TRANSCENDENTAL TRANSCENDENCE OF SOLUTIONS OF SCHRÖDER'S EQUATION ASSOCIATED WITH FINITE BLASCHKE PRODUCTS

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Dedicated to George Piranian

1. Introduction and statement of results. Let S be a finite Blaschke product with $S(0)=0$ and $S'(0)=s$, $0<|s|<1$. Let $\phi\not\equiv 0$ be a function which is meromorphic in the unit disk D and satisfies Schröder's functional equation there:

$$\text{(S)}\qquad \phi(S(z))=s\phi(z).$$

Kuczma ([7], [8]) has a survey of the extensive literature on this equation. More recent work may be found in [5]. Fatou [6] showed that if a certain countable collection of disks is removed from D, then $|\phi(z)|$ approaches infinity as z approaches the unit circle C through what remains of D. This remainder contains circles $|z|=r$ with r arbitrarily close to 1; ϕ is what is now called a strongly annular function. The situation with ϕ is reminiscent of the meromorphic Tsuji function constructed by Collingwood and Piranian [4], and of infinite product annular functions in [1], [2], and [3].

A function ϕ meromorphic in D is said to be a *differentially algebraic* function if it satisfies an algebraic differential equation (ADE)

$$P(z,w(z),w'(z),\dots,w^{(n)}(z))=0,$$

where $P(z,w_0,w_1,\dots,w_n)$ is a polynomial in all its variables. If ϕ satisfies no nontrivial ADE, then ϕ is *transcendentally transcendental*. See [11].

In a private communication, L. A. Rubel asked whether an annular function can be differentially algebraic. The author knows of no such example, and undertook the present research in an attempt to settle the question. The main result of this paper is that every nontrivial solution of (S) is transcendentally transcendental.

THEOREM. *Let S be a finite Blaschke product with* $S(0)=0$, $S'(0)=s$, $0<|s|<1$. *Let ϕ be a function meromorphic in D and not identically* 0, *satisfying Schröder's equation*

$$\text{(S)}\qquad \phi(S(z))=s\phi(z),\quad (z\in D).$$

Then ϕ satisfies no algebraic differential equation.

Two simple examples will illustrate the method of proof.

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EXAMPLE 1. Let $A(z) \not\equiv 0$, $B(z) \not\equiv 0$ be polynomials and suppose that the solution $\phi \not\equiv 0$ of (S) satisfies the equation

$$A(z)\phi'(z) + B(z)\phi^2(z) = 0. \tag{1.1}$$

Then (1.1) still holds with z replaced by $S(z)$. From (S) we have

$$\phi'(S)(z) = \frac{s\phi'(z)}{S'(z)}$$

so that the new equation becomes

$$\frac{A(S(z))s}{S'(z)}\phi'(z) + B(S(z))s^2\phi^2(z) = 0. \tag{1.2}$$

Elimination of $\phi'(z)$ between (1.1) and (1.2) yields

$$\left\{\frac{A(S(z))B(z)s}{S'(z)} - A(z)B(S(z))s^2\right\}\phi^2(z) = 0.$$

That is, the rational function $\Psi = A/B$ must satisfy the functional equation

$$\Psi(S(z)) = sS'(z)\Psi(z). \tag{1.3}$$

If it were known that the only rational solution of (1.3) were $\Psi \equiv 0$, then ϕ could not satisfy (1.1).

EXAMPLE 2. Assume it is known that the solution $\phi \neq 0$ of (S) satisfies no first order ADE. Consider the differential equation

$$\phi''(z) + A(z)\phi'(z) + B(z)\phi(z) = 0, \tag{1.4}$$

where A and B are rational functions. We replace z by $S(z)$ in (1.4) and use the identities

$$\phi'(S(z)) = \frac{s\phi'(z)}{S'(z)}, \qquad \phi''(S(z)) = \frac{s\phi''(z)S'(z) - s\phi'(z)S''(z)}{(S'(z))^3}.$$

We then eliminate $\phi''(z)$ between (1.4) and the new equation. As a result, we obtain a first order differential equation. Since ϕ satisfies no such equation, the coefficient of $\phi'(z)$ must be zero. The reader may verify that this requires A to satisfy the functional equation

$$A(S(z))(S'(z))^2 = A(z)S'(z) + S''(z). \tag{1.5}$$

The proof of the theorem takes up the remainder of this paper. In Section 2 we prove that the assumption that ϕ satisfies an ADE of order 0 or 1 (resp., of order 2 or more) requires that a certain functional equation (E) (resp., (F)) have a nontrivial rational solution (resp., have a rational solution). Equation (1.3) is a special case of (E) while (1.5) is a special case of (F). The methods used in Section 2 have their roots in classical papers in the subject ([9], [10], [12]). In Section 3 we prove that, in fact, (E) has no nontrivial rational solution and (F) has no rational solution.

