\int_{U_2}^{U_1}\frac{dU}{\sin 2U}$$

when U_1, U_2 are related by (4.11) and R_1, R_2 are fixed.

To do this we make a substitution

$$\frac{2U}{\pi-2U}=e^x,\qquad \frac{2U_j}{\pi-2U_j}=e^{x_j},\quad j=1,2.$$

Thus

$$x_1-x_2=\log\frac{R_2}{R_1}=c,$$

say, where c is a positive constant. Also

$$I=\frac{\pi}{2}\int_{x_2}^{x_1}\frac{e^x\,dx}{(1+e^x)^2\sin\{\pi e^x/(1+e^x)\}}=\frac{\pi}{8}\int_{x_2}^{x_1}\phi(x)\,dx.$$

Here

$$\phi(x)=\frac{1}{\cosh^2(x/2)\cos(\pi/2\tanh(x/2))}=\frac{1-t^2}{\cos((\pi/2)t)},$$

where $t=\tanh(x/2)$. Evidently $\phi(x)$ is an even symmetric function of x for $-\infty<x<+\infty$. Next

$$\frac{\cos((\pi/2)t)}{1-t^2}=\prod_{n=1}^{\infty}\left\{1-\left(\frac{t}{2n+1}\right)^2\right\}$$

and the right-hand side evidently decreases with increasing t for $0<t<1$. Thus $\phi(x)$ increases with x when x is positive. From these facts it is clear that if x_1-x_2 is constant and equal to c, the integral

$$\int_{x_2}^{x_1}\phi(x)\,dx$$

has its minimum value when $x_2=-\frac{1}{2}c$ and $x_1=\frac{1}{2}c$. Translating back into the U plane, this corresponds to

$$\frac{4U_1U_2}{(\pi-2U_1)(\pi-2U_2)}=1, \quad \text{i.e.,} \quad U_1+U_2=\frac{\pi}{2},$$

and thus leads to $r=\frac{1}{2}(R_1R_2)^{1/2}$, as required. Also in this case

$$U_2=\frac{\pi}{2+2(R_2/R_1)^{1/2}}, \qquad U_1=\frac{\pi}{2+2(R_1/R_2)^{1/2}}.$$

We write

$$U_2=\frac{\pi}{4}-h, \quad U_1=\frac{\pi}{4}+h, \quad \text{where } h=\frac{\pi}{4}\frac{(R_2/R_1)^{1/2}-1}{(R_2/R_1)^{1/2}+1},$$

and, using (4.6), we have

$$d(R_1,R_2,D_r)=d(U_1,U_2,S)=\int_{\pi/4-h}^{\pi/4+h}\frac{dU}{\sin(2U)}$$

$$=\int_0^{2h}\frac{dt}{\cos t}=\log\left(\frac{1+\tan h}{1-\tan h}\right).$$

This proves Lemma 3. □

5. Proof of Theorem 1 and of (2.6). Suppose now that $F(z)\in Q$, and that $F(z)\neq w_0$. Then D_F lies in a domain given by (3.1). After a rotation and translation in the w plane we may assume that D_F lies in one of the domains D_r. Also since the hyperbolic metric decreases with expanding domain and since functions mapping Δ onto D_r belong to Q, we may assume that $D_F=D_r$. We now write $D=D_F=D_r$.

Suppose then that $F(z)=c_0+c_1z+\cdots$ maps Δ onto D and that $F(z)\neq w_0$, where $|w_0+r|=r$. We write $|c_0-w_0|=R_1$, so that $|c_0+r|\leq|w_0+r|+|c_0-w_0|\leq r+R_1$. Thus Lemma 2 and (4.1) show that

$$\sigma_D(c_0)=\frac{1}{|c_1|}\geq\frac{\pi}{8R_1}, \quad \text{i.e.,} \quad R_1=|c_0-w_0|\geq\frac{\pi}{8}|c_1|,$$

and this yields (2.1) when we set $w_0=0$. Also, by Lemma 2, equality holds when $D_F=D_r$ and $r=\frac{1}{2}c_0$.

