ON REMOVAL OF PERIODS OF CONJUGATE FUNCTIONS IN MULTIPLY CONNECTED DOMAINS

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1. Introduction. Let G be a domain in $\mathbf{C}$ bounded by n analytic Jordan curves $\gamma_1, \dots, \gamma_n$, $\Gamma = \bigcup_1^n \gamma_j$. Recall the following classical result of M. Heins—see [9].

For each $(n-1)$-tuple of real numbers $\Lambda_1, \dots, \Lambda_{n-1}$ there exist points $\zeta_1, \dots, \zeta_n$ on Γ and a positive harmonic function $u(z)$ in G such that $u(\zeta)=0$ for all $\zeta \in \Gamma \setminus \{\zeta_1, \dots, \zeta_n\}$, $u(\zeta_i) = +\infty$, $i=1, \dots, n$ and the periods of the conjugate function of $u(z)$ along $\gamma_1, \dots, \gamma_{n-1}$ equal to $\Lambda_1, \dots, \Lambda_{n-1}$ respectively.

The various refinements and applications of this result can be found in [5], [8], [9]. Also, see [5], [9], [10], [11] for the discussion concerning the corresponding statement for finite Riemann surfaces and its applications.

In this paper (in §3, Lemmas 1 and 2) we generalize the Heins result in the following sense.

For each $(n-1)$-tuple of real numbers $\Lambda_1, \dots, \Lambda_{n-1}$ and each positive Borel measure μ on Γ satisfying $\mu(\gamma_i)>0$, $i=1, \dots, n-1$ $(i=1, \dots, n)$ there exist real (real positive numbers) $\lambda_1, \dots, \lambda_{n-1}$ $(\lambda_1, \dots, \lambda_n)$ such that the periods of the conjugate of the harmonic function defined by the Poisson integral of the measure $\tilde{\mu}: \tilde{\mu}|_{\gamma_i} \equiv \lambda_i \mu|_{\gamma_i}$, along $\gamma_1, \dots, \gamma_{n-1}$ are equal to $\Lambda_1, \dots, \Lambda_{n-1}$.

Let us give a brief description of the contents of the paper. In §2 we recall some basic facts of the function theory in multiply connected domains. For more details we send the reader to [3], [4], [13], [14].

In §3 we prove Lemmas 1 and 2. In §4, using Lemmas 1 and 2, we construct the analogs of the Schwarz kernel for the multiply connected domain G which allow us to reproduce analytic functions in G by means of the boundary values of their real parts. These kernels are different from those constructed in [2], [13], [16].

Finally, in §5 we consider certain applications of the results obtained in the previous sections. In particular, we show the existence in multiply connected domains of an analytic function in a given class (e.g., Nevanlinna's class, Hardy classes, etc.) with prescribed modulus of boundary values.

This problem has been studied in [7], [15]. Also, see [8, Ch. 4, §4]. But the functions constructed there are essentially different from those we obtain here.

Unfortunately, we have been unable to obtain an appropriate generalization of the results described above to finite Riemann surfaces. The fact is that the statements analogous to Lemmas 1 and 2 on a Riemann surface are much more complicated. The reason for that is that there are two kinds of periods on a surface, that is, periods along the boundary curves and around the handles. Therefore, one cannot expect formulas for the Schwarz type kernels on the surfaces to be as simple and clear as in multiply connected domains.

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2. Preliminaries. Let G be a domain in $\mathbf{C}$ with an analytic boundary $\Gamma=\bigcup_{j=1}^{n}\gamma_j$. $g(z,\zeta)$ is the Green function of G with pole at ζ. $\partial g(\zeta,z)/\partial n_\zeta$, $\zeta\in\Gamma$, $z\in G$ is the analog of the Poisson kernel for G (Green's kernel). Here, $\partial/\partial n_\zeta$ denotes the derivative in the direction of the inner normal at ζ. Let $u(z)$ be a harmonic function in G and $v(z)$ denote its harmonic conjugate. In general $v(z)$ may be multivalued. Let $\gamma_j'\subset G$ be an analytic curve homologous to γ_j in G. Then, the period of $v(z)$ around γ_j can be expressed in the following form:

$$\Delta_{\gamma_j}v=\int_{\gamma_j'}\frac{\partial v}{\partial s}\,ds=-\int_{\gamma_j'}\frac{\partial u}{\partial n}\,ds,$$

where ds is Lebesgue measure on γ_j'. Now, assume that $u(z)$ is represented by the Green–Stieltjes integral of a Borel measure μ supported on Γ, that is

$$u(z)=\frac{1}{2\pi}\int_\Gamma\frac{\partial g(\zeta,z)}{\partial n_\zeta}\,d\mu(\zeta).$$

Consider the sequence of n-connected domains $\{D^i\}$ such that $\bigcup_{i=1}^{\infty}D^i=G$, $D^i\subset D^{i+1}$, $\partial D^i=\Gamma^i=\bigcup_{j=1}^{n}\gamma_j^i$. Let $\omega_j(z)$ denote the harmonic measure of γ_j, that is, ω_j is harmonic in G, continuous in $\bar G$ and $\omega_j|_{\gamma_j}\equiv 1$, $\omega_j|_{\gamma_k}\equiv 0$ for $k\neq j$, $j=1,\dots,n-1$. ω_j^i denotes the harmonic measure of γ_j^i with respect to D^i. Using Green's formula and Fubini's theorem we can compute the periods of $v(z)$ as follows:

$$\begin{aligned}\Delta_{\gamma_j}v&=-\int_{\gamma_j^i}\frac{\partial u}{\partial n}\,ds=-\int_{\Gamma^i}\omega_j^i\frac{\partial u}{\partial n}\,ds=-\int_{\Gamma^i}u\frac{\partial\omega_j^i}{\partial n}\,ds\\&=-\int_{\Gamma^i}\left\{\frac{1}{2\pi}\int_\Gamma\frac{\partial g(\zeta,z)}{\partial n_\zeta}\,d\mu(\zeta)\right\}\frac{\partial\omega_j^i}{\partial n}\,ds\\&=-\int_\Gamma\left\{\int_{\Gamma^i}\frac{1}{2\pi}\frac{\partial g(\zeta,z)}{\partial n_\zeta}\cdot\frac{\partial\omega_j^i}{\partial n}\,ds\right\}d\mu(\zeta).\end{aligned}$$

As $i\to\infty$,

$$\frac{1}{2\pi}\frac{\partial g(\zeta,z)}{\partial n_\zeta}\,ds\Big|_{\Gamma_i}$$

converges to δ-measure at ζ and

$$\frac{\partial\omega_j^i(z)}{\partial n}\to\frac{\partial\omega_j(z)}{\partial n}$$

uniformly. So, the inner integral converges to $(\partial\omega_j/\partial n_\zeta)(z)$ pointwise. Therefore, applying the Lebesgue dominated convergence theorem we obtain that

$$\Delta_{\gamma_j}v=-\int_\Gamma\frac{\partial\omega_j}{\partial n}(\zeta)\,d\mu(\zeta). \tag{1}$$

3. The main lemmas.

LEMMA 1. *Let $\mu>0$ (or $\mu<0$) be a Borel measure on Γ such that $\mu(\gamma_j)\neq 0$, $j=1,\dots,n-1$. Then, for arbitrary real numbers $\Lambda_1,\dots,\Lambda_{n-1}$ there exists a* *unique vector* $(\lambda_1, \dots, \lambda_{n-1}) \in \mathbf{R}^{n-1}$ *such that* $\Lambda_1, \dots, \Lambda_{n-1}$ *are the periods along* $\gamma_1, \dots, \gamma_{n-1}$ *respectively of the function conjugate to*

$$u(z) = \frac{1}{2\pi} \int_\Gamma \frac{\partial g(\zeta, z)}{\partial n_\zeta} \, d\bar{\mu}(\zeta),$$

where $\bar{\mu}|_{\gamma_n} \equiv \mu|_{\gamma_n}$, $\bar{\mu}|_{\gamma_j} \equiv \lambda_j \mu|_{\gamma_j}$, $j = 1, \dots, n-1$.