The author wishes to thank the referee for his encouragement to persevere in eliminating the stubborn exceptional cases which weakened the first version of this paper.

2. Implications for the equations (E) and (F) if ϕ satisfied an ADE. A polynomial $P(z, w_0, \ldots, w_n)$ is a sum of terms (for distinct α)

$$A_\alpha(z)w^\alpha = A_\alpha(z)w_0^{a_0}w_1^{a_1}\cdots w_n^{a_n},$$

where α is the vector $(a_0, a_1, \ldots, a_n)$ and A_α is a polynomial in z. We call α the *index* of the term. The *dimension* of the term is $d(\alpha) = a_0 + a_1 + \cdots + a_n$, and its *weight* is $w(\alpha) = a_1 + 2a_2 + \cdots + na_n$. If $\beta = (b_0, b_1, \ldots, b_n)$ is another index, we say that $\beta < \alpha$, or α is a *higher* index than β, if

(i) $d(\beta) < d(\alpha)$; or
(ii) $d(\beta) = d(\alpha)$, but $w(\beta) < w(\alpha)$; or
(iii) $d(\beta) = d(\alpha)$ and $w(\beta) = w(\alpha)$, but $b_n < a_n$; or
(iv) $d(\beta) = d(\alpha)$, $w(\beta) = w(\alpha)$, and $b_n = a_n$, but $b_{n-1} < a_{n-1}$; etc.

The *principal term* in P is the term of highest index. Given two polynomials P and Q, we write $P > Q$ if the principal term in P has higher index than the principal term in Q.

Let ϕ be a nontrivial solution of (S) and let $\mathfrak{P}$ be the collection of all polynomials $P(z, w_0, \ldots, w_n)$ such that $P(z, \phi(z), \phi'(z), \ldots, \phi^{(n)}(z)) = 0$. The point at issue is to show that $\mathfrak{P}$ contains only the zero polynomial. If $\mathfrak{P}$ contains a polynomial $P \neq 0$, then it contains infinitely many. The standard procedure is to assume that $\mathfrak{P}$ contains a $P \neq 0$, get a $P \in \mathfrak{P}$ which is minimal in the sense of our ordering, and use the special properties of ϕ to get a polynomial Q in $\mathfrak{P}$ which is smaller. Q must then be the zero polynomial. We are ready to proceed with the first result of this section.

Suppose that ϕ satisfies an ADE of order 0 or 1, so that $\mathfrak{P}$ contains nontrivial polynomials $P(z, w_0, w_1)$ for which $P(z, \phi(z), \phi'(z)) = 0$. Among these we choose a smallest P. Let

$$P(z, w_0, w_1) = A_\alpha(z)w_0^{a_0}w_1^{a_1} + \textstyle\sum_1 B_\beta(z)w_0^{b_0}w_1^{b_1} + \sum_2 B_\beta(z)w_0^{b_0}w_1^{b_1}.$$

The sum $\sum_1$ has β's with $d(\beta) = d(\alpha)$ and $w(\beta) < w(\alpha)$, while $\sum_2$ has β's with $d(\beta) < d(\alpha)$. Since $P(z, \phi(z), \phi'(z)) = 0$, we also have

$$P(S(z), \phi(S(z)), \phi'(S(z))) = 0,$$

or, in view of (S),

$$P(S(z), s\phi(z), s\phi'(z)/S'(z)) = 0.$$

A term $B_\beta(z)w_0^{b_0}w_1^{b_1}$, for instance, is replaced by

$$B_\beta(S(z))\frac{s^{d(\beta)}}{(S'(z))^{w(\beta)}}w_0^{b_0}w_1^{b_1}.$$

The coefficients are now rational functions in z rather than polynomials, but that does not matter in what follows. We know that ϕ satisfies the differential equation associated with the polynomial

$$\frac{A_\alpha(S(z))s^{d(\alpha)}}{(S'(z))^{w(\alpha)}}P(z,w_0,w_1)-A_\alpha(z)P\left(S(z),sw_0,\frac{sw_1}{S'(z)}\right).$$

