UNIVALENT MULTIPLIERS OF THE DIRICHLET SPACE

Sheldon Axler and Allen L. Shields

To George Piranian with fondness and respect

Let G be a connected open set in the complex plane and fix a distinguished point $z_0 \in G$. The Dirichlet space $D(G)$ is the Hilbert space of analytic functions g on G such that $g(z_0)=0$ and

$$\|g\|_{D(G)}^2 = \int_G |g'|^2 \, dA < \infty,$$

where dA denotes the usual area measure. The Dirichlet norm squared of g is just the area of the image of G under g, counting multiplicity. The condition $g(z_0)=0$ insures that no nonzero function has norm zero. Changing the point z_0 gives a space which is obtained from the original by subtracting a suitable constant from each function. An analytic function ϕ on G is called a multiplier of $D(G)$ if $\phi D(G) \subset D(G)$.

The Bergman space $B(G)$ is the Hilbert space of analytic functions g on G such that

$$\|g\|_{B(G)}^2 = \int_G |g|^2 \, dA < \infty.$$

For the special case of the open unit disk, which we denote by U, the Dirichlet space (with the distinguished point equal to zero) and the Bergman space can be described in terms of Taylor coefficients; namely,

$$\|g\|_{D(U)}^2 = \pi \sum n|a_n|^2,$$
$$\|g\|_{B(U)}^2 = \pi \sum |a_n|^2/(n+1),$$

where $g(z) = \sum a_n z^n$.

From the Taylor coefficient formulas for the norms in $D(U)$ and $B(U)$, it is clear that $D(U)$ is contained in $B(U)$. In this paper we consider the question of when the Dirichlet space $D(G)$ is contained in the Bergman space $B(G)$. Our results deal primarily with the case where G is bounded and simply connected. We show that if G is bounded and starlike, then $D(G)$ is contained in $B(G)$ (Theorem 3). Theorem 1 shows that a Riemann map ϕ of the unit disk U onto G is a multiplier of $D(U)$ precisely when $D(G) \subset B(G)$. Corollary 7 shows that if ϕ' is in H^p for some $p>1$, then $D(G) \subset B(G)$. We show that this conclusion may fail when $p=1$. In Theorem 10 we construct a Jordan region G with a rectifiable boundary such that $D(G)$ is not contained in $B(G)$. For this region G, the function z is not a multiplier of $D(G)$. Theorem 11 identifies the essential spectrum of multipliers on $D(G)$.

Received July 5, 1984. Revision received November 19, 1984.
Both authors were supported in part by the National Science Foundation.
Michigan Math. J. 32 (1985).

Stegenga [16] has given a necessary and sufficient condition for a function ϕ to be a multiplier of $D(U)$. However, it seems that Stegenga's condition cannot be used to derive the results of this paper.

For a simply connected region G and a Riemann map ϕ of U onto G, where $\phi(0)$ equals the distinguished point z_0, we let C_ϕ denote the composition operator defined by $C_\phi(g) = g \circ \phi$. It is easy to verify that C_ϕ is an isometry from $D(G)$ onto $D(U)$.

For G simply connected, we define the integral operator V by the formula

$$(Vg)(z) = \int_{z_0}^{z} g(w)\,dw.$$

The following theorem establishes basic equivalences that will be used throughout the paper.

THEOREM 1. *Let G be a bounded simply connected domain and let ϕ be a Riemann map of the unit disk U onto G. The following are equivalent.*

(1) $D(G) \subset B(G)$.

(2) $zD(G) \subset D(G)$.

(3) $\phi D(U) \subset D(U)$.

(4) $\phi' D(U) \subset B(U)$.

(5) *V maps $B(G)$ into $B(G)$.*

Proof. For g in $D(G)$, $(zg)' = zg' + g$. Since g' is in $B(G)$ and z is bounded on G, the function zg' is in $B(G)$. Thus the equation above shows that zg is in $D(G)$ if and only if g is in $B(G)$. Hence (1) and (2) are equivalent.

Choose the distinguished point z_0 to be $\phi(0)$. Note that the validity of (1) and (2) are independent of the choice of z_0.

Now suppose that (2) holds. To verify (3), let f be in $D(U)$. Since $\phi f = C_\phi(zC_\phi^{-1} f)$, we see that ϕf is in $D(U)$, as required.

Conversely, suppose (3) holds. Let g be in $D(G)$. Since $zg = C_\phi^{-1}(\phi C_\phi(g))$, we see that zg is in $D(G)$, and (2) holds.

To see the equivalence of (3) and (4), let f be in $D(U)$. Then $(\phi f)' = \phi' f + \phi f'$. Since ϕ is bounded and f' is in $B(U)$, we see that ϕf is in $D(U)$ if and only if $\phi' f$ is in $B(U)$, so (3) and (4) are equivalent. (For later use, note that the univalence of ϕ is not used in proving that (3) and (4) are equivalent, although the boundedness of ϕ is required.)

Thus (1) through (4) are all equivalent. Finally, since V maps $B(G)$ onto $D(G)$, (1) is equivalent to (5), completing the proof of Theorem 1. □

Even if G is unbounded, it is not hard to verify that conditions (1), (4), and (5) of Theorem 1 are equivalent.

It is easy to see that for each compact set $K \subset G$ there is a constant c such that $|f(z)| \le c\|f\|_{B(G)}$ for every z in K. To see that point evaluations are continuous on the Dirichlet space, let $z \in G$ and let Γ be a rectifiable path in G from the distinguished point z_0 to z. Then

$$|g(z)| = \left| \int_{z_0}^{z} g'(w)\,dw \right| \le (\text{length of } \Gamma) \sup_{w \in \Gamma} |g'(w)| \le c\|g'\|_{B(G)} = c\|g\|_{D(G)}.$$

Using the continuity of point evaluations and the Closed Graph Theorem, we see that if condition (1) of Theorem 1 holds, then the inclusion map from $D(G)$ into $B(G)$ is continuous. Similarly, the multiplication operators in (2), (3), and (4) of Theorem 1 (and the integral operator V in condition (5)) are continuous.