Next we prove the right-hand inequality in (2.2). We suppose again that $F(z)\in Q$, $F(z)\neq w_0$ and $F(0)=c_0$, and estimate

$$M(\rho)=\sup_{|z|=\rho}|F(z)-c_0|.$$

Again we may assume that $F(z)$ maps Δ onto the domain D_r given by (4.4). We write

$$(5.1) \qquad w_1=c_0, \quad w_2=F(\rho e^{i\theta}), \quad \text{where } |w_2-c_0|=M(\rho).$$

Then the conformal invariance of hyperbolic distance shows that

$$(5.2) \qquad d(w_1,w_2,D_r)=d(0,\rho e^{i\theta},\Delta)=\frac{1}{2}\log\frac{1+\rho}{1-\rho}.$$

Also if $|c_0-w_0|=R_1$ and $R_2=M+R_1$, we have

$$|w_1+r|\le|w_0+r|+R_1=r+R_1 \quad\text{and}\quad |w_2-w_1|=M=R_2-R_1.$$

Thus Lemma 3 and (5.1) show that

$$d(w_1,w_2,D_r)=\frac{1}{2}\log\frac{1+\rho}{1-\rho}\ge\log\frac{1+\tan h}{1-\tan h}=\frac{1}{2}\log\left(\frac{1+\sin 2h}{1-\sin 2h}\right),$$

where

$$h=\frac{\pi}{4}\frac{(R_2/R_1)^{1/2}-1}{(R_2/R_1)^{1/2}+1}.$$

Thus

$$\rho\ge\sin 2h=\sin\left\{\frac{\pi}{2}\frac{(R_2/R_1)^{1/2}-1}{(R_2/R_1)^{1/2}+1}\right\}=\cos\left(\frac{\pi}{(R_2/R_1)^{1/2}+1}\right),$$

so that

(5.3) $$R_2\le R_1\left(\frac{\pi}{\cos^{-1}\rho}-1\right)^2,$$

and

(5.4) $$M=R_2-R_1\le\frac{\pi}{\cos^{-1}\rho}\left(\frac{\pi}{\cos^{-1}\rho}-2\right).$$

Thus using (5.1) we deduce that for $|z|=\rho$,

$$|F(\rho e^{i\theta})-w_0|\le|F(0)-w_0|+|F(\rho e^{i\theta})-F(0)|\le R_1+M=R_2$$
$$\le R_1\left(\frac{\pi}{\cos^{-1}\rho}-1\right)^2.$$

Setting now $w_0=0$ and $R_1=|c_0|=|F(0)|$, we obtain the right-hand inequality in (2.2). Also, since $F\neq 0$ and $F\in Q$, $1/F\in Q$, and by applying the above inequality to $1/F(z)$ we obtain the left-hand inequality in (2.2). Equality holds for the right-hand inequality of (2.2) only if $F(z)/c_0$ maps Δ onto the domain D_r given by (4.4) with

$$2r=(R_2/R_1)^{1/2}=(\pi/\cos^{-1}\rho-1).$$

For the left-hand inequality we must have

$$2r=(R_1/R_2)^{1/2}=(\cos^{-1}\rho)/(\pi-\cos^{-1}\rho).$$

This completes the proof of Theorem 1. □

We finish this section by proving (2.6). Here we need to use Barnard and Schober's Theorem B. They prove in effect that for $f\in\hat{K}$, the maximum value of $M(\rho,f)$ occurs when

$$f(z)=\frac{F(z)-F(0)}{F'(0)},$$

where $F(z)$ maps Δ onto one of the domains D_r given by (4.4), and $F(0)>0$. For these functions we deduce from (5.4) that

$$|F(z)-F(0)| \le F(0)\frac{\pi}{\cos^{-1}\rho}\left(\frac{\pi}{\cos^{-1}\rho}-2\right).$$