Proof. We consider $\mu > 0$. For $\mu < 0$ the proof is the same. In view of (1) it suffices to prove that the following system of linear equations always has a unique solution

$$(*) \quad \sum_{j=1}^{n-1} \lambda_j a_{ij} = \Lambda_i + \int_{\gamma_n} \frac{\partial \omega_i}{\partial n} \, d\mu, \quad i = 1, \dots, n-1, \quad \text{where } a_{ij} = -\int_{\gamma_j} \frac{\partial \omega_i}{\partial n} \, d\mu.$$

So, we have to show that $\det(a_{ij})_1^{n-1} = \det[(a_{ij})_1^{n-1}]^T \neq 0$. Let $A: \mathbf{R}^{n-1} \to \mathbf{R}^{n-1}$ be a linear operator defined by the matrix $[(a_{ij})_1^{n-1}]^T$. Let $\|x\| = \max_i(|x_i|)$ for all $x = (x_1, \dots, x_{n-1}) \in \mathbf{R}^{n-1}$. Note that it suffices to show that $\exists \lambda \neq 0$ such that $(I - \lambda A)$ is a contraction operator, that is, $\|(I - \lambda A)x\| \leq \theta \|x\|$, $0 < \theta < 1$ for all $x \in \mathbf{R}^{n-1}$. (Here, I denotes the identity map on $\mathbf{R}^{n-1}$.) In reality, if $x \in \ker A$, then $\|x\| \leq \theta \|x\|$. Hence, $x = 0$.

It is convenient to single out the following assertion.

ASSERTION.

$$a_{jj} > \sum_{\substack{i=1 \\ i \neq j}}^{n-1} |a_{ij}| \quad \text{for all } j = 1, \dots, n-1.$$

Proof of the assertion. Since

$$\left. \frac{\partial \omega_i}{\partial n} \right|_{\gamma_j} > 0, \quad i \neq j \quad \text{and} \quad \left. \frac{\partial \omega_j}{\partial n} \right|_{\gamma_j} < 0,$$

we have

$$\sum_{\substack{i=1 \\ i \neq j}}^{n-1} |a_{ij}| - a_{jj} = \sum_{\substack{i=1 \\ i \neq j}}^{n-1} \left| -\int_{\gamma_j} \frac{\partial \omega_i}{\partial n} \, d\mu \right| + \int_{\gamma_j} \frac{\partial \omega_j}{\partial n} \, d\mu = \sum_{i=1}^{n-1} \int_{\gamma_j} \frac{\partial \omega_i}{\partial n} \, d\mu$$

$$= \int_{\gamma_j} \left(\sum_{i=1}^{n-1} \frac{\partial \omega_i}{\partial n} \right) d\mu = \int_{\gamma_j} \frac{\partial}{\partial n_\zeta} \left(\sum_{i=1}^{n-1} \omega_i \right) d\mu.$$

The function $\sum_1^{n-1} \omega_i(z)$ is equal to 1 on $\gamma_1, \dots, \gamma_{n-1}$, equal to 0 on γ_n, and harmonic in G. Hence, $(\partial/\partial n)(\sum_1^{n-1} \omega_i(\zeta))|_{\gamma_j} < 0$, by the maximum principle. So

$$\int_{\gamma_j} \frac{\partial}{\partial n_\zeta} \left(\sum_1^{n-1} \omega_i(\zeta) \right) d\mu(\zeta) < 0$$

and the assertion is proved.