We now consider the question of when these maps are compact. First note that the operators in (2) and (3) of Theorem 1 can never be compact because the spectrum of these operators is uncountable; see the remarks preceding Theorem 11. The following theorem shows that if any one of the remaining three maps is compact, then all three are compact. Sufficient conditions for this to hold are given in Theorems 5 and 6 and Corollary 7. Consequences of compactness are given in Theorems 8 and 9.

THEOREM 2. *Let G be a simply connected domain and let ϕ be a Riemann map of the unit disk U onto G. Suppose that* (1) *of Theorem* 1 *holds. Then the following are equivalent.*

(1) *The inclusion map of $D(G)$ into $B(G)$ is compact.*
(2) *Multiplication by ϕ' is a compact operator from $D(U)$ into $B(U)$.*
(3) *V is a compact operator from $B(G)$ into $B(G)$.*

Proof. Let $M_{\phi'}$ be the operator of multiplication by ϕ' from $D(U)$ into $B(U)$; by the Closed Graph Theorem and condition (4) of Theorem 1, $M_{\phi'}$ is a bounded operator. Let I denote the inclusion map from $D(G)$ into $B(G)$, and let W be the unitary map from $B(G)$ onto $B(U)$ defined by $W(g) = \phi' C_\phi(g)$. We see that $M_{\phi'} C_\phi = WI$. Since C_ϕ and W are both unitary, (1) is equivalent to (2).

Let V_1 be the unitary map from $B(G)$ onto $D(G)$ defined by $V_1 g = Vg$ (we are thinking of V as mapping $B(G)$ into $B(G)$, so V_1 and V differ only in the norms on the range space). Then $V = IV_1$, which shows that (1) and (3) are equivalent. □

A region G is called starlike if there is a point w_0 in G such that for each point w in G, the line segment connecting w_0 and w lies in G. The following theorem shows that the conditions of Theorem 1 hold for bounded starlike regions. For unbounded starlike regions, it is not necessarily true that $D(G) \subset B(G)$. For example, let G be the right half plane. Then the function $1/(z+1)$ is in $D(G)$ but not in $B(G)$.

THEOREM 3. *Let G be a bounded starlike domain and let ϕ be a Riemann map of the unit disk U onto G. Then ϕ multiplies $D(U)$ into itself.*

Proof. We can assume without loss of generality that G is starlike with respect to the origin and that the distinguished point z_0 of G is also the origin. We will prove the theorem by showing that condition (1) of Theorem 1 holds.

Let g be a function in $D(G)$. Then

$$g(z) = \int_0^z g'(w)\,dw = z\int_0^1 g'(tz)\,dt.$$

Thus, by Cauchy's inequality,

$$|g(z)|^2 \le |z|^2 \int_0^1 |g'(tz)|^2\,dt \le c\int_0^1 |g'(tz)|^2\,dt,$$

where c is a constant such that $|z|^2 < c$ for all z in G. Hence

$$\text{(1)}\qquad \begin{aligned}\int_G |g(z)|^2\,dA(z) &\le c\int_0^1 \int_G |g'(tz)|^2\,dA(z)\,dt\\ &= c\int_0^1 t^{-2}\int_{tG} |g'(w)|^2\,dA(w)\,dt.\end{aligned}$$

Since the origin is an interior point of G and G is bounded, there is a positive number $s<1$ such that the closure of sG is contained in G. Thus there is a constant k such that $|g'|^2 < k$ on sG. Hence for $0<t<s$, we have

$$t^{-2}\int_{tG} |g'(w)|^2\,dA(w) < k|G|,$$

where $|G|$ denotes the area of G. So

$$\text{(2)}\qquad \int_0^s t^{-2}\int_{tG} |g'(w)|^2\,dA(w)\,dt < \infty.$$

Also,

$$\text{(3)}\qquad \begin{aligned}\int_s^1 t^{-2}\int_{tG} |g'(w)|^2\,dA(w)\,dt &\le \int_s^1 t^{-2}\int_G |g'(w)|^2\,dA(w)\,dt\\ &= (s^{-1}-1)\|g\|^2_{D(G)} < \infty.\end{aligned}$$

Now (1), (2), and (3) show that g is in $B(G)$, which completes the proof. □

Let $M(D(U))$ denote the set of multipliers of $D(U)$, so $M(D(U))$ consists of the analytic functions ϕ on U such that $\phi D(U) \subset D(U)$. As noted before Theorem 2, each ϕ in $M(D(U))$ induces a bounded multiplication operator on $D(U)$. Giving each function in $M(D(U))$ the operator norm of the corresponding multiplication operator makes $M(D(U))$ into a normed space. It is known that if ϕ is a multiplier of $D(U)$, then ϕ is bounded on U and the supremum of $|\phi|$ is less than or equal to the corresponding operator norm; see [6, Lemma 11].

The set of multipliers of $B(U)$ into $B(U)$ is just H^∞, and the operator norm coincides with the supremum norm. Thus the set of multipliers of $B(U)$ into $B(U)$ is nonseparable. Even though $M(D(U))$ is strictly smaller than H^∞, the following corollary shows that $D(U)$ also has enough multipliers to be nonseparable.

COROLLARY 4. *$M(D(U))$ is nonseparable.*

Proof. For w a boundary point of the unit disk U, let $R(w)$ be the radial segment defined by

$$R(w) = \{rw : \tfrac{1}{2} \le r < 1\}.$$

Let w_n be any sequence of distinct points of ∂U converging to 1. Let

$$G = U \sim [R(1) \cup R(w_1) \cup R(w_2) \cup \cdots].$$

Since G is the unit disk with countably many radial slits removed, G is starlike with respect to the origin. Let ϕ be a Riemann map of the unit disk U onto G. By Theorem 3, ϕ is a multiplier of $D(U)$. For $w \in \partial U$, let ϕ_w be the rotation of ϕ defined by $\phi_w(z) = \phi(wz)$. Since $D(U)$ is rotation invariant, it is clear that ϕ_w is also a multiplier of $D(U)$.

