# CYCLIC VECTORS OF BOUNDED CHARACTERISTIC IN BERGMAN SPACES

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**1. Introduction.** Let $D$ be the unit disk in the complex plane. $H^p$ denotes the usual class of functions analytic on $D$. Let $A^2$ be the Bergman space of analytic functions $f$ such that

$$\|f\|_{A^2}^2=\int_0^{2\pi}\int_0^1 |f(re^{i\theta})|^2 r\,dr\,d\theta<\infty.$$

If $f\in A^2$, let $[f]$ denote the smallest closed subspace of $A^2$ which contains $\{z^n f\}_{n=0}^{\infty}$. If $S$ is the unilateral shift $Sf=zf$, then $[f]$ is the smallest closed subspace of $A^2$ containing $f$ which is invariant under $S$.

A function $f\in A^2$ is called *cyclic* if $[f]=A^2$. It is an open problem to give a useful characterization of the cyclic functions in $A^2$. On the other hand, if $f\in H^1$ much more is known.

Recall that any function $f\in H^1$ has the canonical factorization $f=BsF$, where $B$ is a Blaschke product, $F$ is an outer function and $s$ is a singular inner function generated by a singular measure $\sigma$:

$$s(z)=s_\sigma(z)=\exp\left(-\int_T \frac{\zeta+z}{\zeta-z}\,d\sigma(\zeta)\right).$$

Here, $\sigma$ is singular with respect to Lebesgue measure on the unit circle $T$.

A closed set $K\subseteq T$ of Lebesgue measure 0 is called a *BCH set* (Beurling-Carleson-Hayman) if

$$\int_T \log\frac{1}{\rho_K(\zeta)}\,|d\zeta|<\infty,$$

where $\rho_K(\zeta)=\inf_{z\in K}|z-\zeta|$. Equivalently, $K$ is a BCH set if (i) $|K|=0$, and (ii) $\sum |I_k|\log(1/|I_k|)<\infty$, where $|E|$ denotes the Lebesgue measure of $E$ and $T\backslash K=\bigcup_{k=1}^{\infty} I_k$ is the canonical decomposition of $T\backslash K$ into disjoint open arcs.

We can now state the following theorem.

THEOREM I. *If $f$ is in $H^1$ and non-vanishing, then $f$ is cyclic for $A^2$ if and only the singular factor of $f$ equals $s_\sigma$ where $\sigma(K)=0$ for all BCH sets $K$.*

The necessity is due to H. S. Shapiro [13]. The sufficiency was established independently by Korenblum [9, 10] and Roberts [14].

Actually the authors above proved necessity and sufficiency in case $f=s_\sigma$. The extension to $H^1$ was noted in [4]. We sketch a proof. If $f\in H^1$, then $\|f\|_{A^2}\leqslant$

Received September 8, 1983. Final revision received February 21, 1984.

The first author was supported in part by a grant from Wayne State University. The second and third authors were supported in part by the National Science Foundation.

Michigan Math. J. 31 (1984).

$c\|f\|_{H^1}$; see [13, p. 325]. If $f=sF$ then $[f]=[s]$. This is true since for $F$ outer there are polynomials $p_n$ such that $Fp_n \to 1$ in $H^1$. Thus $fp_n \to s$ in $H^1$ and hence in $A^2$. Thus $s \in [f]$ and $[s] \subseteq [f]$. The other containment is obvious.

In this note we find a necessary condition for $f$ to be cyclic which applies to all functions $f$ in $A^2$ which generalizes the Shapiro condition. This condition turns out to be sufficient provided $[f] \cap N$ contains a non-vanishing function. Here, $N$ is the Nevanlinna class. In particular, we characterize cyclic vectors $f$ which lie in both $A^2$ and $N$. As a corollary we obtain the following result.

THEOREM. *Let $f$ be a non-vanishing function in $A^2 \cap N$. Let the canonical factorization of $f$ be $f=(s_1/s_2)F$, where $F$ is outer and $s_1$ and $s_2$ are singular inner functions. Then $s_2$ is always cyclic and $f$ is cyclic if and only if $s_1$ is cyclic.*

COROLLARY. *If $f \in A^2 \cap N$ and $f^{-1} \in A^2$, then $f$ is cyclic.*

This answers a question of Aharonov, H. S. Shapiro, and Shields [1]. It is still unknown if the corollary is true for an arbitrary $f$ invertible in $A^2$.