Note that we can actually assume that

$$\theta \cdot a_{jj} > \sum_{\substack{i=1 \\ i \neq j}}^{n-1} |a_{ij}| \quad \text{for some } \theta < 1 \text{ and all } j.$$

Now, let $x=(x_1,\dots,x_{n-1})$, $(I-\lambda A)x=x'=(x_1',\dots,x_{n-1}')$. Then, according to our assertion, for all $j=1,\dots,n-1$ we have:

$$|x_j'|=\left|x_j-\lambda\sum_{i=1}^{n-1}a_{ij}x_i\right|\le|x_j|\,|1-\lambda a_{jj}|+|\lambda|\sum_{i\ne j}|a_{ij}|\,|x_i|$$
$$\le\|x\|(|1-\lambda a_{jj}|+\theta|\lambda|\,|a_{jj}|).$$

Taking $\lambda>0$ small enough that $\lambda a_{jj}<1$ for all j we obtain

$$|x_j'|<\|x\|(1-|\lambda|\,|a_{jj}|(1-\theta))=\theta_0\|x\|,$$

where $0<\theta_0<1$. Lemma 1 is proved. □

The proof of the following statement is very similar to the one of Lemma 1 and we will omit it (see [8], [11], [12]).

LEMMA 2. *Let $\mu>0$ (or $\mu<0$) be a Borel measure on Γ such that $\mu(\gamma_j)\ne 0$ $\forall j=1,\dots,n$. Then, for arbitrary real numbers $\Lambda_1,\dots,\Lambda_{n-1}$ there exist* positive *numbers $\lambda_1,\dots,\lambda_n$ such that $\Lambda_1,\dots,\Lambda_{n-1}$ are the periods of the function conjugate of*

$$u(z)=\frac{1}{2\pi}\int_\Gamma\frac{\partial g(\zeta,z)}{\partial n_\zeta}\,d\bar\mu(\zeta),\quad\text{where }\bar\mu|_{\gamma_j}\equiv\lambda_j\mu|_{\gamma_j}\text{ for }j=1,\dots,n$$

around $\gamma_1,\dots,\gamma_{n-1}$ respectively. Moreover, all such $\bar\lambda=(\lambda_1,\dots,\lambda_{n-1})$ fill out a ray in $\mathbf{R}^n$.

4. Reproducing kernels. Fix $z_0\in G$. Let $\zeta_j\in\gamma_j$, $j=1,\dots,n-1$ be arbitrary fixed points. Let δ_{ζ_j} be a δ-mass at ζ_j. Then in view of Lemma 1 applied to $\sum_1^{n-1}\delta_{\zeta_j}$,

$$\det\left[\left(\frac{\partial\omega_i}{\partial n_\zeta}(\zeta_j)\right)_1^{n-1}\right]\ne 0.$$

Let $(t_{ij})_1^{n-1}$ be the inverse matrix of $((\partial\omega_i/\partial n_\zeta)(\zeta_j))_1^{n-1}$.

THEOREM 1. *There exists a function $P_1(z,\zeta)$ satisfying the following.*

(i) *$P_1(z,\zeta)$ is continuous on $G\times\Gamma$ and analytic in G for each fixed $\zeta\in\Gamma$.*

(ii) *For any $z\in G$,*

$$P_1(z,\zeta_j)=0,\ j=1,\dots,n-1,\quad\textit{and}\quad\int_\Gamma P_1(z,\zeta)\,ds=1.$$

(iii) *For any function $f(z)=u(z)+iv(z)$ analytic in G and such that $u(z)$ is representable by the Green–Stieltjes integral of the measure μ, the following holds: $f(z)=\int_\Gamma P_1(z,\zeta)\,d\mu(\zeta)+iv(z_0)$.*