Note that ϕ cannot be extended to be continuous on the closed unit disk. Indeed, if there were such an extension, then ϕ would map the unit circle onto the boundary of G; but no continuous function can do this. (The theory of prime ends can be used to show that ϕ has precisely one discontinuity on ∂U; see for example [9, Chapter 9].) Thus the collection $\{\phi_w : w \in \partial U\}$ is not separable in the H^∞ norm; for example, see [10, Theorem 1]. But as noted earlier, the operator norm of a multiplier dominates the H^∞ norm, and so the result follows. □

Theorem 3 showed that if G is a bounded starlike domain, then $D(G) \subset B(G)$. The following theorem (in conjunction with Theorem 1) shows that if a slightly stronger geometric condition is satisfied by G, then the inclusion map is compact. Note that this condition is satisfied by every bounded convex region.

THEOREM 5. *Let G be a bounded domain such that the closure of tG is contained in G for $0 < t < 1$. Let ϕ be a Riemann map of the unit disk U onto G. Then multiplication by ϕ' is a compact operator from $D(U)$ into $B(U)$.*

Proof. We will prove the theorem by proving that condition (1) of Theorem 2 holds. Let $\{g_n\}$ be a sequence in $D(G)$ such that g_n goes to 0 weakly in $D(G)$. To show that the inclusion map of $D(G)$ into $B(G)$ is compact, it is enough to show that g_n goes to 0 in norm in $B(G)$.

For $0 < t < 1$, let

$$I_n(t) = t^{-2} \int_{tG} |g_n'(w)|^2 \, dA(w).$$

From inequality (1) in the proof of Theorem 3, we see that

$$\int_G |g_n(z)|^2 \, dA(z) \le c \int_0^1 I_n(t) \, dt.$$

We will show that first, $I_n(t) \to 0$ as $n \to \infty$ for all $t < 1$, and second, there is a constant K such that $I_n(t) < K$ for all n and t. The result will then follow from the bounded convergence theorem.

Since g_n tends to 0 weakly in $D(G)$, we know that g_n' tends to zero uniformly on each compact subset of G (for example, see [2, Corollary to Proposition 1]). Since the closure of tG is contained in G, this implies that $I_n(t) \to 0$ as $n \to \infty$ for all $t < 1$.

Since g_n' tends to zero uniformly on $\frac{1}{2}G$, there is a constant k such that $|g_n'|^2 < k$ on $\frac{1}{2}G$ for all n. Thus $I_n(t) < k|G|$ for $t < \frac{1}{2}$.

For $\frac{1}{2} \le t \le 1$, we have

$$I_n(t) \leq 4 \int_G |g_n'|^2 \, dA < \text{constant},$$

since every weakly convergent sequence is norm bounded. We are done. $\square$

An infinite matrix is called Hilbert-Schmidt if the entries are square summable. Every Hilbert-Schmidt matrix represents a compact (hence bounded) linear map on Hilbert space, and this is often the easiest way to verify that an operator is compact. The following proof exploits this idea. The first part of the following theorem is Proposition 18(b) of [2]; the proof in [2] uses different ideas. The second conclusion holds even if ϕ is unbounded.

THEOREM 6. *Let $\phi = \sum a_n z^n$ be a bounded analytic function defined on the unit disk U such that*

$$\sum n(\log n)|a_n|^2 < \infty.$$

Then ϕ is a multiplier of $D(U)$ into itself. Furthermore, multiplication by ϕ' is a Hilbert-Schmidt operator from $D(U)$ into $B(U)$.

Proof. In what follows, all summations run from 1 to ∞. Since $z^m \phi' = \sum n a_n z^{n+m-1}$, we have

$$\begin{aligned}
\sum_m \|m^{-1/2} z^m \phi'\|_B^2 &= \sum_m \sum_n n^2 |a_n|^2 / [(n+m)m] \\
&= \sum_n n^2 |a_n|^2 \sum_m 1/[(n+m)m] \\
&= \sum_n n|a_n|^2 \sum_m [m^{-1} - (n+m)^{-1}] \\
&= \sum n|a_n|^2 (1 + \tfrac{1}{2} + \cdots + n^{-1}) \\
&\leq \sum n|a_n|^2 (1 + \log n) < \infty.
\end{aligned}$$

Since $\{m^{-1/2} z^m\}$ is an orthonormal basis for $D(U)$, the inequality above shows that multiplication by ϕ' is a Hilbert-Schmidt operator from $D(U)$ into $B(U)$.

The proof that conditions (4) and (3) of Theorem 1 are equivalent now shows that ϕ is a multiplier of $D(U)$ into itself, completing the proof. $\square$

For the case where $p = 2$, the following corollary was noted in [2, Remark 5 following Proposition 3]. The following corollary is sharp in the sense that we will see later (Theorem 10) that there is a bounded univalent function ϕ such that $\phi' \in H^1$ but ϕ is not a multiplier of $D(U)$.

COROLLARY 7. *Let ϕ be an analytic function on the unit disk U such that ϕ' is in H^p for some $p > 1$. Then ϕ multiplies $D(U)$ into itself. Furthermore, multiplication by ϕ' is a Hilbert-Schmidt operator from $D(U)$ into $B(U)$.*

Proof. Without loss of generality, assume that $1 < p < 2$. Let $\phi(z) = \sum a_n z^n$ be the Taylor series of ϕ. Let p' be the index conjugate to p, so $(1/p) + (1/p') = 1$, and $p' > 2$. Let t be the index conjugate to $p'/2$, so $t > 1$. Applying Holder's inequality with indices t and $p'/2$ we have:

$$\sum n(\log n)|a_n|^2 = \sum n^{-1}(\log n)|na_n|^2$$
$$\leq [\sum (n^{-1}\log n)^t]^{1/t}(\sum |na_n|^{p'})^{2/p'} < \infty,$$

where the last term is finite by the Hausdorff–Young Inequality ([19, Chapter XII, Theorem 2.3]).

Finally, ϕ' is in H^1 so an inequality of Fejér, Hardy, and Littlewood (sometimes called "Hardy's inequality") shows that ϕ is bounded (see [5, Sec. 3.6, p. 49]; for the history of this inequality see [15, p. 476]). Theorem 6 now gives us the desired result. □

Let $B_0(G)$ denote the subspace of $B(G)$ of codimension one consisting of all functions in $B(G)$ which vanish at the distinguished point z_0. Theorems 5 and 6, and Corollary 7, give conditions under which the inclusion map of $D(G)$ into $B(G)$ is compact (this holds, for example, if G is convex). Ideas similar to those used in the proof of the following theorem appear in [13].

