EIGENVALUE ESTIMATES FOR CERTAIN NONCOMPACT MANIFOLDS

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1. Introduction. Suppose that X is a complete Riemannian manifold. The Laplacian Δ of X is essentially self adjoint on the space of smooth compactly supported functions. This means that Δ has a unique self adjoint extension to L^2X. In general, Δ may have both point and continuous spectrum. We say that ω is an eigenvalue of Δ if there exists a square integrable $\phi \in L^2X$ with $\Delta\phi = \omega\phi$. The symbol $N(\lambda)$ will denote the number of eigenvalues of Δ, which are less than λ.

For a general noncompact manifold X, standard techniques do not yield any estimate of $N(\lambda)$. In particular, the usual Neumann comparison only applies if λ is below the essential spectrum of Δ. The presence of continuous spectrum also causes serious difficulties in applying the heat kernel method.

In this paper, we study two specific classes of noncompact Riemannian manifolds. These are the manifolds with cylindrical ends, and the manifolds whose ends are isometric to the ends in locally symmetric spaces of rational rank one. Using the explicit metric structure on the ends of these manifolds, we employ a modified Neumann comparison to estimate $N(\lambda)$. The main result is the following.

THEOREM 1.1. *Suppose that X is a complete Riemannian manifold having a finite number of ends. Moreover, assume that either* (i) *each end is cylindrical or* (ii) *each end is isometric to an end in a locally symmetric space of Q-rank* 1. *Then $N(\lambda)$ has at most polynomial growth in λ.*

If $X = K\backslash G/\Gamma$ is a locally symmetric space of Q-rank one, then it is also interesting to consider the Casimir operator acting on a non-trivial K-type. Our method extends easily to prove the following.

COROLLARY 1.2. *Let X be a locally symmetric space of Q-rank one. Suppose that $N(\lambda)$ is the number of eigenvalues of the Casimir operator, belonging to a fixed K-type, which are less than λ. Then $N(\lambda)$ has at most polynomial growth in λ.*

Part (ii) of Theorem 1.1 resolves the trace class dilemma, for Q-rank one, as formulated by Borel and Garland [4] and Osborne and Warner [11, 12]. These authors proved that $N(\lambda)$ is finite for fixed λ. That is, there are no accumulation points of the set of eigenvalues. However, their method, which is based on the theory of Eisenstein systems, only yielded a growth estimate when X is a locally symmetric space of real rank one. The importance and applications of our bound on $N(\lambda)$ are clearly described in [11] and [12].

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2. Neumann comparisons. We begin by summarizing some known techniques in the spectral analysis of differential operators. For further details, the reader may consult [6], [8], and [13].

Let X be a complete Riemannian manifold. We suppose that X has finitely many ends. Thus $X = W \cup_i X_i$, where W is a compact manifold with boundary and X_i, $i = 1, \ldots, k$, are the ends of X. Let ∂W_i, $i = 1, \ldots, k$, be the components of the boundary of W, denoted ∂W. One assumes that each X_i is attached to W along $\partial W_i = \partial X_i$.

The Laplacian Δ of X is a second order differential operator defined on smooth compactly supported functions. Since X is complete, Δ extends uniquely to an unbounded self adjoint operator acting on $L^2 X$. This self adjoint operator will also be denoted by Δ. We need to consider another self adjoint operator $\bar{\Delta}$ which is the Laplacian acting on $L^2 W \oplus_i L^2 X_i$, with Neumann boundary conditions on each component of ∂W and each ∂X_i, $i = 1, \ldots, k$. Clearly, $L^2 X$ and $L^2 W \oplus_i L^2 X_i$ are isomorphic Hilbert spaces. However, Δ and $\bar{\Delta}$ have different domains of definition. Thus, these two operators need not have the same spectrum.

Recall that the essential spectrum of a self adjoint operator, acting on a given Hilbert space, consists of cluster points of its spectrum and eigenvalues of infinite multiplicity. Equivalently, the complement of the essential spectrum is the set of isolated eigenvalues having finite multiplicity. A special case of the decomposition principle of [6] and [8] is the following.