The function $P_1(z,\zeta)$ satisfying (i)–(iii) *is unique.*

(iv) *Moreover, the kernel $P_1(z,\zeta)$ can be written in the form $P_1(z,\zeta)=R_1(z,\zeta)+i\tilde R_1(z,\zeta)$, where*

$$R_1(z,\zeta)=\frac{1}{2\pi}\left\{\frac{\partial g(\zeta,z)}{\partial n_\zeta}-\sum_1^{n-1}\Lambda_j(\zeta)\frac{\partial g(\zeta_j,z)}{\partial n_\zeta}\right\}\quad \textit{and}$$

$$\Lambda_j(\zeta)=-\sum_{i=1}^{n-1}t_{ji}\frac{\partial\omega_i}{\partial n_\zeta}(\zeta).$$

Proof. (i) Fix $\zeta\in\Gamma$. According to Lemma 1 applied to $\sum_1^{n-1}\delta_{\zeta_j}$, there exist $\Lambda_1(\zeta),\dots,\Lambda_{n-1}(\zeta)$ such that the function

$$R_1(z,\zeta)=\frac{1}{2\pi}\left\{\frac{\partial g(\zeta,z)}{\partial n_\zeta}-\sum_1^{n-1}\Lambda_j(\zeta)\frac{\partial g(\zeta_j,z)}{\partial n_\zeta}\right\}$$

has a single-valued conjugate $\tilde R_1(z,\zeta)$ in G. Set $\tilde R_1(z_0,\zeta)=0$. From the linear system (*) one can see that $\Lambda_j(\zeta)$, $j=1,\dots,n-1$ are continuous on Γ. Putting $P_1(z,\zeta)=R_1(z,\zeta)+i\tilde R_1(z,\zeta)$ we complete the proof of (i).

(ii) Let μ be a measure on Γ. Form the integrals

$$u(z)=\frac{1}{2\pi}\int_\Gamma\frac{\partial g(\zeta,z)}{\partial n_\zeta}\,d\mu(\zeta);\tag{2}$$

(3)

$$u_1(z)=\int_\Gamma R_1(z,\zeta)\,d\mu(\zeta)=u(z)-\sum_1^{n-1}\lambda_j\frac{\partial g(\zeta_j,z)}{\partial n_{\zeta_j}},\quad\text{where } \lambda_j=\int_\Gamma\Lambda_j(\zeta)\,d\mu(\zeta);$$

$$F(z)=\int_\Gamma P_1(z,\zeta)\,d\mu(\zeta)=u_1(z)+iv_1(z),\tag{4}$$

($v_1(z)$ is a conjugate of $u_1(z)$). In view of Lemma 1 the numbers λ_j, $j=1,\dots,n-1$ are uniquely determined from the system (*). Take

$$u(z)=\frac{\partial g(\zeta_j,z)}{\partial n_{\zeta_j}},\quad 1\le j\le n-1.$$

Set $\lambda_1=0,\dots,\lambda_j=1,\lambda_{j+1}=0,\dots,\lambda_{n-1}=0$. Then, $u_1(z)\equiv 0$. On the other hand, according to (i), $u_1(z)=R_1(z,\zeta_j)$. So, $P_1(z,\zeta_j)\equiv 0+ic$. But $\tilde R_1(z_0,\zeta_j)=0$, hence $P_1(z,\zeta_j)\equiv 0$. Letting $u(z)\equiv 1$ and putting $\lambda_1=\dots=\lambda_{n-1}=0$, we obtain that $u_1(z)\equiv 1$. Therefore

$$\int_\Gamma P_1(z,\zeta)\,ds=u_1+iv_1\equiv 1+ic.$$

As above we verify that $c=0$.

(iii) Let $f(z)=u(z)+iv(z)$ be an analytic and single-valued function in G such that $u(z)$ is representable in the form (2). Then, all $\lambda_j=0$, $j=1,\dots,n-1$. Hence, $f(z)=F(z)+iv(z_0)$.