But this polynomial has 0 as the coefficient in its α-term, and therefore is smaller in our ordering than the minimal polynomial. It must then be identically zero. For all other indices β, then, we have

$$\frac{A_\alpha(S(z))s^{d(\alpha)}B_\beta(z)}{(S'(z))^{w(\alpha)}}=\frac{A_\alpha(z)B_\beta(S(z))s^{d(\beta)}}{(S'(z))^{w(\beta)}}.$$

Therefore the rational function $\Psi(z)=B_\beta(z)/A_\alpha(z)$ satisfies the functional equation

$$\Psi(S(z))=s^l(S'(z))^m\Psi(z), \tag{E}$$

where $l=d(\alpha)-d(\beta)$ and $m=w(\beta)-w(\alpha)$. In the next section we shall show that unless $l=m=0$, the only rational solution Ψ of (E) is $\Psi\equiv 0$. In the present situation, either $l>0$ or else $l=0$ and $m<0$. Assuming the result from Section 3, we see that $B_\beta(z)=0$ for all $\beta<\alpha$, so that the differential equation satisfied by ϕ is simply

$$A(z)\phi(z)^{a_0}(\phi'(z))^{a_1}=0 \quad (A(z)\neq 0).$$

This requires that ϕ be constant, and, from (S), the constant must be 0. Hence ϕ satisfies no ADE of order less than 2.

The situation for equations of order two or greater is complicated by the fact that the higher derivatives of $\phi(S(z))$ are no longer monomials.

LEMMA 2.1. *Let ϕ be meromorphic and satisfy* (S) *in D. Then for $j=0,1,\dots,$ we have*

$$\phi^{(j)}(S(z)(S'(z))^j+\frac{j(j-1)}{2}\phi^{(j-1)}(S(z))(S'(z))^{j-1}S''(z) \\ +(\textit{terms involving } \phi^{(j-2)},\dots,\phi')=s\phi^{(j)}(z) \tag{2.1}$$

and

$$\phi^{(j)}(S(z))=\frac{s\phi^{(j)}(z)}{(S'(z))^j}-\frac{j(j-1)sS''(z)\phi^{(j-1)}(z)}{2(S'(z))^{j+1}} \\ +(\textit{terms involving } \phi^{(j-2)},\dots,\phi'). \tag{2.2}$$

Proof. Equation (2.1) is proved by induction, and equation (2.2) follows from using (2.1) for j and $j-1$. □

DEFINITION. Let $\beta=(b_0,\dots,b_k,\dots,b_n)$ be an index with $b_k\geq 1$ for some $k\geq 2$. A related index β^* is obtained as follows. Let k, $2\leq k\leq n$, be the smallest integer greater than 1 for which $b_k\geq 1$. Then β^* is obtained from β by replacing b_k by b_k-1 and b_{k-1} by $b_{k-1}+1$. (We note that $d(\beta^*)=d(\beta)$ and $w(\beta^*)=w(\beta)-1$.)

LEMMA 2.2. *Let* $n \geq 2$ *and let* (2.2) *be applied to a term of index* β;

$$(\phi(S(z)))^{b_0} \ldots (\phi^{(k)}(S(z)))^{b_k} \ldots (\phi^{(n)}(S(z)))^{b_n}. \tag{2.3}$$

There is obtained a homogeneous polynomial of degree $d(\beta)$ *in the variables* $\phi(z), \phi'(z), \ldots, \phi^{(n)}(z)$. *Its highest index is* β *and its next lower index is* β^*. *The coefficients of these terms are*

$$\frac{s^d}{(S'(z))^w} \quad \text{and} \quad \frac{-s^d k(k-1) b_k S''(z)}{2(S'(z))^{w+1}} \tag{2.4}$$

respectively, where we have used d *for* $d(\beta)$, *and* w *for* $w(\beta)$, *and* k *is as in the definition of* β^*.