THEOREM 8. *Let G be a simply connected region with finite area such that $D(G) \subset B(G)$ and the inclusion map of $D(G)$ into $B(G)$ is compact. Then there is an orthonormal basis of $D(G)$ which is also an orthogonal basis for $B_0(G)$.*

Proof. Let I denote the inclusion map of $D(G)$ into $B_0(G)$. Thus I^*I is a compact self-adjoint operator on $D(G)$. By the spectral theorem for compact self-adjoint operators on Hilbert space ([11, Theorem VI.16]), there is an orthonormal basis $\{g_n\}$ for $D(G)$ consisting of eigenvectors of I^*I. The following computation with inner products in $B(G)$ and $D(G)$ shows that $\{g_n\}$ is also an orthogonal sequence in $B(G)$:

$$(f_n, f_m)_{B(G)} = (If_n, If_m)_{B(G)} = (I^*If_n, f_m)_{D(G)}$$
$$= (t_n f_n, f_m)_{D(G)} = 0 \quad \text{if } n \neq m.$$

To show that $\{g_n\}$ is a basis for $B_0(G)$, we will verify that $D(G)$ is dense in $B_0(G)$. Let $M_{\phi'}$ be the operator of multiplication by ϕ' from $D(U)$ into $B_0(U)$. Let W be the unitary map from $B_0(G)$ onto $B_0(U)$ defined by $W(g) = \phi' C_\phi(g)$. Since $M_{\phi'}C_\phi = WI$, to show that $D(G)$ is dense in $B_0(G)$, it suffices to show that $\phi' D(U)$ is dense in $B_0(U)$. However, it follows from [14, Proposition 41], that the linear span of $\{\phi' z^n : n = 1, 2, \ldots\}$ is dense in $B_0(U)$, and so we are done. □

As noted before Theorem 2, point evaluations are continuous linear functionals on the Hilbert spaces $D(U)$ and $B(U)$, and hence are represented by inner products with suitable elements of these spaces. To be specific, for $z \in U$, let

$$k_w(z) = \log(1/(1-\bar{w}z)) \quad \text{and} \quad K_w(z) = 1/(1-\bar{w}z)^2.$$

Then

$$f(z) = (f, k_z)_D \quad \text{for all } f \in D(U),$$
$$h(z) = (h, K_z)_B \quad \text{for all } h \in B(U).$$

The functions k_z and K_z are called the reproducing kernels for their respective spaces. We have

$$\|k_z\|_D^2 = (k_z, k_z)_D = k_z(z) = \log(1/(1-|z|^2)),$$
$$\|K_z\|_B^2 = (K_z, K_z)_B = K_z(z) = 1/(1-|z|^2)^2.$$

The normalized kernels $k_z/\|k_z\|_D$ tend weakly to 0 in $D(U)$ as $|z| \to 1$. This can be proved by taking inner products with bounded functions, which are dense in $D(U)$.

The first part of the following theorem was proved in Theorem 1 of [17] in a more general context. For a stronger conclusion which applies to univalent mappings onto starlike regions, see [18, Theorem 1].

THEOREM 9. *Let ϕ be an analytic function on the unit disk U such that ϕ is a multiplier of $D(U)$. Then there exists a constant c such that*

$$|\phi'(z)| \le c(1-r)^{-1}[\log 1/(1-r)]^{-1/2}$$

for all z in U, where $r = |z|$. Furthermore, if multiplication by ϕ' is a compact operator from $D(U)$ into $B(U)$, then

$$\phi'(z)(1-r)[\log 1/(1-r)]^{1/2} \to 0$$

as $|z| \to 1$ in U.

Proof. Condition (4) of Theorem 1 and the Closed Graph Theorem imply that the multiplication operator $M_{\phi'}$ is a bounded operator from $D(U)$ to $B(U)$. For each z in U we have

$$|\phi'(z)|\,\|k_z\|_D^2 = |\phi'(z)|\,(k_z, k_z)_D = |\phi'(z)k_z(z)| = |(M_{\phi'}k_z, K_z)_B|$$
$$\le \|M_{\phi'}\|\,\|k_z\|_D\|K_z\|_B,$$

and the first part of the theorem follows.

Now suppose that $M_{\phi'}$ is a compact operator from $D(U)$ to $B(U)$ and that $|z| \to 1$ in U. As noted above, $k_z/\|k_z\|_D$ tends weakly to 0. The compactness of $M_{\phi'}$ thus implies that $M_{\phi'}(k_z/\|k_z\|_D)$ tends to 0 in norm in $B(U)$. Hence

$$(M_{\phi'}k_z/\|k_z\|_D, K_z/\|K_z\|_B) \to 0 \quad \text{as } |z| \to 1.$$

Since

$$\phi'(z)\|k_z\|_D/\|K_z\|_B = (M_{\phi'}k_z/\|k_z\|_D, K_z/\|K_z\|_B),$$

we obtain the desired result. □

Corollary 7 implies that if G is a bounded simply connected region with smooth boundary, then a Riemann map of U onto G is a multiplier of $D(U)$. If G is any bounded simply connected region and ϕ is a Riemann map of U onto G, then the condition that ϕ' be in H^1 is equivalent to the condition that G have a rectifiable boundary; see [5, Theorem 3.12]. Thus the following theorem implies that there is a bounded univalent function ϕ such that $\phi' \in H^1$ but ϕ is not a multiplier of $D(U)$. By Theorem 1, the function z is not a multiplier of the Dirichlet space of the region constructed in the following theorem.

For an arbitrary region G, one can define $H^2(G)$ by means of harmonic majorants; see [12]. If G is simply connected, then $D(G) \subset H^2(G)$ because this statement holds for the disk and both spaces are conformally invariant. Hence the following theorem also gives an example of a rectifiable Jordan region G such that $H^2(G)$ is not contained in $B(G)$.