PROPOSITION 2.1 (Decomposition Principle). *The operators Δ and $\bar{\Delta}$ have the same essential spectrum.*

We now turn to the Neumann comparisons. Suppose that γ is a positive lower bound on the essential spectrum of $\bar{\Delta}$. If $\lambda < \gamma$, let $\bar{N}(\lambda)$ be the number of eigenvalues of $\bar{\Delta}$, counted to multiplicity, which are less than λ. Clearly, $\bar{N}(\lambda)$ is finite, since $\lambda < \gamma$. Similarly, let $N(\lambda)$ be the number of eigenvalues of Δ, which are less than λ. A special case of the Neumann comparison of [8] and [13] is the following.

PROPOSITION 2.2 (Neumann Comparison). *Suppose that γ is a positive lower bound on the essential spectrum of $\bar{\Delta}$. If $\lambda < \gamma$, then one has $N(\lambda) \leq \bar{N}(\lambda)$.*

For general manifolds with finitely many ends, we cannot achieve a significant modification or improvement of Proposition 2.2. However, the ends of the manifolds studied in this paper have specific additional geometric structure. Given a real number $t \geq 0$, we will formulate certain conditions $\mathcal{P}_t$ defined by the vanishing of a finite number of integrals. These conditions are $\bar{\Delta}$-invariant and so they determine a $\bar{\Delta}$-invariant subspace H_t of $L^2 W \oplus_i L^2 X_i$. Let $\gamma(t)$ be a positive lower bound for the essential spectrum of $\bar{\Delta}$ in H_t. If $\lambda < \gamma(t)$, let $\bar{N}_t(\lambda)$ be the number of eigenvalues of $\bar{\Delta}$, corresponding to eigenfunctions which satisfy the conditions $\mathcal{P}_t$, and are less than λ. Similarly, let $N_t(\lambda)$ be the number of eigenvalues of Δ which correspond to eigenfunctions satisfying the conditions $\mathcal{P}_t$, and are less than λ. The Neumann comparison of [8] and [13], in this context is the following.

PROPOSITION 2.3 (Modified Neumann Comparison). *Suppose that $\gamma(t)$ is a positive lower bound for the essential spectrum of $\bar{\Delta}$ acting in H_t. If $\lambda < \gamma(t)$, then one has $N_t(\lambda) \leq \bar{N}_t(\lambda)$.*

Modified Neumann comparisons in similar geometric settings were used in [5], [7], and [9].

3. Manifolds with cylindrical ends. Let W^m be a compact Riemannian manifold with boundary. Here m is the dimension of W. We suppose that the metric of W is a product near each component Z_i, $i=1,\dots,k$ of the boundary. One may attach a semi-infinite cylinder $\mathbf{R}^+ \times Z_i$ to each boundary component, where $\mathbf{R}^+$ is the positive real line. The union $X = W \cup_i \mathbf{R}^+ \times Z_i$ is called a manifold with cylindrical ends. The spectral theory of these manifolds played an important role in [1].

Suppose that Δ and $\bar{\Delta}$ are as in Section 2. Here $X_i = \mathbf{R}^+ \times Z_i$. The following proposition is readily verified.

PROPOSITION 3.1. *The essential spectrum of Δ, acting in L^2X, is the entire half line* $[0,\infty)$.

Proof. The decomposition principle, Proposition 2.1, states that Δ and $\bar{\Delta}$ have the same essential spectrum. Since W is compact, $\bar{\Delta}$ has no essential spectrum in L^2W ([10]). Thus Δ has the same essential spectrum as $\bar{\Delta}$ acting on the disjoint union $L^2(\mathbf{R}^+ \times Z_i)$, $i=1,\dots,k$, with Neumann boundary conditions on each $0 \times Z_i$. The result now follows from an elementary computation using the product metric structure of $\mathbf{R}^+ \times Z_i$. □

The Neumann comparison of Proposition 2.2 only applies below the essential spectrum. Thus, Proposition 3.1 indicates some difficulty in studying the point spectrum of Δ. We will therefore formulate a modified Neumann comparison and use Proposition 2.3.