Proof. Since $\phi(S(z)) = s\phi(z)$ and $\phi'(S(z)) = s\phi'(z)/S'(z)$, the powers b_0 and b_1 are unchanged by the substitution. Since $b_j = 0$ for $2 \leq j \leq k-1$, the corresponding derivatives do not appear in (2.3). The factor $(\phi^{(k)}(S(z)))^{b_k}$ leads, by Lemma 2.1, to a homogeneous polynomial of degree b_k in the variables $\phi'(z), \phi''(z), \ldots, \phi^{(k)}(z)$. Its two largest indices are

$$(0, \ldots, 0, b_k) \quad \text{and} \quad (0, \ldots, 0, 1, b_k - 1)$$

and the respective coefficients are

$$\frac{s^{b_k}}{(S'(z))^{kb_k}} \quad \text{and} \quad -\frac{(1/2)k(k-1)s^{b_k} b_k S''(z)}{(S'(z))^{kb_k+1}}.$$

In the factors from $\phi^{(k+1)}(S(z)), \ldots, \phi^{(n)}(S(z))$, only the largest term is of consequence. The remaining assertions follow from computations.

We are now ready to show, assuming the results from Section 3, that a minimal algebraic differential equation for ϕ cannot be of order 2 or more. Assume the conclusion false, let P be a minimal polynomial, and let α be the index of its principal term. Let $A(z)$ and $B(z)$ be the coefficients in the terms of index α and α^* respectively. Let z and $\phi(z)$ be replaced by $S(z)$ and $\phi(S(z))$ in the differential equation and the new polynomial considered. By Lemma 2.2, the coefficients in the terms of index α and α^* become, respectively, the rational functions

$$\frac{A(S(z))s^d}{(S'(z))^w} \quad \text{and} \quad \frac{-s^d k(k-1) b_k S''(z)}{2(S'(z))^{w+1}} A(S(z)) + B(S(z)) \frac{s^d}{(S'(z))^{w-1}}.$$

From minimality, the polynomial corresponding to

$$\begin{aligned} &\frac{A(S(z))s^d}{(S'(z))^w} P(z, \phi(z), \ldots, \phi^{(n)}(z)) \\ &\quad - A(z) P(S(z), \phi(S(z)), \ldots, \phi^{(n)}(S(z))) \end{aligned} \tag{2.5}$$

must be identically zero. In particular, the coefficient in the term of index α^* is identically zero. But this says

$$\frac{A(S(z))s^d}{(S'(z))^w} B(z) = \frac{A(z)s^d}{(S'(z))^w} \left\{ B(S(z))S'(z) - \frac{A(S(z))\lambda S''(z)}{S'(z)} \right\},$$

where we have written λ for $b_k k(k-1)/2$. Simplifying, and writing $\Psi(z)$ for the rational function $B(z)/A(z)$, we find that Ψ must satisfy

$$\text{(F)} \qquad \Psi(S(z))(S'(z))^2 = \Psi(z)S'(z) + \lambda S''(z) \quad (\lambda \neq 0).$$

Since (F) will be shown to have no rational solutions, ϕ satisfies no ADE.

3. The functional equations (E) and (F). Let S be a finite Blaschke product

$$\text{(3.1)} \qquad S(z) = z \prod_{\nu=1}^{p} \left(\frac{z-a_\nu}{1-\bar{a}_\nu z}\right)^{k_\nu} \quad (p \geq 1),$$

where $0, a_1, \ldots, a_p$ are distinct points in D, $k_1, \ldots, k_p$ are positive integers, and the integer $n \geq 1$ is defined by $n = \sum_{\nu=1}^{p} k_\nu$. Then $S(0) = 0$, $S'(0) = s$ with $0 < |s| < 1$, S maps D onto itself $n+1$ times, and the derivative S' has $2n$ zeros in the complex plane. In fact, a theorem of Walsh [13] gives a version for Blaschke products of the Gauss–Lucas theorem; it shows that S' has exactly n zeros in D, that if $z_0 \in D$ and $S'(z_0) = 0$ then $|z_0| \leq \max|a_\nu|$, and $|z_0| < \max|a_\nu|$ unless z_0 is a zero of S.