To prove the uniqueness suppose that $\exists P(z,\zeta)$ satisfying (i)–(iii). Take an arbitrary measure μ on Γ. Form $u(z),u_1(z),F(z)$ as in (2)–(4). Using the properties (i)–(iii), we obtain:

$$F(z)=\int_\Gamma P(z,\zeta)\left[d\mu(\zeta)-\sum_1^{n-1}\lambda_j\delta_{\zeta_j}\right]$$
$$=\int_\Gamma P(z,\zeta)\,d\mu(\zeta)-\sum_1^{n-1}\lambda_j P(z,\zeta_j)=\int_\Gamma P(z,\zeta)\,d\mu(\zeta).$$

At the same time since $P_1(z,\zeta)$ also satisfies (i)-(iii) we can verify that

$$F(z)=\int_\Gamma P_1(z,\zeta)\,d\mu(\zeta).$$

Since μ was arbitrary, we conclude that $P\equiv P_1$.

(iv) Again, let us take an arbitrary measure ψ on Γ. Form the integrals (2) and (3). Then, $u_1(z)$ has a single-valued conjugate. Therefore, $\lambda_1\dots\lambda_{n-1}$ in (3) can be obtained as the solutions of the system (*) with $\mu=\sum_1^{n-1}\delta_{\zeta_j}$. In view of (1), $-\int_\Gamma(\partial\omega_i/\partial n_\zeta)\,d\psi$, $i=1,\dots,n-1$ play the role of Λ_i in (*). Therefore,

$$\lambda_j=-\sum_{i=1}^{n-1}t_{ji}\int_\Gamma\frac{\partial\omega_i}{\partial n_\zeta}\,d\psi(\zeta),\quad j=1,\dots,n-1,\quad\text{where }((t_{ij}))=\left(\left(\frac{\partial\omega_i}{\partial n_{\zeta_j}}(\zeta_j)\right)\right)^{-1}.$$

On the other hand, $\lambda_j=\int_\Gamma\Lambda_j(\zeta)\,d\psi(\zeta)$. Hence, for an arbitrary ψ, we have

$$\int_\Gamma\left[\Lambda_j(\zeta)+\sum_{i=1}^{n-1}t_{ji}\frac{\partial\omega_i}{\partial n_\zeta}(\zeta)\right]d\psi(\zeta)=0$$

and the statement follows. The theorem is proved. □

Using Lemma 2 instead of Lemma 1 one can obtain the following theorem. We shall omit the proof.

THEOREM 2. *Fix arbitrary points $\zeta_1,\dots,\zeta_n$ on $\gamma_1,\dots,\gamma_n$ respectively. There exists a unique function $P_2(z,\zeta)$ satisfying the following properties.*

(i) *$P_2(z,\zeta)$ is continuous on $G\times\Gamma$ and analytic single-valued in G as a function of z.*

(ii) $$\operatorname{Re}P_2(z,\zeta)=R_2(z,\zeta)=\frac{1}{2\pi}\left\{\frac{\partial g(\zeta,z)}{\partial n_\zeta}+\sum_1^n\Lambda_j'(\zeta)\frac{\partial g(\zeta_j,z)}{\partial n_{\zeta_j}}\right\},$$

where all $\Lambda_j'(\zeta)>0$ on Γ. $P_2(z_0,\zeta_j)=1$ for $j=1,\dots,n$ (z_0 is a fixed point in G).

(iii) *If $f(z)=u(z)+iv(z)$ is a single-valued analytic function in G and $u(z)$ is representable in the form* (2) *with the measure μ, then*

$$f(z)=\int_\Gamma P_2(z,\zeta)\,d\mu(\zeta)+iv(z_0).$$

REMARK. Using the measure $\mu'=-\sum_1^n\delta_{\zeta_j}$ in Lemma 2, one can construct the kernel $P_2'(z,\zeta)$ having the same reproducing properties as the kernel $P_2(z,\zeta)$ in Theorem 2 and such that

$$\operatorname{Re}P_2'(z,\zeta)=\frac{1}{2\pi}\left\{\frac{\partial g(\zeta,z)}{\partial n_\zeta}+\sum_1^n\Lambda_j''(\zeta)\frac{\partial g(\zeta_j,z)}{\partial n_{\zeta_j}}\right\}$$

where all $\Lambda_j''(\zeta)<0$.