Let $\phi \in L^2X$ be an eigenfunction of Δ, with eigenvalue ω. Our basic construction is motivated by considering the restriction of ϕ to each end of X. Suppose that $\psi_{i,j}$ are the eigenfunctions and $\mu_{i,j}$ are the eigenvalues of the Laplacian on the compact manifold Z_i.

One may expand ϕ, on each end $\mathbf{R}^+ \times Z_i$, in a series:

$$\phi(r,z) = \sum_{j=1}^{\infty} a_j(r)\psi_{i,j}(z).$$

Since $\Delta\phi = \omega\phi$, the $a_j(r)$ must satisfy the ordinary differential equations

$$\frac{-d^2}{dr^2}a_j + (\mu_{i,j} - \omega)a_j = 0.$$

Now $a_j(r) \in L^2(\mathbf{R}^+)$, and this forces $a_j = 0$ for $\mu_{i,j} \leq \omega$. Thus, we may write

$$\phi(r,z) = \sum_{\mu_{i,j} > \omega} a_j(r)\psi_{i,j}(z) \tag{3.2}$$

for $(r,z) \in \mathbf{R}^+ \times Z_i$.

Equation (3.2) suggests the formulation of certain integral conditions $\mathcal{P}_t$, for any $t \geq 0$. Suppose that $f \in L^2X = L^2W \oplus_i L^2(\mathbf{R}^+ \times Z_i)$ is written as $f = f_W \oplus_i f_i$. We say that f satisfies the conditions $\mathcal{P}_t$ if the following integrals vanish.

$$\int_{Z_i} f_i(r,z)\psi_{i,j}(z)\,d\mathrm{vol}(z) = 0, \quad \text{if } \mu_{i,j} \leq t. \tag{3.3}$$

A priori, these conditions are only defined for smooth functions. However, they extend by continuity to all of L^2X.

Consider the Laplacian $\bar{\Delta}$ acting on $L^2W \oplus_i L^2(\mathbf{R}^+ \times Z_i)$ with Neumann boundary conditions. The additional side conditions define a closed subspace H_t of $L^2W \oplus_i L^2(\mathbf{R}^+ \times Z_i)$. Moreover, H_t is $\bar{\Delta}$ invariant.

Let $\mu(t)$ be the smallest eigenvalue $\mu_{i,j}$ which is greater than t. That is, $\mu(t) = \min_i \min_j\{\mu_{i,j} \mid \mu_{i,j} > t\}$. One has the following.

LEMMA 3.4. *The essential spectrum of $\bar{\Delta}$, acting in the space H_t, is the half line $[\mu(t), \infty)$. Moreover, all eigenfunctions of $\bar{\Delta}$ are contained in L^2W.*

Proof. Since W is compact, the Laplacian Δ has pure point spectrum in L^2W, when Neumann boundary conditions are imposed ([10]). The spectrum on each end $\mathbf{R}^+ \times Z_i$ is computed using the product metric structure. The conclusions of Lemma 3.4 follow directly. □

We now use the modified Neumann comparison of Proposition 2.3. This leads to the following.

LEMMA 3.5. *Let $M(\lambda, t)$ be the number of eigenvalues of Δ, acting in L^2X, which are less than λ and contained in the interval $[t, \mu(t))$. For $\lambda > 0$, one has $M(\lambda, t) \leq C_1 \lambda^{m/2}$. Here C_1 is independent of t.*

Proof. Let $N_t(\lambda)$ be the number of eigenvalues of Δ, acting in L^2X, which are less than λ and correspond to eigenfunctions satisfying the conditions $\mathcal{P}_t$. The equation (3.2) shows that $M(\lambda, t) \leq N_t(\lambda)$. That is, any eigenfunction corresponding to an eigenvalue $\omega \in [t, \mu(t)]$ must satisfy the conditions $\mathcal{P}_t$ of (3.3). By Proposition 2.3 and Lemma 3.4, we know that $N_t(\lambda) \leq \bar{N}_t(\lambda)$. Here $\bar{N}_t(\lambda)$ is the number of eigenvalues of $\bar{\Delta}$, acting in H_t, which are less than λ. Since all eigenfunctions of $\bar{\Delta}$ are contained in L^2W, we have $\bar{N}_t(\lambda) \leq C_1 \lambda^{m/2}$ by the standard asymptotic formula [10] for the compact manifold W. In summary, $M(\lambda, t) \leq N_t(\lambda) \leq \bar{N}_t(\lambda) \leq C_1 \lambda^{m/2}$. This proves Lemma 3.5. □

The main result of this section is the following.