Two special results from complex variables will be used in the study of (E) and (F). The first is

$$\text{(3.2)} \qquad |S'(z)| > 1 \quad \text{for } |z| = 1.$$

From (3.1) we have

$$\frac{S'(z)}{S(z)} = \frac{1}{z} + \sum_{\nu=1}^{p} k_\nu \left(\frac{1}{z-a_\nu} - \frac{\bar{a}_\nu}{\bar{a}_\nu z - 1}\right).$$

Then

$$\frac{e^{i\phi}S'(e^{i\phi})}{S(e^{i\phi})} = 1 + \sum_{\nu=1}^{p} k_\nu \frac{e^{i\phi}(-1+|a_\nu|^2)}{(e^{i\phi}-a_\nu)(\bar{a}_\nu e^{i\phi}-1)} = 1 + \sum_{\nu=1}^{p} k_\nu \frac{(1-|a_\nu|^2)}{|e^{i\phi}-a_\nu|^2}$$

and inequality (3.2) follows.

The next result handles the most difficult case, which arises in showing that (F) has no rational solution.

PROPOSITION 3.1. *In* (3.1), *let* $p \geq 2$, $k_1 = 1$, *and* $k_2 = \cdots = k_p = 2$, *so that*

$$\text{(3.3)} \qquad S(z) = \frac{z(z-a_1)}{(1-\bar{a}_1 z)} \prod_{\nu=2}^{p} \frac{(z-a_\nu)^2}{(1-a_\nu z)^2}.$$

Let $S^{-1}(a_1) = \{z_1, \ldots, z_l\}$ *(z_j's distinct). Then it is impossible to have*

$$\text{(3.4)} \qquad S'(z_j) = 0 \quad \text{and} \quad S''(z_j) \neq 0 \quad \text{for } j = 1, \ldots, l.$$

Proof. For each j, the fact that $S(z_j) \neq 0$ and the assumption that $S'(z_j) = 0$ leads, via the theorem of Walsh, to the conclusion that

$$\text{(3.5)} \qquad |z_j| < \max_{2 \leq \nu \leq p} |a_\nu|, \quad (j = 1, \ldots, l).$$

(Since $a_1 = S(z_j)$, we have $|a_1| < |z_j|$.) Let

$$L(w)=\frac{w-a_1}{1-\bar{a}_1 w}, \qquad F(z)=L(S(z)).$$

Then F is a finite Blaschke product, since it is a rational function and $|F(z)|=1$ for $|z|=1$. The numerator in F is a polynomial of degree $n+1$, and the zeros of F are the points of $S^{-1}(a_1)$. Since

(3.6) $$F'(z)=L'(S(z))S'(z) \text{ and } F''(z)=L''(S(z))(S'(z))^2+L'(S(z))S''(z),$$

it follows from (3.4) that $F'(z_j)=0$ and $F''(z_j)\neq 0$. That is, each z_j is a zero of F of order two:

$$F(z)=c\prod_{j=1}^{l}\frac{(z-z_j)^2}{(1-\bar{z}_j z)^2}, \qquad |c|=1.$$

If $2\leq \nu\leq p$, then $a_\nu\notin\{z_1,\ldots,z_l\}$, and (3.4) and (3.6) show that $F'(a_\nu)=0$. Appealing again to Walsh's theorem, we conclude that

(3.7) $$|a_\nu|<\max_{1\leq j\leq l}|z_j|, \quad (\nu=2,\ldots,p).$$

But (3.7) contradicts (3.5), and the proposition is proved. □

We are now ready to start work on the functional equations (E) and (F).

LEMMA 3.1. *Let S be a finite Blaschke product* (3.1). *Let l and m be integers and let Ψ be a rational function which satisfies*

(E) $$\Psi(S(z))=s^l(S'(z))^m\Psi(z).$$

Then Ψ is a constant, and unless $l=m=0$, Ψ is identically zero.

REMARK. The situation in Section 2 had $l=d(\alpha)-d(\beta)$ and $m=w(\beta)-w(\alpha)$. Hence, either $l\geq 1$ or else $l=0$ and $m\leq -1$.