This paper is divided into five sections. In the second section we prove a theorem which may be of independent interest about the restriction of singular functions to subdomains of $D$. In the third section we give a necessary condition for cyclicity. In section four we show that the condition of the third section is sufficient under additional hypotheses. We then prove the results stated in the introduction. In section five we indicate how the results of the previous sections extend to the familiar weighted Bergman spaces $A^{2,p}$, which we define in that section.

We adopt the convention of using the letter $c$ to denote a constant which may change its value each time it appears. The symbol $\bar{D}$ denotes the closed unit disc. A positive measure $\sigma$ is said to *live on a set $X$* if $\|\sigma\|=\sigma(X)$.

**2. Restrictions of singular inner functions.** In this section we prove a theorem which will be necessary to obtain the main results. Essentially, we answer the following question: If $\Omega$ is a subregion of $D$ and $s$ is a singular inner function, when does the restriction of $s$ to $\Omega$ have a singular factor? For a related result see [7, Proposition 1].

Let $\Omega \subseteq D$ be a simply connected domain whose boundary is a Jordan curve. Let $\varphi: D \to \Omega$ be a Riemann map of the disc onto $\Omega$ and let $\psi$ be its inverse. If $\zeta \in T$ and $|\varphi(\zeta)|=1$, then define the *angular derivative* of $\varphi$ at $\zeta$ by $\varphi'(\zeta)=\lim_{r\to 1}\varphi'(r\zeta)$ if the limit exists, and $\infty$ otherwise. It is well known that

$$|\varphi'(\zeta)|=\lim_{r\to 1}\frac{1-|\varphi(r\zeta)|}{1-r}$$

and that the latter limit exists; in fact $0<|\varphi'(\zeta)|\leqslant\infty$. See [3, pp. 7–12] and [2] for details.

Now let $e^{i\theta}\in T$. Then

$$\frac{1-|\varphi(z)|^2}{|e^{i\theta}-\varphi(z)|^2}=\int_T\frac{1-|z|^2}{|1-\bar{\eta}z|^2}\,d\mu_\theta(\eta)$$

for a positive measure $d\mu_\theta$, since the left hand side is a positive harmonic function. Let $d\mu_\theta = d\nu_\theta + d\sigma_\theta$ be the Lebesgue decomposition of $d\mu_\theta$, where $\nu_\theta$ is singular and $\sigma_\theta$ is absolutely continuous with respect to $d\theta$.

LEMMA 2.1. *Suppose $e^{i\theta} \notin \varphi(T)$. Then $d\mu_\theta = d\sigma_\theta$.*

*Proof.* Let $\zeta \in T$ and set $z = (1-\epsilon)\zeta$. Then

$$\frac{1-|\varphi(z)|^2}{|e^{i\theta}-\varphi(z)|^2} = \int_T \frac{1-|z|^2}{|1-\bar{\eta}z|^2}\, d\mu_\theta \geqslant \frac{c}{\epsilon}\mu_\theta(I(\zeta,\epsilon)),$$

where $I(\zeta,\epsilon)$ is the arc on $T$ centered at $\zeta$ of length $2\epsilon$. Since $\varphi(\zeta) \neq e^{i\theta}$,

$$\infty > \frac{1-|\varphi(\zeta)|^2}{|e^{i\theta}-\varphi(\zeta)|^2} \geqslant c \cdot \varlimsup_{\epsilon\to 0} \frac{\mu_\theta(I(\zeta,\epsilon))}{\epsilon}.$$

Since this is true for any $\zeta \in T$ it follows that the "lower cubical derivative" of $\mu_\theta$ is finite for each $\zeta \in T$ and that $\mu_\theta \ll d\theta$. See [11, Theorem 8.10, p. 169] for details.