THEOREM 10. *There exists a bounded domain G whose boundary is a rectifiable Jordan curve, such that if ϕ is a Riemann map of the unit disk U onto G, then ϕ is not a multiplier of $D(U)$.*

Proof. We will prove the theorem by constructing a domain G which fails to satisfy condition (1) of Theorem 1. We will choose sequences $\{a_n\}$ and $\{t_n\}$ such that

$$a_0 > 1 > a_1 > a_2 > a_3 > \cdots, \tag{4}$$

$$a_n \to 0, \tag{5}$$

$$0 < t_{2n+1} \le \tfrac{1}{4} \quad \text{and} \quad t_{2n} = \tfrac{1}{4} \quad \text{for all } n \ge 0; \tag{6}$$

further conditions on $\{a_n\}$ and $\{t_n\}$ will be specified later. Let

$$G_n = \{re^{it} : a_{n+1} \le r \le a_n \text{ and } t \le t_n\},$$

and let G be the interior of the union (for $n \ge 0$) of the G_n (see Figure 1). We choose for our distinguished point $z_0 = 1$.

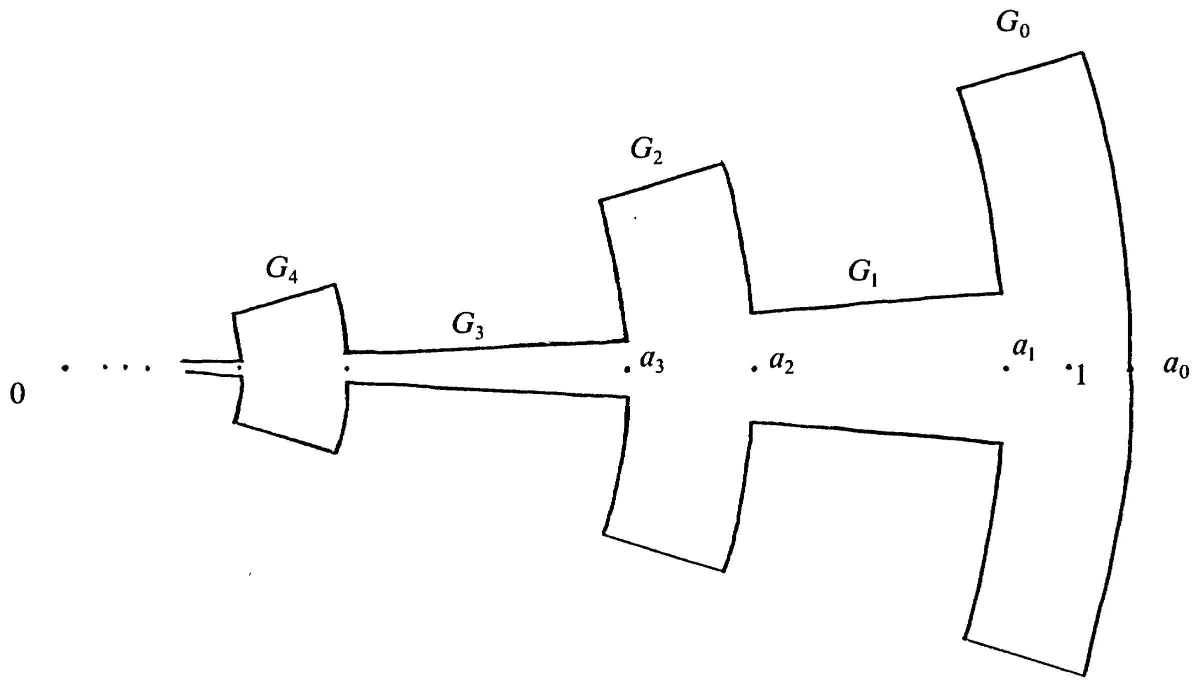

Figure 1

Let $g_m(z)=(1-z^2)^m$. Note that $g_m(z_0)=0$, so each g_m is in $D(G)$. If $D(G)$ were contained in $B(G)$, then the Closed Graph Theorem would show that the inclusion map from $D(G)$ into $B(G)$ is bounded. We will show that this is not the case by showing that

$$\sup_m \|g_m\|_{B(G)}/\|g_m\|_{D(G)}=\infty. \tag{7}$$

The image of G_j under the map $z\mapsto z^2$ is the annular slice

$$G_j^2=\{re^{it}: a_{j+1}^2\le r\le a_j^2 \text{ and } t\le 2t_j\}.$$

Thus for $j>0$, the function $|g_m|$ attains its minimum on G_j at a_j. Hence

$$\int_G |g_m|^2\ge\int_{G_j}|g_m|^2\ge(1-a_j^2)^{2m}|G_j|, \tag{8}$$

where all integrations are with respect to the area measure dA, and $|G_j|$ denotes the area of G_j.

The $\{a_n\}$ and $\{t_n\}$ will be chosen inductively. We begin by letting $a_0=1.1$ and $a_1=.9$. Then $|G_0|<1$ and for each z in G_0 we have $|z|<2$ and $|1-z^2|<.7$. Thus

$$\int_{G_0}|g_m'|^2\le 16m^2(.7)^{2m-2}. \tag{9}$$

Note that $a_1^2<\cos\frac{1}{2}$. Hence by (4) and (6), for $j\ge 1$, we will have $a_j^2\le\cos 2t_j$, and so the furthest point from 1 in G_j^2 is the point $a_{j+1}^2\exp(i2t_j)$; see Figure 2. The distance from 1 to $a_{j+1}^2\exp(i2t_j)$ is less than or equal to the distance from 1 to a_{j+1}^2 plus the arc length along the circle; see Figure 2. Thus

$$\begin{aligned}|1-a_{j+1}^2\exp(i2t_j)|&\le 1-a_{j+1}^2+a_{j+1}^2 2t_j\\ &\le 1-\tfrac{1}{2}a_{j+1}^2.\end{aligned}$$

Hence for $j\ge 1$, we have

$$\int_{G_j}|g_m'|^2\le 4m^2a_j^2(1-\tfrac{1}{2}a_{j+1}^2)^{2m-2}|G_j|. \tag{10}$$