THEOREM 3.6. *Let X^m be an m-dimensional Riemannian manifold with cylindrical ends. Suppose that $N(\lambda)$ is the number of eigenvalues of Δ, acting in L^2X, which are less than λ. One has $N(\lambda) \leq C_2 \lambda^{m-1/2}$, for $\lambda > 0$.*

Proof. Choose an increasing sequence t_l so that the intervals $[t_l, \mu(t_l))$ cover the half line $[0, \infty)$. The standard asymptotic formula [10] for the compact manifolds Z_i allows us to require that $C_3 l^{2/(m-1)} \leq t_l < \mu(t_l) \leq C_4 l^{2/(m-1)}$, for some constants C_3 and C_4. The result now follows from Lemma 3.5 and an elementary counting argument. The number of intervals $[t_l, \mu(t_l))$ which intersect $[0, \lambda)$ is bounded by a constant multiple of $\lambda^{(m-1)/2}$. Lemma 3.5 guarantees that the number of eigenvalues in any single interval $[t_l, \mu(t_l))$ is bounded by a constant multiple of $\lambda^{m/2}$. Thus $N(\lambda) \leq C_2 \lambda^{m/2} \lambda^{(m-1)/2}$, which proves Theorem 3.6. □

4. Symmetric spaces of rational rank one. Let $X = K\backslash G/\Gamma$ be a locally symmetric space of Q-rank one. Here G is a semisimple Lie group and Γ is an arithmetic subgroup. Alternatively, we may assume that G is reductive, Γ is a rank one lattice in G, and the pair (G, Γ) satisfies the conditions of [12, pp. 62–63]. It is a basic fact [2] that X is diffeomorphic to the interior of a compact manifold with boundary. The finitely many ends of X are called the cusps. To prove the polynomial growth of $N(\lambda)$, we will employ an argument similar to the method used for cylindrical manifolds. In particular, we slice along the individual cusps and define suitable Neumann comparisons.

More generally, we need only assume that X has finitely many ends and that each end is isometric to a cusp in a locally symmetric space of rational rank one. The cusps need not be isometric to each other and may come from different pairs (G, Γ). We prove everything in this greater generality to indicate the basic structure of our method. The required assumptions about X will now be precisely described. Suppose that $X = W \cup_i X_i$, where W is a compact manifold with boundary and X_i, $i = 1, \ldots, k$, are the ends of X. Each X_i is required to be isometric to a cusp in some locally symmetric space of rational rank one, which may depend upon i. Let ∂W_i, $i = 1, \ldots, k$, be the components of ∂W, the boundary of W. One assumes that each X_i is attached to W along $\partial W_i = \partial X_i$.

Each cusp X_i is diffeomorphic to $\mathbf{R}^+ \times \partial W_i$. The metric along the cusp is explicitly described in [3, p. 247]. Specifically, there is a fiber bundle

$$\begin{array}{ccc} N_i/\Gamma_i \cap N_i & \rightarrow & \mathbf{R}^+ \times \partial W_i \\ & & \downarrow \\ & & \mathbf{R}^+ \times Z_i . \end{array}$$

Here Z_i and $N_i/\Gamma_i \cap N_i$ are compact manifolds. Associated with this fibering, one has a local decomposition of the metric

$$ds^2 = dr^2 + dz^2 + e^{-2b_1 r}\, dn_1^2(z) + e^{-4b_1 r}\, dn_2^2(z). \tag{4.1}$$