Now suppose $e^{i\theta} = \varphi(e^{ix})$. According to [3, p. 11],

$$\frac{1-|\varphi(z)|^2}{|e^{i\theta}-\varphi(z)|^2} = \frac{1-|z|^2}{|e^{ix}-z|^2}\frac{1}{|\varphi'(e^{ix})|} + \int_T \frac{1-|z|^2}{|1-\bar{\eta}z|^2}\, d\tau_\theta(\eta),$$

where $\tau_\theta(\{e^{ix}\}) = 0$. □

LEMMA 2.2. *The measure $\tau_\theta$ is absolutely continuous and equals $\sigma_\theta$.*

*Proof.* Let $e^{iy} \in T$ and suppose $e^{iy} \neq e^{ix}$. We repeat the argument of Lemma 2.1: For $z = (1-\epsilon)e^{iy}$,

$$\frac{1-|\varphi(z)|^2}{|e^{i\theta}-\varphi(z)|^2} \geqslant \frac{c}{\epsilon}\tau_\theta(I(e^{iy},\epsilon)).$$

Since $e^{i\theta} \neq \varphi(e^{iy})$, letting $\epsilon \to 0$ shows that

$$\varlimsup_{\epsilon\to 0} \frac{\tau_\theta(I(e^{iy},\epsilon))}{\epsilon} < \infty.$$

Since $\tau_\theta(\{e^{ix}\}) = 0$ it follows that $\tau_\theta \ll d\theta$. □

We are ready to prove the main result of this section. Let $\Gamma = \varphi(T)$ and let $w \in \Gamma$. If $w \in T$ and $\varphi(\zeta) = w$, let

$$|\psi'(w)| \equiv \frac{1}{|\varphi'(\zeta)|}.$$

Let $\Gamma_0$ be the subset of $\Gamma \cap T$ on which $|\psi'(w)| > 0$; that is, $|\varphi'(\zeta)| < \infty$.

THEOREM 2.1. *Suppose $s$ is a singular inner function generated by the measure $\mu$. Then $s \circ \varphi$ has a singular factor if and only if $\mu(\Gamma_0) > 0$.*

*Proof.* We have

$$-\log|s(\varphi(z))| = \int_T \frac{1-|\varphi(z)|^2}{|1-e^{-i\theta}\varphi(z)|^2}\, d\mu(\theta)$$
$$= \int_{\Gamma\cap T} + \int_{T\setminus(\Gamma\cap T)}.$$

By Lemmas 2.1 and 2.2,

$$-\log|s\circ\varphi(z)| = \int_{\Gamma_0} \frac{1-|z|^2}{|\psi(e^{i\theta})-z|^2}\, |\psi'(e^{i\theta})|\, d\mu(\theta) + \int_T \int_T \frac{1-|z|^2}{|\eta-z|^2}\, d\sigma_\theta(\eta)\, d\mu(\theta).$$

Call the first integral $u(z)$ and the second $v(z)$. We claim that $u(z)$ is the Poisson integral of a singular measure and that $v(z)$ is the Poisson integral of an $L^1$ function. To see the second statement, observe that

$$v(z) = \int_T \frac{1-|z|^2}{|e^{i\theta}-z|^2}\, d\nu(\theta),$$

where $\nu$ is the measure defined by

$$\nu(E) \equiv \int_T \int_E d\sigma_\theta(\eta)\, d\mu(\theta).$$

Lemmas 2.1 and 2.2 imply that $\nu \ll d\theta$, and this proves the second statement.

For the first statement, changing variables yields

$$u(z) = \int_{\varphi^{-1}(\Gamma_0)} \frac{1-|z|^2}{|e^{ix}-z|^2}\, \frac{1}{|\varphi'(e^{ix})|}\, d\hat{\mu}(x),$$

where $\hat{\mu}$ is the measure defined by

$$\hat{\mu}(E) \equiv \mu(\varphi(E)).$$

Since $\mu$ lives on a set of Lebesgue measure 0 and since the inverse image under $\varphi$ of such a set also has measure 0 [15, Theorem VIII.30, p. 322], $\hat{\mu}$ lives on a set of measure 0 and is singular. Since

$$\int_E \frac{1}{|\varphi'(e^{ix})|}\, d\hat{\mu} = \int_{\varphi(E)} |\psi'(e^{i\theta})|\, d\mu(\theta),$$

$u(z)$ is non-trivial if and only if the right hand term is positive for some $E$. This proves the theorem. □

If $\Omega \subseteq D$ is a domain as above then we may define $H^p(\Omega)$ to be all functions $g$ analytic on $\Omega$ such that $g\circ\varphi \in H^p$. Equivalently, $g \in H^p(\Omega)$ if and only if $|g|^p$ has a harmonic majorant on $\Omega$. See [6, Chapter 10].

It is clear that if $g \in H^p$ then $g \in H^p(\Omega)$. If $g = BsF$ is the canonical factorization of $g$ in $H^p$ one can ask about the canonical factorization of $g$ in $H^p(\Omega)$.