We now proceed with the inductive construction of $\{a_n\}$ and $\{t_n\}$. Suppose that $a_0, a_1, \dots, a_{2n-1}$ and $t_0, t_1, \dots, t_{2n-2}$ have been chosen (recall from (6) that $t_j=\frac{1}{4}$ for all even j). First we estimate the Dirichlet integral over the regions already determined (except for G_0). For $j\ge 1$, we have $|G_j|<\frac{1}{4}$ and $a_j<1$. From (10) we have

$$\begin{aligned}\sum_{j=1}^{2n-1}\int_{G_j}|g_m'|^2&\le\sum_{j=1}^{2n-1} m^2(1-\tfrac{1}{2}a_{j+1}^2)^{2m-2}\\ &\le(2n-2)m^2(1-\tfrac{1}{2}a_{2n-1}^2)^{2m-2}.\end{aligned} \tag{11}$$

We will fix an integer m (which depends upon n) that will satisfy certain conditions to be specified later. Let $a_{2n}=1/m^2$, where as a first condition on m we require that $1/m^2<a_{2n-1}$. Now choose t_{2j-1} so that $|G_{2n-1}|$ is small enough so that

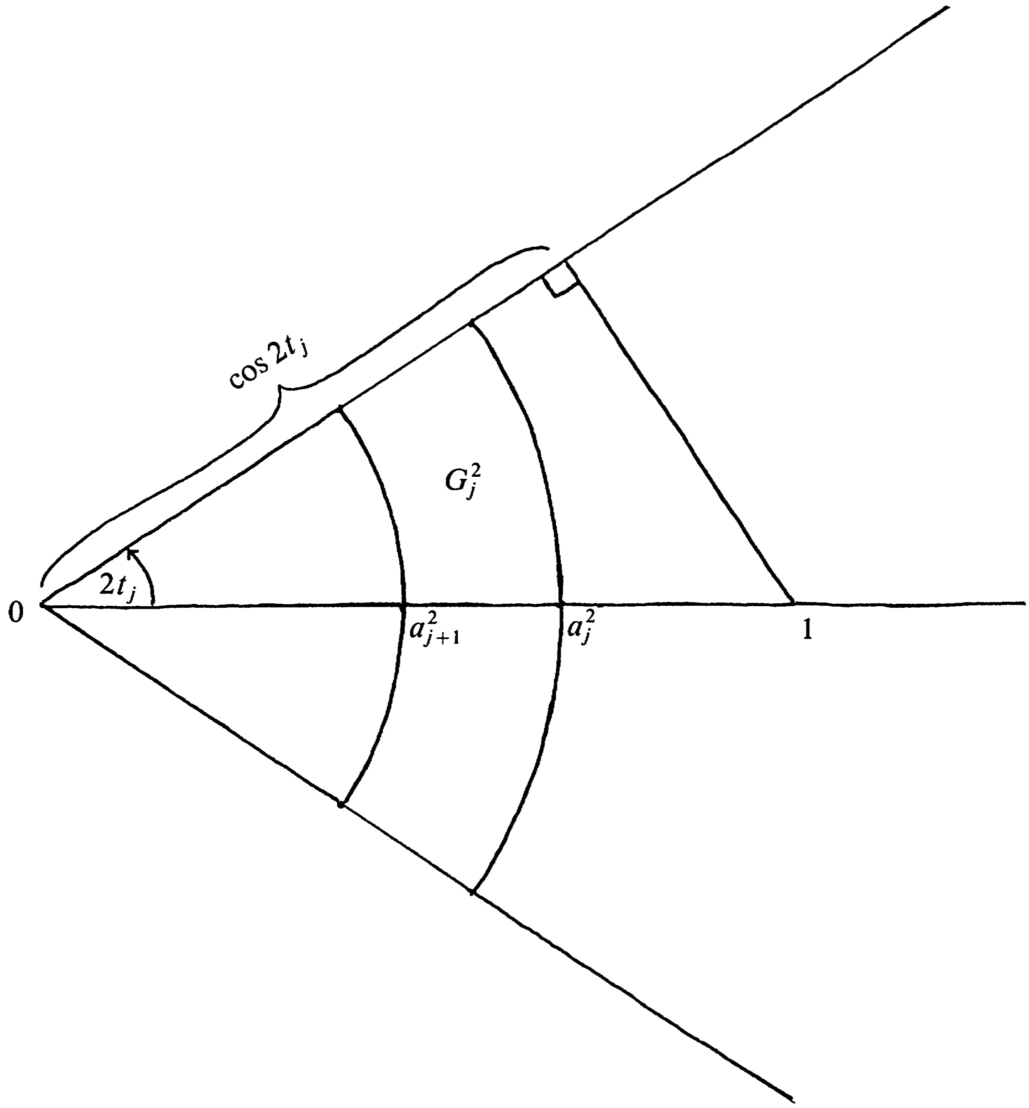

Figure 2

$$\int_{G_{2n-1}} |g_m'|^2 \le 1/m^6; \tag{12}$$

see (10) and recall that m, a_{2n-1}, and a_{2n} have now been fixed.

Regardless of how a_{2n+1} is chosen, we have

$$|G_{2n}| \le \tfrac{1}{4}a_{2n}^2 = \tfrac{1}{4}m^{-4},$$

and so from (10) we get

$$\int_{G_{2n}} |g_m'|^2 \le 4m^2 a_{2n}^2 |G_{2n}| < a_{2n}^2/m^2 = 1/m^6. \tag{13}$$

Let $a_{2n+1} = 1/m^4$. Regardless of how a_j and t_{j-1} are chosen for $j > 2n+1$, for these values of j we will have

$$G_j \subset \{re^{it} : 0 < r \le a_{2n+1} \text{ and } |t| \le \tfrac{1}{4}\},$$

where the latter set has area $\frac{1}{4}a_{2n+1}^2 = \frac{1}{4}m^{-8}$. Since the interiors of the G_j are disjoint, we have

$$\sum_{j=2n+1}^{\infty} |G_j| \leq \tfrac{1}{4}m^{-8}.$$

Hence from (10) we get

$$\sum_{j=2n+1}^{\infty} \int_{G_j} |g_m'|^2 \leq 4m^2 \sum_{j=2n+1}^{\infty} |G_j| \leq 1/m^6. \tag{14}$$

We now specify the conditions that m must satisfy (n is still fixed and m will depend upon n). Choose m large enough so that:

(a) the right-hand side of (9) is less than $1/m^6$;
(b) the right-hand side of (11) is less than $1/m^6$;
(c) $(1-m^{-4})^{2m+1}$ is greater than $\frac{1}{2}$; and
(d) $m > n$.