On the other hand, since D_r always contains the half plane Re $w>0$, we deduce from subordination that $F'(0)>2F(0)$. Thus

$$|F(z)-F(0)| < \frac{1}{2}F'(0)\frac{\pi}{\cos^{-1}\rho}\left(\frac{\pi}{\cos^{-1}\rho}-2\right),$$

and this proves (2.6). □

6. Proof of Theorem 2 and completion of proof of Theorem 3. We suppose now that $F(z)$ is a general function in Q. Suppose that w_0 is a point outside the image of D_F. Then Theorem 4 shows that there is a line L outside D_F. If w_1 is a point on L, we deduce that $\phi(z)=(F(z)-w_1)^2$ is univalent and $\phi(z)\neq 0$. Thus

$$F_1(z)=F(z)-w_1=\phi^{1/2}(z)=c_0-w_1+\sum_1^\infty c_n z^n$$

is circumferentially mean $\frac{1}{2}$-valent. Now we deduce from [3, Chapter 5, Theorems 5.1 and 5.7] that $((1-\rho)/(1+\rho))M(\rho,F_1)$ is a decreasing function of ρ, for $0\le\rho<1$, and so tends to a finite limit $\frac{1}{2}\alpha$, say as $\rho\to 1$. Also,

$$\frac{1-\rho}{1+\rho}M(\rho,F)=\frac{1-\rho}{1+\rho}M(\rho,F_1)+O(1-\rho)\to\tfrac{1}{2}\alpha$$

and this is (2.4). The limiting relation (2.5) is also a consequence of the above mentioned theorems. This proves Theorem 2.

We also note that from the above monotonicity we have

$$\begin{aligned} M(\rho,F) &\ge M(\rho,F_1)-|w_1| \\ &\ge \frac{1}{2}\alpha\frac{1+\rho}{1-\rho}-|w_1|=\frac{\alpha}{1-\rho}-\frac{1}{2}\alpha-|w_1|, \quad 0<\rho<1. \end{aligned} \tag{6.1}$$

We can now complete the proof of Theorem 3. We write

$$\cos^{-1}\rho=\theta, \qquad \rho=\cos\theta=1-\tfrac{1}{2}\theta^2+O(\theta^4),$$

so that, as $\rho\to 1$, we have

$$\theta=\{2(1-\rho)\}^{1/2}\{1+O(1-\rho)\}.$$

Thus

$$\frac{\pi}{\cos^{-1}\rho}=\frac{\pi}{\theta}=\frac{\pi}{\{2(1-\rho)\}^{1/2}}\{1+O(1-\rho)\},$$

and

$$\left(\frac{\pi}{\cos^{-1}\rho}-1\right)^2=\frac{\pi^2}{2(1-\rho)}-\frac{\pi\sqrt{2}}{(1-\rho)^{1/2}}+O(1).$$

Thus for $F\in\hat{K}$ we deduce from (2.6) that as $\rho\to 1$ we have

$$M(\rho,F)<\frac{\pi^2}{4(1-\rho)}-\frac{\pi}{\{2(1-\rho)\}^{1/2}}+O(1).$$

This inequality contradicts (6.1) for ρ sufficiently near to one if $\alpha \geq \frac{1}{4}\pi^2$, and this gives (2.7).

Similarly we deduce from the right-hand inequality of (2.2) that if $F \in Q$ and $F(z) \neq 0$, we have as $\rho \to 1$

$$M(\rho, f) \leq |c_0| \left\{ \frac{\pi^2}{2(1-\rho)} - \frac{\pi\sqrt{2}}{(1-\rho)^{1/2}} \right\} + O(1).$$

This again contradicts (6.1) if $\alpha \geq \pi^2 |c_0|/2$. This proves (2.8).

It remains to show that (2.7) and (2.8) are sharp. To see this we investigate maps $w = F(z)$ of Δ onto the domain D_r given by (4.4) when r is large but fixed and F is real on the real axis. We see as in Section 4 that such a map is given by

$$W = \frac{\pi r}{w + 2r} = \frac{\pi}{4} - \tan^{-1} z,$$

that is,

$$z = \tan\left\{ \frac{\pi}{4} \frac{w - 2r}{w + 2r} \right\}.$$

In this map $z = 0$ corresponds to $w = 2r$. To make $z = 0$ correspond to $w = 1$, we compose with a map of Δ onto itself, so that our final map is given by

$$\xi = \frac{z - t}{1 - tz} = \tan\left\{ \frac{\pi}{4} \left(\frac{w - 2r}{w + 2r} \right) \right\}, \quad \text{where } \tan\left\{ \frac{\pi}{4} \frac{1 - 2r}{1 + 2r} \right\} = -t. \tag{6.2}$$