THEOREM 2.2. *Let $g = BsF \in H^p$. Then $g$ has a singular factor in $H^p(\Omega)$ if and only if $s$ has a singular factor in $\Omega$.*

*Proof.* It is well known that $F$ is outer if and only if $FH^\infty$ is dense in $H^p$. But if $FH^\infty$ is dense in $H^p$ it follows that $FH^\infty(\Omega)$ is dense in $H^p(\Omega)$. Thus $F$ is outer in $H^p(\Omega)$.

It is also well known that a univalent map has no singular factor. See [6, Theorem 3.17, p. 51]. Since $B(\varphi)$ is the product of univalent maps, $B$ can have no singular factor in $H^p(\Omega)$. This proves the theorem. □

Thus Theorem 2.1 provides a method of determining whether a function $g \in H^p$ has a singular factor when restricted to $\Omega$. We conclude this section with the following corollary of Theorem 2.2.

COROLLARY 2.1. *Suppose $\Omega$ is a simply connected subset of $D$ bounded by the Jordan curve $\Gamma$. Let $s$ be the singular inner function generated by the measure $\mu$. Then if $\mu(\Gamma \cap T) = 0$, $s$ is outer in $H^\infty(\Omega)$.*

**3. A necessary condition for cyclicity.** In this section we generalize Theorem 2 of Shapiro [13].

Let $\mathcal{C}$ denote the collection of all BCH sets. To each $K$ in $\mathcal{C}$ we will associate a region $\Omega_K$ and a function $F_K$. Without loss of generality we assume that the arc length of each interval of $T \backslash K$ has length less than $\pi$. This can be accomplished by adding at most two points to $K$.

Let $G$ be the outer function (depending on $K$) defined by

$$G(z) = \exp\left(-\int_T \log \frac{1}{\rho_K(\zeta)} \frac{\zeta + z}{\zeta - z} |d\zeta|\right).$$

$G$ is well-defined because $\log(1/\rho_K) \in L^1$. It is easy to see that $|G|$ is continuous on $\bar{D}$ and $|G| \equiv 0$ on $K$.

Let $T \backslash K = \bigcup_n (a_n, b_n)$, where the $(a_n, b_n)$ are the open arcs comprising the components of $T \backslash K$. Define $\gamma\colon [0, 2\pi] \to \bar{D}$ by

$$\gamma(\theta) = \begin{cases} \left(1 - \left| \dfrac{(e^{i\theta} - a_n)(e^{i\theta} - b_n)}{a_n - b_n} \right|^2 \right) e^{i\theta}, & e^{i\theta} \in (a_n, b_n) \\ e^{i\theta}, & e^{i\theta} \in K. \end{cases}$$

The following proposition may be verified by a routine argument; we omit the proof.

PROPOSITION. *The curve $\gamma$ is $C^{1+\epsilon}$ for any $\epsilon$, $0 < \epsilon < 1$. In fact, for an absolute constant $c$, $|\gamma'(e^{i\theta}) - \gamma'(e^{it})| \leqslant c|e^{i\theta} - e^{it}|$.*

Since $\gamma$ is a Jordan curve it bounds a simply connected subregion of $D$. Call this subregion $\Omega_K$ and let $\Gamma_K$ denote its boundary.

LEMMA 3.1. *There are constants $A$ and $c$ such that*

$$|G(z)| \leqslant A(1 - |z|)^c \tag{3.1}$$

*for $z \in \Gamma_K \cap D$.*

*Proof.* For $z \in \Gamma_K \cap D$,

$$
\begin{aligned}
|G(z)| &= \exp\left(-\int_T \log \frac{1}{\rho_K(\zeta)} \frac{1-|z|^2}{|\zeta-z|^2}\,|d\zeta|\right) \\
&\leqslant \exp\left(-\int_{J_z} \log \frac{1}{\rho_K(\zeta)} \frac{1-|z|^2}{|\zeta-z|^2}\,|d\zeta|\right),
\end{aligned}
$$

where $J_z$ is the arc on $T$ with center $z/|z|$ and length $1-|z|$. By the definition of $\Gamma_K$, for $\zeta \in J_z$, $\rho_K(\zeta) \leqslant c_2\sqrt{(1-|z|)}$ for an absolute constant $c_2$ independent of $z$. Since