Adding equations (9), (11), (12), (13), and (14), we get

$$\int_G |g_m'|^2 \leq 5/m^6. \tag{15}$$

From (8) we have

$$\begin{aligned} \int_G |g_m|^2 &\geq \int_{G_{2n}} |g_m|^2 \geq (1-a_{2n}^2)^{2m} |G_{2n}| \\ &= \tfrac{1}{4}(1-a_{2n}^2)^{2m}(a_{2n}^2 - a_{2n+1}^2) = \tfrac{1}{4}(1-m^{-4})^{2m}(m^{-4}-m^{-8}) \\ &= \tfrac{1}{4}(1-m^{-4})^{2m+1}/m^4 \\ &\geq m^{-4}/8. \end{aligned} \tag{16}$$

Dividing (16) by (15), we see that

$$\|g_m\|_{B(G)} / \|g_m\|_{D(G)} \geq (m^2/40)^{1/2} > n/7.$$

Thus (7) holds and hence $D(G)$ is not contained in $B(G)$.

It only remains to show that G has a rectifiable boundary. From Figure 1 it clearly suffices to show that $\sum a_n < \infty$. However,

$$\sum a_n \leq 2 \sum a_{2n} \leq 2 \sum 1/n^2 < \infty.$$

This completes the proof. □

If G is a bounded region and ϕ is a multiplier of $D(U)$, then the spectrum of the multiplication operator induced by ϕ is the closure of $\phi(G)$; see the addendum concerning question 6 in [14]. Recall that an operator T on a Hilbert space H is called Fredholm if the kernel of T and H/TH are both finite dimensional vector spaces. (These conditions imply that T has closed range; see [3, Cor. 3.2.5].) The essential spectrum of T is defined to be the set of complex numbers w such that $T-w$ is not Fredholm.

The following theorem determines the essential spectrum of a multiplication operator on the Dirichlet space. We require the notion of cluster set. If ϕ is an analytic function on G, then the cluster set of ϕ on ∂G, denoted $\mathrm{cl}(\phi; \partial G)$, is the set of complex numbers w such that there exists a sequence $\{z_n\}$ in G such that z_n tends to the boundary of G and $\phi(z_n) \to w$.

THEOREM 11. *Let G be a bounded simply connected region, and let ϕ be an analytic function on G such that ϕ is a multiplier of $D(G)$. Then the essential spectrum of the multiplication operator*

$$M_\phi : D(G) \to D(G)$$

equals $\mathrm{cl}(\phi; \partial G)$.

Proof. A Riemann map from the unit disk U onto G establishes (by composition) a unitary correspondence between multipliers of $D(G)$ and multipliers of $D(U)$. Since essential spectra and cluster sets are preserved under this correspondence, we can assume that $G = U$ and $z_0 = 0$.

Thus assume that ϕ is a multiplier of $D(U)$. We first will show that $\mathrm{sp}_e(M_\phi) \subset \mathrm{cl}(\phi; \partial U)$. It suffices to show that if 0 is not in $\mathrm{cl}(\phi; \partial U)$, then M_ϕ is Fredholm. So suppose that ϕ is bounded away from 0 near ∂U. Let $z_1, \ldots, z_n$ be the zeroes of ϕ in U, repeated according to multiplicity. Let S denote the subspace of $D(U)$ consisting of all functions f in $D(U)$ such that f vanishes on $\{0, z_1, \ldots, z_n\}$ with appropriate multiplicities. Let f be a function in S. Then f/ϕ is analytic on U and $(f/\phi)(0) = 0$. To see that f/ϕ is in $D(U)$, observe that

$$(f/\phi)' = (f'\phi - f\phi')/\phi^2.$$

Since ϕ is a multiplier of $D(U)$, we know that ϕ is bounded and that ϕ' multiplies $D(U)$ into $B(U)$. Thus the numerator above is square integrable. We need only check that $(f/\phi)'$ is square integrable near ∂U. This follows from the equation above, since ϕ is bounded away from 0 near ∂U. Now that we know that f/ϕ is in $D(U)$, it follows that $f = M_\phi(f/\phi)$, so f is in the range of M_ϕ. Thus S is contained in the range of M_ϕ. But S is the intersection of the kernels of finitely many linear functionals, so S has finite codimension. Hence the range of M_ϕ has finite codimension. Since the kernel of M_ϕ is zero, we can conclude that M_ϕ is Fredholm. Thus $\mathrm{sp}_e(M_\phi) \subset \mathrm{cl}(\phi; \partial U)$.

To prove the converse inclusion, suppose that 0 is in $\mathrm{cl}(\phi; \partial U)$. Let $\{z_n\}$ be a sequence in U such that $\phi(z_n) \to 0$ and $|z_n| \to 1$. Let k_n denote the reproducing kernel for the point z_n, and let f_n be the normalized reproducing kernel: $f_n = k_n / \|k_n\|_{D(U)}$. Recall from the discussion preceding Theorem 9 that f_n tends weakly to 0 in $D(U)$. Suppose that M_ϕ were Fredholm. Then there would be a bounded operator T such that $1 - M_\phi T$ is compact. Thus $\|(1 - M_\phi T) f_n\| \to 0$, so

$$1 - \phi(z_n)(Tf_n, f_n) = ((1 - M_\phi T) f_n, f_n) \to 0,$$

but $1 - \phi(z_n)(Tf_n, f_n) \to 1$, a contradiction. Thus M_ϕ is not Fredholm and we are done. □

We end the paper by raising a few questions. For which regions G is $D(G) \subset B(G)$? When is the inclusion map compact? Which Schatten p-classes contain the inclusion map? (For the disk U, the inclusion map is in the Schatten p-class precisely when $p > 1$.)