The constant $b_1 > 0$ may depend upon i. Of course, both $dn_1^2(z)$ and $dn_2^2(z)$ are supported along the fiber $N_i/\Gamma_i \cap N_i$. The volume element is given by

$$d\mathrm{vol} = e^{-b_2 r}\, dr\, d\mathrm{vol}(z)\, d\mathrm{vol}(n). \tag{4.2}$$

Again, the constant b_2 may depend upon i. The symbols $d\mathrm{vol}(z)$ and $d\mathrm{vol}(n)$ represent r-independent volume elements along the base and fiber. Note that the volume element depends on z only from the term $d\mathrm{vol}(z)$. Since $d\mathrm{vol}(n)$ is independent of z, the volume element, $d\mathrm{vol}$, has a simpler z-dependence than the metric ds^2.

Let Δ be the Laplacian of X and $\Delta_{Z,i}$ the Laplacian of Z_i with its induced metric. We record two elementary lemmas for future reference.

LEMMA 4.3. *Suppose that $h(r,z)$ is a function, on the cusp, $\mathbf{R}^+ \times \partial W_i$, which depends only upon r and z. Then one has*

$$\Delta h = -e^{b_2 r} \frac{\partial}{\partial r}\left(e^{-b_2 r} \frac{\partial h}{\partial r}\right) + \Delta_{Z,i} h.$$

Proof. If d is the exterior derivative, with adjoint d^*, then, for the Laplacian on functions of any Riemannian manifold, one may write $\Delta = d^* d$. The lemma follows from this general fact and the explicit form of the metric (4.1) and the volume element (4.2). □

Consider the cylinder $\mathbf{R}^+ \times Z_i$ with the standard product metric $dr^2 + dz^2$. Its Laplacian is

$$\Delta_{\mathbf{R}^+ \times Z, i} = -\frac{d^2}{dr^2} + \Delta_{Z,i}.$$

A useful consequence of Lemma 4.3 is the following.

LEMMA 4.4. *Suppose that $h(r,z)$ satisfies $\Delta h = \omega h$. Set $g(r,z) = e^{-b_2 r/2} h(r,z)$. Then*

$$\Delta_{\mathbf{R}^+ \times Z, i}\, g = (\omega + b_{3,i}) g.$$

Here $b_{3,i} < 0$ is independent of ω. If $h \in L^2(\mathbf{R}^+ \times \partial W_i)$, with the L^2 structure induced by the metric (4.1), *then $g \in L^2(\mathbf{R}^+ \times Z_i)$ with the L^2 structure subordinate to the standard product metric.*

Proof. This follows by an elementary calculation using Lemma 4.3. □

Suppose that $\phi \in L^2 X$ is an eigenfunction of Δ with eigenvalue ω. For $x \in \mathbf{R}^+ \times \partial W_i$, one may define

(4.5) $$T_i \phi(x) = \int_{N_i/\Gamma_i \cap N_i} \phi(xn)\, d\mathrm{vol}(n).$$

The Haar measure on N_i may be normalized so that $\mathrm{Vol}(N_i/\Gamma_i \cap N_i) = 1$. Clearly $T_i \phi$ depends only upon r and z. Moreover, since N_i acts isometrically, one has $\Delta(T_i \phi) = \omega T_i \phi$. Now set $h_i = T_i \phi$ and $g_i = e^{-b_2 r/2} h_i$. Applying Lemma 4.4, we find that $\Delta_{\mathbf{R}^+ \times Z, i}\, g_i = (\omega + b_{3,i}) g_i$.

Let $\psi_{i,j}$ be the eigenfunctions and $\mu_{i,j}$, $j = 1, 2, \ldots$, the corresponding eigenvalues of the Laplacian $\Delta_{Z,i}$ on the compact manifold Z_i. One may expand the function g_i in a series:

$$g_i(r,z) = \sum_{j=1}^{\infty} a_j(r) \psi_{i,j}(z).$$

Applying the argument used for cylindrical manifolds, in Section 3, we see that $a_j(r) = 0$ when $\mu_{i,j} \le \omega + b_{3,i}$. Thus one may write

(4.6) $$g_i(r,z) = \sum_{\mu_{i,j} > \omega + b_{3,i}} a_j(r) \psi_{i,j}(z)$$

for $(r,z) \in \mathbf{R}^+ \times Z_i$.