$$
\int_{J_z} \frac{1-|z|^2}{|\zeta-z|^2}\,|d\zeta| \geqslant \delta > 0,
$$

where $\delta$ does not depend on $z$,

$$
\begin{aligned}
|G(z)| &\leqslant \exp\left(-\delta \log \frac{1}{c_2\sqrt{(1-|z|)}}\right) \\
&= \delta_1 \cdot (1-|z|)^{\delta/2},
\end{aligned}
$$

where $\delta_1 = e^{-\delta \log(1/c_2)}$. This proves the lemma. □

Now let $f \in A^2$. It is well known that for $c>0$ independent of $z$,

$$
|f(z)| \leqslant c \frac{1}{1-|z|} \|f\|_{A^2} \tag{3.2}
$$

for all $z \in D$. By taking a high enough power of the $G$ of Lemma 3.1 we may assume that $F_K \equiv G^N$ satisfies

$$
|F_K(z)| \leqslant c(1-|z|) \tag{3.3}
$$

for $z \in \Gamma_K$.

LEMMA 3.2. *Let $f \in A^2$. Then $F_K f \in H^\infty(\Omega_K)$, and for a c independent of f*

$$
\|F_K f\|_{\infty, \Omega_K} \leqslant c \|f\|_{A^2}.
$$

*Proof.* Let $p$ be a polynomial. Then

$$
\sup_{z \in \Gamma_K} |p(z) F_K(z)| \leqslant c \|p\|_{A^2}
$$

by inequalities (3.3) and (3.2). The maximum principle implies that

$$
\|F_K p\|_{\infty, \Omega_K} \equiv \sup_{z \in \Omega_K} |p(z) F(z)| \leqslant c \|p\|_{A^2}.
$$

Since polynomials are dense in $A^2$, and since convergence in $A^2$ implies convergence on compact subsets of $D$, the proof may be completed by standard arguments. □

We now state the promised necessary condition for cyclicity.

DEFINITION. Let $f \in A^2$. Then $f$ is called $\mathcal{C}$-*outer* if $F_K f$ is an outer function in $H^2(\Omega_K)$ for all $K \in \mathcal{C}$.

THEOREM 3.1. *Let f be cyclic for $A^2$. Then f is $\mathcal{C}$-outer.*

*Proof.* Since $[f]=A^2$, we can find polynomials $p_n$ such that

$$\lim_{n\to\infty} \|p_n f-1\|_{A^2}=0.$$

By Lemma 3.2,

$$\|F_K p_n f-F_K\|_{H^2(\Omega_K)}\leq c\|p_n f-1\|_{A^2},$$

and thus the closure in $H^2(\Omega_K)$ of $F_K fH^\infty(\Omega_K)$ contains the outer function $F_K$. By Beurling's theorem, $F_K f$ is also outer. □

Observe now that by combining Theorems 3.1 and 2.1 we can obtain the result of Shapiro.

COROLLARY 3.1 (H. S. Shapiro). *If $\sigma$ is a singular measure and $\sigma(K)>0$ for a $K\in\mathcal{C}$, then $s_\sigma$ is not cyclic.*

*Proof.* It follows from Kellogg's theorem [15, Theorem IX.7, p. 361] that if $\varphi:D\to\Omega_K$ is the conformal map from $D$ onto $\Omega_K$ then $|\varphi'|$ extends to a continuous function on $\bar{D}$. By Theorem 2.1, $s_\sigma$ has a singular factor in $H^\infty(\Omega_K)$. By Theorem 3.1, $s_\sigma$ cannot be cyclic for $A^2$.

Observe next that the Korenblum–Roberts theorem implies that the condition $\mathcal{C}$-outer is also sufficient that $f$ be cyclic, in case $f$ is a singular inner function.

COROLLARY 3.2. *Let $s=s_\sigma$ be a singular inner function. Then $s$ is cyclic if and only if $s$ is $\mathcal{C}$-outer.*

*Proof.* We need only show the sufficiency. If $s$ is $\mathcal{C}$-outer then $s$ is outer in $H^2(\Omega_K)$ for all $K\in\mathcal{C}$. Applying Kellogg's theorem again and Theorem 2.1 we see that $\sigma(K)=0$ for all $K\in\mathcal{C}$. Therefore $s=s_\sigma$ is cyclic. □

REMARK. The domains $\Omega_K$, $K\in\mathcal{C}$ were used by Korenblum in [8]. Functions similar to the $F_K$'s were also used by Korenblum in [8] and [9] and Shapiro [13].