Let ϕ be a multiplier of $D(U)$. What is the relation between ϕ being continuous on the closure of U and multiplication by ϕ' being a compact operator from $D(U)$ to $B(U)$?

Is there a bounded region G such that $M(D(G))$ is separable? (By Corollary 4, such a region could not be simply connected.)

For nonsimply connected regions, identify the essential spectrum of multiplication operators on the Dirichlet space; see Theorem 11.

If ϕ is a univalent condition on U that satisfies a Lipschitz condition of order less than or equal to $\frac{1}{2}$, then must ϕ be a multiplier of $D(U)$? If the Lipschitz order is greater than $\frac{1}{2}$, then this is true even if ϕ is not univalent; see [2, Proposition 12(b)].

If ϕ is a multiplier of $D(U)$ and M_ϕ denotes the corresponding multiplication operator on $D(U)$, then must the self-commutator $M_\phi^* M_\phi - M_\phi M_\phi^*$ be compact? trace class?

NOTE. Since this paper was written we have learned of several additional references that are relevant to the problems discussed here. As pointed out in the discussion following the proof of Theorem 1, if G is a region for which $D(G) \subset B(G)$, then the inclusion map is a bounded linear transformation. Thus there is a constant $c = c(G; z_0)$ such that

$$\int_G |f|^2 \, dA \leq c \int_G |f'|^2 \, dA \tag{17}$$

for all f holomorphic in G such that $f(z_0) = 0$.

This inequality may be viewed as an analogue for analytic functions of the Poincaré inequalities in the theory of elliptic partial differential equations. One form of these inequalities (see [7, (7.45), p. 157]) states that if G is a bounded convex open set in the plane, then there is a constant c such that

$$\int |u|^2 \, dA \leq c \int |Du|^2 \, dA$$

for all $u \in C^1(G)$ with $\int u \, dA = 0$, and u_x, u_y in $L^2(G)$, where Du denotes the gradient vector.

In [4, Chap. VII, §8.1] Courant and Hilbert establish this inequality for a broader class of bounded domains, including those that are finite unions of convex subdomains. They also give an example of a Jordan region with rectifiable boundary for which the inequality fails [4, §8.2, p. 521].

The analytic inequality (17) seems to have been considered first by Hummel [8], who showed that it fails when G is a ribbon inside the unit disc spiralling out to the boundary. The problem of describing the regions G for which (17) is valid was posed by David Hamilton at a conference in Durham in 1983 (see [1, Problem 8.18]). He has informed us that he has obtained several results on this problem. In particular, Theorem 6 of the present paper was known to him. Also, if $\phi' \in L^p(dA)$ for some $p > 2$, then (17) is valid (here ϕ denotes the conformal map of the unit disc onto G). This last result is also a consequence of Proposition 19 of [2], together with Theorem 1 of the present paper.

Added in proof. We mention one further result (compare Corollary 7). Here $M_p(\phi', r)$ denotes the mean of order p of ϕ' on the circle $|z| = r$.

THEOREM 12 ([2, Proposition 19b]). *If ϕ is holomorphic in U and if*

$$M_p(\phi', r) \in L^2(dr) \quad \textit{for some } p > 2,$$

then $\phi \in M(D(U))$.

REFERENCES

1. K. Barth, D. Brannan, and W. Hayman, *Research problems in complex analysis*, preprint.
2. Leon Brown and Allen L. Shields, *Cyclic vectors in the Dirichlet space*, Trans. Amer. Math. Soc. 285 (1984), 269-304.
3. S. R. Caradus, W. E. Pfaffenberger, and Bertram Yood, *Calkin algebras and algebras of operators on Banach spaces*, Dekker, New York, 1974.
4. R. Courant and D. Hilbert, *Methoden der mathematischen Physik*, Vol. II, Springer, Berlin, 1937.
5. Peter L. Duren, *Theory of H^p spaces*, Academic Press, New York, 1970.
6. P. L. Duren, B. W. Romberg, and A. L. Shields, *Linear functionals on H^p spaces with* $0 < p < 1$, J. Reine Angew. Math. 238 (1969), 32-60.
7. D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Springer, Berlin, 1977.
8. J. A. Hummel, *Counterexamples to the Poincaré inequality*, Proc. Amer. Math. Soc. 8 (1957), 207-210.
9. Ch. Pommerenke, *Univalent functions*, Vandenhoeck & Ruprecht, Göttingen, 1975.
10. H. Porta, L. A. Rubel, and A. L. Shields, *Separability of orbits of functions on locally compact groups*, Studia Math. 48 (1973), 89-94.
11. Michael Reed and Barry Simon, *Methods of modern mathematical physics. I. Functional analysis*, Academic Press, New York, 1972.
12. W. Rudin, *Analytic functions of class H_p*, Trans. Amer. Math. Soc. 78 (1955), 46-66.
13. H. S. Shapiro, *Stefan Bergman's theory of doubly-orthogonal functions. An operator-theoretic approach*, Proc. Roy. Irish Acad., Sect. A, 79 (1979), 49-58.
14. Allen L. Shields, *Weighted shift operators and analytic function theory. Topics in operator theory*, 49-128, Math. Surveys, Vol. 13, Amer. Math. Soc., Providence, R.I., second printing, 1979.
15. ———, *An analogue of a Hardy-Littlewood-Fejér inequality for upper triangular trace class operators*, Math. Z. 182 (1983), 473-484.
16. David A. Stegenga, *Multipliers of the Dirichlet space*, Illinois J. Math. 24 (1980), 113-139.

17. G. D. Taylor, *Multipliers on D_α*, Trans. Amer. Math. Soc. 123 (1966), 229–240.
18. J. B. Twomey, *On the derivative of a starlike function*, J. London Math. Soc. (2) 2 (1970), 99–110.
19. A. Zygmund, *Trigonometric series*, 2nd ed., Cambridge Univ. Press, London, 1968.

Department of Mathematics
Michigan State University
East Lansing, Michigan 48824-1027

and

Department of Mathematics
University of Michigan
Ann Arbor, Michigan 48109-1003