Proof. If $m\geq 1$ and $l\leq 0$, then (E) and (3.2) show that there is an $M>1$ such that

$$|\Psi(S(z))|\geq M|\Psi(z)| \quad \text{for } |z|=1.$$

Unless Ψ is identically zero, there is a sequence $\{z_j\}$ on the unit circle C with $\Psi(z_j)\to\infty$, so that Ψ has a pole at some point z_0, $|z_0|=1$. Successive appeals to (E) show that Ψ has a pole at each of the $n+1$ points of $S^{-1}(z_0)$ (distinct, since $|S'(z)|\neq 0$ on C), at the $(n+1)^2$ points of $S^{-2}(z_0)$, etc. This is impossible, so that $\Psi\equiv 0$ if $m\geq 1$ and $l\leq 0$. The same argument works if $m\geq 0$ and $l\leq -1$, and a parallel argument (with zeros of Ψ instead of poles) works if $m\leq 0$ and $l\geq 1$ or $m\leq -1$ and $l\geq 0$.

If $l=m=0$, so that $\Psi(S(z))=\Psi(z)$, then differentiation gives

$$\Psi'(S(z))=[S'(z)]^{-1}\Psi'(z).$$

This is (E) with $l=0$ and $m=-1$, so that $\Psi'\equiv 0$ and Ψ is constant.

Finally, we consider the situation where $lm\geq 1$. We let $|a_p|=\max|a_\nu|$, write a and k instead of a_p and k_p, expand Ψ, S and S' about 0, and S and S' about a.

$$(3.8) \qquad \Psi(z) = bz^r(1+O(|z|)), \quad b \neq 0, \quad r \text{ an integer}$$

$$(3.9) \qquad S(z) = sz(1+O(|z|)), \qquad S'(z) = s(1+O(|z|))$$

$$(3.10) \qquad \begin{aligned} S(z) &= c(z-a)^k(1+O(|z-a|)), \quad c \neq 0 \\ S'(z) &= ck(z-a)^{k-1}(1+O(|z-a|)). \end{aligned}$$

Substitution of (3.8) and (3.9) into (E) shows that

$$b(sz)^r = s^l s^m bz^r$$

so that

$$(3.11) \qquad r = l+m.$$

Substitution of (3.10) and (3.8) into (E) shows that, as $z \to a$,

$$b(c(z-a)^k)^r = s^l(ck(z-a)^{k-1})^m \Psi(z)(1+O(|z-a|)),$$

$$\Psi(z) = d(z-a)^{kr+m-km}(1+O(|z-a|)),$$

where $d \neq 0$. By (3.11) we have, then,

$$\Psi(z) = d(z-a)^{kl+m}(1+O(|z-a|)),$$

as $z \to a$. Since $lm \geq 1$ and $k \geq 1$, Ψ has either a zero or pole at a. From $|a| = \max|a_\nu|$, it follows that $S'(z) \neq 0$ for all z in $S^{-1}(a) \cup S^{-2}(a) \cup \cdots$. If a is a zero (resp., pole) for Ψ, then (E) shows that each point of $S^{-1}(a) \cup S^{-2}(a) \cup \cdots$ is a zero (resp., pole) for Ψ. Since Ψ is rational, Ψ must be identically 0. The proof of Lemma 3.1 is complete. □

LEMMA 3.2. *Let S be a finite Blaschke product* (3.1) *and let λ be a nonzero constant. Then there is no rational function Ψ such that the equation*

$$(\text{F}) \qquad \Psi(S(z))(S'(z))^2 = \Psi(z)S'(z) + \lambda S''(z)$$

holds in D.

REMARKS. The proof is based on the consideration and elimination of cases until the only case left is that which Proposition 3.1 shows is impossible. We suppose that there is a rational solution Ψ, ask about the location and order of its poles, and what is implied about S and especially the zeros of S'.

To see why this interplay between Ψ and S is relevant, consider the following situation. Let there be a point $z_0 \in D$ such that Ψ is analytic at both z_0 and $S(z_0)$, and such that $S'(z_0) = 0$. If $S''(z_0) \neq 0$, then (F) is instantly contradicted. Even if S' has a zero of order $m \geq 2$ at z_0, the equation arising from $m-1$ differentiations in (F) cannot hold at z_0. Therefore, the only way that (F) can hold on D is that, *for every z_0 in D where $S'(z_0) = 0$, either z_0 or $S(z_0)$ is a pole of Ψ.* It follows immediately that (F) *fails if Ψ* is *analytic everywhere on D.* Less immediately, (F) *fails if the only pole of Ψ in D occurs at $z = 0$.* For S' takes the value zero n times in D but only $n-p$ times on $\{a_1, \ldots, a_p\}$. Hence there is a $z_0 \in D$ such that $S'(z_0) = 0$ but $S(z_0) \neq 0$, so that neither z_0 nor $S(z_0)$ is a pole.