**4. Cyclic functions in the Nevanlinna class.** We show in this section that with an additional hypothesis on $f$, the converse of Theorem 3.1 is true.

We will need to identify the dual of $A^2$ in a particular way. It is easy to verify that $A^2=\{f: f=\sum a_n z^n$, where $\sum_{n=0}^\infty |a_n|^2/(n+1)<\infty\}$. Thus $(A^2)^*$ may be identified with the space $D_1=\{g: g=\sum b_n z^n$, where $\sum |b_n|^2(n+1)<\infty\}$. If $f\in A^2$ and $g\in D_1$ then $\langle f,g\rangle\equiv\sum a_n\bar{b}_n$ accounts for all bounded linear functionals on $A^2$. Also observe that

$$\langle f,g\rangle=\sum a_n\bar{b}_n=\lim_{r\to 1}\frac{1}{2\pi}\int_T f(r\zeta)\overline{g(r\zeta)}\,|d\zeta|.$$

In particular, if $f\in H^2$,

$$\langle f,g\rangle=\frac{1}{2\pi}\int_T f(\zeta)\overline{g(\zeta)}\,|d\zeta|,$$

since $g\in D_1\subseteq H^2$.

We are ready for our main result.

THEOREM 4.1. *Let $f$ be non-vanishing in $A^2$, and suppose $[f] \cap N$ contains a non-vanishing function. Then $f$ is cyclic if and only if $f$ is $\mathcal{C}$-outer.*

*Proof.* We must show that if $[f] \neq A^2$ then $F_K f$ has a singular factor in $H^\infty(\Omega_K)$ for some $K \in \mathcal{C}$. Since $[f] \neq A^2$, $[f]^\perp \neq \{0\}$, where $[f]^\perp = \{g : g \in D_1, \langle h, g\rangle = 0, h \in [f]\}$. Let $X = [f]^\perp$. Since $[f]$ is invariant under the shift $S$, $X$ is *-invariant, that is, invariant under the operator $S^*$, where

$$S^* g = \frac{g(z) - g(0)}{z}.$$

Let $\bar{X}$ be the closure of $X$ in $H^2$. Then $\bar{X}$ is also *-invariant. By Beurling's theorem, either $\bar{X} = H^2$ or $\bar{X} = (sH^2)^\perp$, where $s$ is an inner function and $(sH^2)^\perp$ is the orthogonal complement in $H^2$ of $sH^2$. By hypothesis, there exists a non-vanishing $g$ such that $g \in [f] \cap N$. Since $g = p/q$ where $p$ and $q$ can be assumed to be non-vanishing bounded functions, and since $qg$ also belongs to $[f]$, we see that $[f]$ contains $p$, a non-vanishing bounded function.

Thus, for $\psi \in X$,

$$0 = \langle p, \psi\rangle = \int_T p(\zeta) \overline{\psi(y)}\, |d\zeta|.$$

If $\bar{X} = H^2$ it would follow that $p \equiv 0$, contrary to hypothesis. Thus $\bar{X} \neq H^2$, and $\bar{X} = (sH^2)^\perp$ for a non-trivial inner function $s$.

We claim that $s$ must be a singular inner function. If not, then $s(a) = 0$ for some $a \in D$ and $(1 - \bar{a}z)^{-1} \in (sH^2)^\perp$. Since there are functions $\varphi_n \in X$ converging in $H^2$ to $(1 - \bar{a}z)^{-1}$,

$$0 = \langle p, \varphi_n\rangle = \int p\bar{\varphi}_n \, d\theta$$

and

$$0 = \lim_{n\to\infty} \int p\bar{\varphi}_n \, d\theta = 2\pi p(a),$$

contradicting the fact that $p$ is non-vanishing.

Thus $\bar{X} = (sH^2)^\perp$, where $s$ is a singular inner function.

By the bi-polar theorem, [12, Theorem 4.7, p. 91], $[f] = {}^\perp X$, where ${}^\perp X = \{h : h \in A^2 \text{ and } \langle h, \psi\rangle = 0, \ \psi \in X\}$. Since $s \in {}^\perp X$ we see that $s \in [f]$ and that $[s] \subseteq [f] \neq A^2$. By the Korenblum–Roberts theorem, it follows that $s = s_\sigma$ where $\sigma(K) > 0$ for some $K \in \mathcal{C}$.