5. Applications. The following theorem is a classical result of Bieberbach and Grunsky (see [6], [8]). For a different approach due to L. Ahlfors, see [1]. Our proof (based on Lemma 2 and Theorem 2), although discovered independently, is almost the same as that due to M. Heins in [11] or H. Grunsky in [8].

THEOREM 3. *Let $\zeta_1, \dots, \zeta_n$ be arbitrary fixed points on $\gamma_1, \dots, \gamma_n$ respectively. Then, for each j, $1 \le j \le n$, $\phi(z) = P_2(z, \zeta_j)$ is the unique function giving a conformal mapping of G onto an n-sheeted right half-plane $\{\operatorname{Re} w > 0\}$ such that $\phi(\zeta_j) = \infty$, $j = 1, \dots, n$, $\phi(z_0) = 1$. Moreover, $\psi(z) = (\phi(z) - 1)/(\phi(z) + 1)$ is the unique function mapping G conformally onto an n-sheeted unit disk such that $\psi(\zeta_j) = 1$, $j = 1, \dots, n$, and $\psi(z_0) = 0$.*

Proof. $\operatorname{Re} P_2(z, \zeta) > 0$, $P_2(z_0, \zeta_j) = 1$ and

$$\operatorname{Re} P_2(z, \zeta_j)\,|_\Gamma = \frac{1}{2\pi}\left\{\sum_1^n \Lambda_j'(\zeta_j)\frac{\partial g(z, \zeta_j)}{\partial n_{\zeta_j}} + \frac{\partial g(z, \zeta_j)}{\partial n_{\zeta_j}}\right\}$$

where $\Lambda_j'(\zeta_j) > 0$, $j = 1, \dots, n$.

$$\frac{\partial g(\zeta, \zeta_j)}{\partial n_{\zeta_j}} = 0 \quad \text{for } \zeta \in \Gamma, \ \zeta \ne \zeta_j$$

(see [6], [13]). A standard argument shows that $w = \phi(z)$ maps each γ_j onto $\operatorname{Re} w = 0$. Hence, for any $w \in \{\operatorname{Re} w > 0\}$, it follows from the argument principle that $\phi(z) - w$ has precisely n zeroes in G. Since for any ϕ' giving such a conformal mapping,

$$\phi' = \frac{1}{2\pi}\sum_1^n \lambda_j\left(\frac{\partial g(z, \zeta_j)}{\partial n_{\zeta_j}} + i\frac{\partial \tilde{g}(z, \zeta_j)}{\partial n_{\zeta_j}}\right)$$

(where all $\lambda_j > 0$), the uniqueness follows immediately from Lemma 2 and the normalization $\phi'(z_0) = 1$. The second statement is a direct corollary of the first. The theorem is proved. □

To discuss further applications of the results in Sections 3 and 4, let us recall the definitions of the basic classes of analytic functions in multiply connected domains. For more detailed information, we refer the reader to [3], [4], [13], [14], and [15].

An analytic function belongs to the class $N(G)$ (Nevanlinna's class) or to $H^p(G)$ (Hardy's class), $0 < p < \infty$ if $\ln^+|f|$ or, respectively, $|f|^p$ have a harmonic majorant in G. $f \in H^\infty(G)$ if f is bounded in G. $f(z)$ belongs to $N^+(G)$ (Smirnov's class) if $f(z) \in N(G)$ and $\{\int_{E \subset \Gamma^i} \ln^+|f|\, ds\}$ are uniformly absolutely continuous ($\Gamma^i = \partial D^i$ are the same as in §2). It is known that $N(G) \supset N^+(G) \supset H^p(G)$ for all $p > 0$.