Proof. We shall prove Lemma 3.2 in the following steps, always assuming that Ψ is rational and that (F) holds.

Step 1. Suppose that $0<|z_0|<1$, *that* z_0 *is not a pole of* Ψ, *but that* $\zeta_0=S(z_0)$ *is a pole. Let* m *be the order of the zero of* $S(z)-\zeta_0$ *at* z_0. *Then*

(3.12) $$S'(z_0)=0, \text{ so that } m\geq 2,$$

(3.13) $$\Psi \text{ has a simple pole at } \zeta_0, \text{ and } \operatorname{Res}(\Psi,\zeta_0)=\lambda(m-1)/m,$$

(3.14) $$S(z)-\zeta_0 \text{ has a zero of order exactly } m \text{ at each point of } S^{-1}(\zeta_0) \text{ which is not a pole of } \Psi.$$

Step 2. Ψ *has at most two poles in* D. *If it has two poles, then one of them is at* 0, *and the other is at one of the zeros of* S, *say* a_1. *In this case we have the conditions that are ruled out by Proposition* 3.1.

Step 3. (F) *cannot hold if* $\zeta_1\neq 0$ *is the only pole of* Ψ *in* D.

Steps 2 and 3 cover all the cases left after the remarks above.

Proceeding to the proof of Step 1, we expand S, S', S'' and Ψ about z_0 and Ψ about ζ_0:

$$S(z)-\zeta_0=c(z-z_0)^m(1+O(|z-z_0|)), \quad (c\neq 0),$$
$$S'(z)=cm(z-z_0)^{m-1}(1+O(|z-z_0|)),$$
$$S''(z)=cm(m-1)(z-z_0)^{m-2}(1+O(|z-z_0|)),$$
$$\Psi(z)=d(z-z_0)^l(1+O(|z-z_0|)), \quad (d\neq 0,\ l\geq 0),$$
$$\Psi(\zeta)=b(\zeta-\zeta_0)^r(1+O(|\zeta-\zeta_0|)), \quad (b\neq 0,\ r<0).$$

If $S'(z_0)\neq 0$, then the left side of (F) has a pole at z_0, while the right side is analytic there. Hence $m\geq 2$; (3.12) holds. Then the term of lowest degree in $(z-z_0)$ on the right side of (F) comes from $\lambda S''(z)$, namely,

$$\lambda cm(m-1)(z-z_0)^{m-2},$$

while on the left the term of lowest degree is

$$b(c(z-z_0)^m)^r c^2m^2(z-z_0)^{2m-2}=bc^{r+2}m^2(z-z_0)^{m(r+2)-2}.$$

Comparing the powers of $z-z_0$ in the two expressions, we conclude that $r=-1$, so that the pole at ζ_0 is simple. Then, equating the coefficients, we find that $\lambda(m-1)=bm$, so that (3.13) holds. But (3.13) determines m as soon as b and λ are known, so that the same m works for all points of $S^{-1}(\zeta_0)$ which are not poles of Ψ. The proof of Step 1 is complete.

For Step 2, we suppose that Ψ has at least two poles in D and let ζ_1 be a pole of largest modulus. Let $S^{-1}(\zeta_1)=\{z_1,\dots,z_l\}$. Then no z_j is a pole, and we may apply the results of Step 1 to each of them. We then have

(3.15) $$ml=n+1,$$

and, of the n zeros of S' available in D, the number used on $\{z_1, \ldots, z_l\}$ is