Following the treatment of the cylindrical manifolds, we formulate integral conditions $\mathcal{P}_t$, for $t \geq 0$. Suppose that $f \in L^2 X = L^2 W \oplus_i L^2(\mathbf{R}^+ \times \partial W_i)$ decomposes as $f = f_W \oplus_i f_i$. We say that f satisfies the conditions $\mathcal{P}_t$ if one has

$$\int_{Z_i} \int_{N_i/\Gamma_i \cap N_i} f_i(r, z, n) \psi_{i,j}(z) \, d\mathrm{vol}(n) \, d\mathrm{vol}(z) = 0 \tag{4.7}$$

for $\mu_{i,j} \leq t + b_{3,i}$. Consider the Laplacian $\bar{\Delta}$ acting on $L^2 W \oplus_i L^2(\mathbf{R}^+ \times Z_i)$ with Neumann boundary conditions. The additional side conditions (4.7) define a closed subspace H_t of $L^2 W \oplus_i L^2(\mathbf{R}^+ \times Z_i)$. Moreover, H_t is $\bar{\Delta}$ invariant.

There is a particularly interesting subspace of H_t. Suppose that $f = f_W \oplus_i f_i$, as above. One says that f is cuspidal if $T_i f_i = 0$, for all i. Here T_i is defined by the formula (4.5). It is proved in [5] that $\bar{\Delta}$ has pure point spectrum when restricted to the closed subspace spanned by the cuspidal functions. Note that a cuspidal function satisfies (4.7), for any t.

Let S_t be the set defined by $S_t = \{\mu_{i,j} \mid \mu_{i,j} > t + b_{3,i}\}$. Suppose that $\mu_l(t) \in S_t$ is an eigenvalue, on some particular Z_l, satisfying

$$\mu_l(t) - b_{3,l} = \min_{i,j} \{\mu_{i,j} - b_{3,i} \mid \mu_{i,j} \in S_t\}.$$

Clearly, such a $\mu_l(t)$ exists and one has $\mu_l(t) - b_{3,l} > t$.

Given these preliminaries, we may derive the following.

LEMMA 4.8. *The essential spectrum of* $\bar{\Delta}$, *restricted to* H_t, *is the half line* $[\mu_l(t) - b_{3,l}, \infty)$.

Proof. If $f = f_W \oplus_i f_i$, then define $\mathcal{O}_i f = e^{-b_2 r/2} T_i f_i$. Clearly $\mathcal{O}_i : L^2 X \to L^2(\mathbf{R}^+ \times Z_i)$ is a bounded linear map. Here $L^2(\mathbf{R}^+ \times Z_i)$ has the L^2 structure subordinate to the standard product metric. Let $\mathcal{K}$ be the intersection of the kernels of the $\mathcal{O}_i$. That is, $\mathcal{K}$ is the space of cuspidal functions. According to [5], $\bar{\Delta}$ has pure point spectrum in $\mathcal{K}$. If f is in the orthocomplement of $\mathcal{K}$, then $f_W = 0$ and $T_i f_i = f_i$ for all i. Thus $\{\mathcal{O}_i\}$ defines an isometric isomorphism from the orthocomplement of $\mathcal{K}$ onto $\oplus_i L^2(\mathbf{R}^+ \times Z_i)$. This intertwines the respective Laplacians, after shifting by the constant $b_{3,i}$, according to Lemma 4.4. Therefore, one is reduced to computing the essential spectrum for the product Laplacian on each $\mathbf{R}^+ \times Z_i$, with the induced conditions. That is, a function $g(r, z)$ must satisfy the boundary conditions $(\partial g/\partial r)(r, 0) = (-b_2/2) g(r, 0)$, on $0 \times Z_i$, and side conditions compatible with (4.7), in order for $g(r, z)$ to be in the domain of the product Laplacian. As observed in the study of cylindrical manifolds, the remaining details are elementary. □

Also, one has the following.