THEOREM 4. *Let $\rho(\zeta) > 0$ be a function on Γ such that*

$$\int_\Gamma |\ln \rho(\zeta)|\, ds < \infty. \tag{5}$$

(i) *$\exists f_1(z) \in N(G)$ such that $|f(\zeta)|\,|_\Gamma \equiv \rho(\zeta)$ a.e. ($f(\zeta)$ denotes the boundary values of f).*

(ii) $\exists f_2(z) \in N^+(G)$ *(different from* $f_1(z)$*) such that* $|f_2(\zeta)| \equiv \rho(\zeta)$ *a.e. on* Γ.
(iii) *If, in addition to* (5), $\rho(\zeta)$ *satisfies*

(6) $$\int_\Gamma \rho^p(\zeta)\,ds < \infty \text{ for a certain } p>0$$

or $\rho(\zeta)$ *is bounded on* Γ, *then* $\exists f \in H^p(G)$ *such that* $|f(\zeta)| = \rho(\zeta)$ *a.e. on* Γ.

Proof. (i) Define

$$f_1(z) = \exp\left(\int_\Gamma P_1(z,\zeta) \ln \rho(\zeta)\,ds\right).$$

Then, in view of the construction of $P_1(z,\zeta)$, $\ln|f_1|$ is representable by a Green–Stieltjes integral in G, that is,

$$\ln|f_1(z)| = \frac{1}{2\pi}\int_\Gamma \frac{\partial g(z,\zeta)}{\partial n_\zeta}\,d\mu(\zeta), \quad \text{where}$$

$$d\mu = \ln \rho\,ds - \sum_1^{n-1} \lambda_j \delta_{\zeta_j}, \quad \lambda_j = \int_\Gamma \Lambda_j(\zeta) \ln \rho(\zeta)\,ds, \ j=1,\dots,n-1.$$

Hence, $|\ln|f_1||$ has a harmonic majorant in G (see [14]). Thus, $f_1 \in N(G)$. According to the version of Fatou's theorem for Green–Stieltjes integrals obtained in [13], we get

$$\ln|f_1(\zeta)| = \frac{d\mu}{ds}(\zeta) = \ln \rho(\zeta) \ \text{ a.e. on } \Gamma.$$

So, $|f_1| \equiv \rho$ a.e.

(ii)–(iii) Let $\ln \rho(\zeta) = \ln^+ \rho(\zeta) - \ln^- \rho(\zeta)$. Define

$$f'(z) = \exp\left(\int_\Gamma P_2'(z,\zeta) \ln^+ \rho(\zeta)\,ds\right),$$

$$f''(z) = \exp\left(-\int_\Gamma P_2(z,\zeta) \ln^- \rho(\zeta)\,ds\right).$$

Then, in view of the construction of the kernels P_2 and P_2' (see Theorem 2 and the following remark), we obtain that $\ln|f'|$ and $\ln|f''|$ are representable in G by the Green–Stieltjes integrals of measures whose singular parts are non-positive. This (see [13, Theorem 1.5]) implies that $f', f'' \in N^+(G)$ and, therefore, $f \in N^+(G)$. Also, $\ln|f| = \ln|f'| + \ln|f''| = \ln^+ \rho - \ln^- \rho = \ln \rho$ a.e. on Γ. If $\rho(\zeta)$ also satisfies (6) or is bounded, then using the results from [13, Theorem 4.3] we obtain that $f \in H^p(G)$ or $f \in H^\infty(G)$. The proof is complete. □

REMARK. The problem of existence of a function in a given class with a given modulus of its boundary values has been investigated by H. Grunsky in [7] and by S. Ya. Khavinson and G. C. Tumarkin in [15]. The functions they constructed could, in general, have $\le (n-1)$ zeroes in G. Also, in their construction they were using the methods of the theory of extremal problems. Required functions appeared as solutions of extremal problems. So, in [7] and in [15] there was not a direct construction of such functions. On the other hand the logarithm of the modulus of functions constructed in [7] and [15] is representable by a Green-Stieltjes integral with an absolutely continuous measure. Functions constructed in Theorem 4 fail to have this property. At the same time they do not vanish in G. So, one can say that Theorem 4 and results in [7] and [15] supplement each other.

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