We now set $z = \rho$ and $w = R$, and let ρ, R tend to 1 and ∞ through real values, while r and t are kept fixed. We have

$$1 - \frac{\rho - t}{1 - t\rho} = \frac{(1-\rho)(1+t)}{1 - t\rho} \sim \frac{1+t}{1-t}(1-\rho), \tag{6.3}$$

while

$$\tan\left\{ \frac{\pi}{4} \left(\frac{R - 2r}{R + 2r} \right) \right\} = \frac{1-s}{1+s}, \quad \text{where } s = \tan\left(\frac{\pi r}{R + 2r} \right).$$

Hence

$$1 - \tan\left\{ \frac{\pi}{4} \left(\frac{w - 2r}{w + 2r} \right) \right\} \sim 2s \sim \frac{2\pi r}{R}. \tag{6.4}$$

Now we deduce from (6.2) to (6.4) that

$$R \sim 2\pi r \frac{1-t}{1+t} \frac{1}{1-\rho} = \frac{c(r)}{1-\rho}$$

say, where

$$c(r) = 2\pi r \frac{1-t}{1+t} = 2\pi r \tan\left\{ \frac{\pi}{4} - \frac{\pi}{4} \left(\frac{2r-1}{2r+1} \right) \right\}$$

$$= 2\pi r \tan\left(\frac{\pi}{4r+2} \right) \to \frac{\pi^2}{2}, \quad \text{as } r \to \infty.$$

Thus given $\epsilon > 0$, there exists $F(z) \in Q$ such that $F(0) = 1$, $F(z) \neq 0$, and

(6.5) $$\lim_{\rho\to 1}(1-\rho)M(\rho,F)>\frac{\pi^2}{2}-\epsilon.$$

This shows that the inequality (2.8) is sharp.

Next we note that for the map $w=F(z)$ constructed above we have

$$F'(z)=\frac{dw}{dz}=\frac{d\xi}{dz}\Bigg/\frac{d\xi}{dw}=\frac{1-t^2}{(1-tz)^2}\Bigg/\left[\sec^2\left\{\frac{\pi}{4}\left(\frac{w-2r}{w+2r}\right)\right\}\frac{\pi r}{(w+2r)^2}\right].$$

Putting $z=0$ and $w=1$, by (6.2) we obtain, for large r,

$$F'(0)=(1-t^2)\cos^2\left\{\frac{\pi}{4}\left(\frac{2r-1}{2r+1}\right)\right\}\frac{(2r+1)^2}{\pi r}\sim\frac{4r(1-t)}{\pi}\to 2$$

as $r\to\infty$. We note that

$$F_0(z)=\frac{F(z)-F(1)}{F'(0)}\in\hat{K},$$

and we deduce from (6.5) that

$$\lim_{\rho\to 1}(1-\rho)M(\rho,F_0)=\frac{1}{F'(0)}\lim_{\rho\to 1}(1-\rho)M(\rho,f)>\frac{1-\epsilon}{2}\left(\frac{\pi^2}{2}-\epsilon\right)$$

if r is sufficiently large. This shows that (2.7) is also sharp, and completes the proof of Theorem 3. □

Comparing Theorem A and Theorem 3, we see that Hall's argument yields about twice the correct bound for the maximum modulus in $\hat{K}$. I am grateful for discussions with R. Hall and T. Sheil-Small while preparing this paper. I am grateful to the referee for a number of corrections and suggestions.

REFERENCES

1. R. W. Barnard and G. Schober, *Möbius transformations of convex mappings*, Complex Variables 3 (1984), 55–69.
2. R. R. Hall, *On a conjecture of Clunie and Sheil-Small*, Bull. London Math. Soc. 12 (1980), 25–28.
3. W. K. Hayman, *Multivalent functions*, Cambridge Univ. Press, Cambridge, 1958.
4. R. Nevanlinna, *Eindeutige analytische Funktionen*, Springer, Berlin, 1936.

Department of Mathematics
Imperial College
London SW7 2BZ, England