$$(m-1)l=(n+1)(m-1)/m \geq (n+1)/2. \tag{3.16}$$

Of the remaining poles of Ψ, let ζ_2 be one of largest modulus, and assume that $\zeta_2 \neq 0$. If $S^{-1}(\zeta_2)$ contained only points of analyticity of Ψ, then the argument for ζ_1 could be repeated, and S' would assume on $S^{-1}(\zeta_1) \cup S^{-1}(\zeta_2)$ at least $(n+1)/2+(n+1)/2=n+1$ zeros, which is impossible. Hence $S^{-1}(\zeta_2)$ must contain a pole, and, from our construction, the pole must be ζ_1. Let $S^{-1}(\zeta_2)=\{\zeta_1, \sigma_1, \ldots, \sigma_N\}$ and let the order of the zeros of $S(z)-\zeta_2$ at ζ_1 be $\mu_1 \geq 1$ and at $\sigma_1, \ldots, \sigma_N$ let it be μ_2. We have $\mu_1+N\mu_2=n+1$ so that

$$2N \leq \mu_2 N=n+1-\mu_1 \leq n, \quad \text{or} \quad N \leq n/2.$$

The number of zeros of S' occurring in $S^{-1}(\zeta_2)$ is then

$$\mu_1-1+N(\mu_2-1)=n-N \geq n/2.$$

In view of (3.16), this is impossible. We continue to assume that Ψ has two poles in D, and now know that they are at $\zeta_1 \neq 0$ and $\zeta_2=0$. We have $S^{-1}(0)=\{0, a_1, \ldots, a_p\}$, where $p \geq 1$. If no a_j is a pole of Ψ, then each a_j is a zero of S of the same order $\mu \geq 2$. We have $p\mu=n$, so that S' assumes on $\{a_1, \ldots, a_p\}$, $p(\mu-1)=n-p \geq n/2$ of its zeros. By (3.16), this is impossible. Therefore one of the a_j, say a_1, is equal to ζ_1. There is at least one point z_1 in $S^{-1}(a_1)=S^{-1}(\zeta_1)$, and $S'(z_1)=0$. We have $|a_1|=|\zeta_1|<|z_1|$. But, by Walsh's theorem, we have $|z_1|<\max|a_\nu|$. Hence p must be at least 2. With this information, we count the zeros of S' once again. Let the order of the zeros of S at a_1 be $\mu_1 \geq 1$ and let the order at each a_j, $2 \leq j \leq p$, be $\mu \geq 2$. Then

$$\mu_1+(p-1)\mu=n, \qquad p=\frac{n-\mu_1}{\mu}+1,$$

and, of the n zeros of S' in D,

$$(\mu_1-1)+(p-1)(\mu-1)=n-p$$

occur on $\{a_1, \ldots, a_p\}$. On $S^{-1}(a_1)$, an additional $(n+1)(m-1)/m$ occur. Therefore we require $n-p+(n+1)(1-1/m) \leq n$.

$$(n+1)\left(1-\frac{1}{m}\right) \leq p=\frac{n-\mu_1}{\mu}+1. \tag{3.17}$$

The left side of (3.17) is at least $(n+1)/2$ and its right side is at most $(n-1)/2+1$. Hence we have equality, so that $m=\mu=2$, and $\mu_1=1$. That is, S is given by (3.3), and each point of $S^{-1}(a_1)$ is a first order zero of S'. This is ruled out by Proposition 3.1 and the proof of Step 2 is complete.

We proceed to the third and final step. Suppose that $\zeta_1 \neq 0$ is the only pole of Ψ in D. Then in (F)

$$\Psi(S(z))(S'(z))^2=\Psi(z)S'(z)+\lambda S''(z), \tag{F}$$

both the left side and $\lambda S''(z)$ are analytic at ζ_1. Hence $\Psi(z)S'(z)$ is analytic there, so that $S'(\zeta_1)=0$. Suppose that, for $z\to\zeta_1$,

$$S(z)-S(\zeta_1)=c(z-\zeta_1)^m(1+O(|z-\zeta_1|)),\quad (c\neq 0,\ m\geq 2).$$

Now the left side of (F) has a zero of order at least $2m-2$ at ζ_1, $\Psi(z)S'(z)$ has a zero of order exactly $m-2$ there (since the pole is simple), and $\lambda S''(z)$ has a zero of order $m-2$ also. The coefficient of $(z-\zeta_1)^{m-2}$ on the right is

$$cm\,\mathrm{Res}(\Psi,\zeta_1)+cm(m-1)\lambda.$$

Since this must be zero, it follows that $\mathrm{Res}(\Psi,\zeta_1)=-\lambda(m-1)$. This is impossible, by (3.13). Step 3 and the proof of Lemma 3.2 are complete. □

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