LEMMA 4.9. *Let* $\bar{N}(\lambda)$ *be the number of eigenvalues of* $\bar{\Delta}$, *acting on* $L^2 W \oplus_i L^2(\mathbf{R}^+ \times Z_i)$ *which are less than* λ. *Suppose that* m *is the dimension of* X. *For* $\lambda > 0$, *one has* $\bar{N}(\lambda) \leq C_1 \lambda^{m/2}$.

Proof. We use the notation of the proof of Lemma 4.8. It was proved in [5] that the number of cuspidal eigenfunctions, with eigenvalues less than λ, is bounded by a constant multiple of $\lambda^{m/2}$. Since $\{\Theta_i\}$ defines an isometric isomorphism from the orthocomplement of $\mathcal{K}$ onto $\oplus_i L^2(\mathbf{R}^+ \times Z_i)$, we are reduced to an elementary computation. Consider the Laplacian of the product metric on $\mathbf{R}^+ \times Z_i$, with the induced boundary conditions $(\partial g/\partial r)(0,z) = (-b_2/2)g(0,z)$, on $0 \times Z_i$. One readily verifies that its eigenfunctions are in the space spanned by functions of the form $e^{-b_2 r/2}\psi_{i,j}(z)$. Since $\dim Z_i < m$, for all i, the result now follows from the standard asymptotic formula [10] on the compact manifolds Z_i. □

By analogy with Proposition 3.5, one may formulate the following.

LEMMA 4.10. *Let $M(\lambda, t)$ be the number of eigenvalues of Δ, acting in L^2X, which are less than λ and contained in the interval $[t, \mu_l(t) - b_{3,l})$. Then, one has $M(\lambda, t) \le C_1 \lambda^{m/2}$, for $\lambda > 0$.*

Proof. If $\phi \in L^2X$ is an eigenfunction of Δ, with eigenvalue $\omega \in [t, \mu_l(t) - b_{3,l})$, then (4.6) shows that ϕ satisfies the conditions $\mathcal{P}_t$ of (4.7). Thus $M(\lambda, t) \le N_t(\lambda)$ in the notation of Proposition 2.3. In other words, every eigenfunction of Δ, with eigenvalue in the interval $[t, \mu_l(t) - b_{3,l})$, must satisfy the conditions $\mathcal{P}_t$. Using Lemma 4.8 and Proposition 2.3, we obtain $N_t(\lambda) \le \bar{N}_t(\lambda)$. However, one clearly has $N_t(\lambda) \le \bar{N}(\lambda)$, since H_t is a subspace of $L^2(W) \oplus_i L^2(\mathbf{R}^+ \times \partial W_i)$. Lemma 4.9 gives $\bar{N}(\lambda) \le C_1 \lambda^{m/2}$. In summary, $M(\lambda, t) \le N_t(\lambda) \le \bar{N}_t(\lambda) \le \bar{N}(\lambda) \le C_1 \lambda^{m/2}$. This proves Lemma 4.10. □

Let p_i be the dimension of Z_i and $p = \max_i p_i$. The main result of this section is the following.

THEOREM 4.11. *Suppose that X has finitely many ends and each end is isometric to the end of a locally symmetric space of rational rank one. Then $N(\lambda) \le C_2 \lambda^{(m+p)/2}$.*

Proof. One follows the proof of Theorem 3.6, but uses Lemma 4.10 instead of Lemma 3.5. □

This proves Theorem 1.1, part (ii). Note that Theorem 4.11 is well-known for symmetric spaces of real rank one, ([7, 11]). In the real rank one case, we have $p = 0$.

To prove Corollary 1.2, one needs to study the Laplacian acting on sections of equivariant vector bundles, associated to representations of K, by a standard reformulation ([4, 5, 11]). Our methods carry over to this vector bundle context without requiring any significant modification.

ADDED IN PROOF. R. P. Langlands has recently obtained another proof of the trace class property, for locally symmetric spaces of Q-rank one. His method, which relies upon earlier work of J. Arthur, is substantially different from ours.

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