Factor $s = s_1 s_2$ where $s_1 = s_{\sigma_1}$ and $s_2 = s_{\sigma_2}$ where $\sigma_1$ lives on $K$ and $\sigma_2(K) = 0$. We claim that $s_2 F_K f \in [s]$.

To establish this, we must show that $\langle s_2 F_K f, \varphi\rangle = 0$ for all $\varphi \in [s]^\perp$. Now for $\varphi \in [s]^\perp \subseteq (sH^2)^\perp$, there is an $h \in H^2$ such that

$$\varphi(e^{i\theta}) = s(e^{i\theta}) \overline{h(e^{i\theta})} e^{-i\theta} \quad \text{a.e. } [d\theta].$$

This is because $\bar{s}\varphi$ is in $L^2 \ominus H^2$. See [5, p. 357] for details.

If $p$ is a polynomial,

$$\langle s_2 F_K p, \varphi\rangle = \frac{1}{2\pi}\int_T s_2 F_K p\bar{\varphi}\,d\theta$$
$$= \frac{1}{2\pi i}\int_T F_K p\frac{h}{s_1}\,dz.$$

Since

$$-\log|s_1(z)| = \int_K \frac{1-|z|^2}{|\zeta-z|^2}\,d\sigma(\zeta)$$

and since for $x\in D\setminus\bar{\Omega}_K$,

$$|\zeta-z|\geqslant c\sqrt{(1-|z|)},\quad \zeta\in K,$$

for $c$ independent of $z$ or $\zeta$ by construction of $\Gamma_K$, it follows that $-\log|s_1(z)|\leqslant c\|\sigma_1\|$ for $z\in D\setminus\bar{\Omega}_K$. We may therefore use Cauchy's theorem and obtain

$$\langle s_2 F_K p, \varphi\rangle = \frac{1}{2\pi i}\int_{\Gamma_K}\frac{F_K p}{s_1}h\,dz.$$

Since $\|F_K p\|_{\infty,\Omega_K}\leqslant c\|p\|_{A^2}$ and since $s_1^{-1}$ is bounded on $\Gamma_K$, a standard argument yields

$$\langle s_2 F_K f, \varphi\rangle = \frac{1}{2\pi i}\int_{\Gamma_K}\frac{F_K fh}{s_1}\,dz$$

for all $\varphi\in[s]^\perp$. Since $X$ is $H^2$ dense in $[s]^\perp\subseteq(sH^2)^\perp$, we may find a sequence $\{\varphi_n\}\subseteq X$ such that $\varphi_n\to\varphi$ in $H^2$. Thus

$$0=\langle s_2 F_K f, \varphi_n\rangle = \int_{\Gamma_K}\frac{F_K fh_n}{s_1}\,dz,$$

where $\varphi_n = s\overline{h_n z}$. As $\varphi_n\to\varphi$, $h_n\to h$ in $H^2$. Since $F_K f/s_1$ is bounded on $\Gamma_K$ we conclude that

$$0=\lim_{n\to\infty}\int_{\Gamma_K}\frac{F_K fh_n}{s_1}\,dz = \int_{\Gamma_K}\frac{F_K fh}{s_1}\,dz = \langle s_2 F_K f,\varphi\rangle.$$

Thus $s_2 F_K f\in{}^\perp([s]^\perp)=[s]$, as claimed.

We now complete the proof. Since $s_2 F_K f\in[s]$, there are polynomials $p_n$ for which $\|p_n s - s_2 F_K f\|_{A^2}\to 0$ as $n\to\infty$. Using Lemma 3.2,

$$\|F_K p_n s - s_2 F_K^2 f\|_{H^2(\Omega_K)}\to 0$$

as $n\to\infty$. By Theorem 2.1, $s$ has a singular factor in $H^2(\Omega_K)$ and therefore so does $s_2 F_K^2 f$. By Theorem 2.2 and Corollary 2.1, $s_2 F_K$ is outer in $H^2(\Omega_K)$. Thus $F_K f$ has a singular factor, and the proof is complete. □

COROLLARY 4.1. *If* $[f]\cap N$ *contains a non-vanishing function and* $f^{-1}\in A^2$, *then* $f$ *is cyclic.*

*Proof.* Let $K\in\mathcal{C}$. Then $(F_K f)(F_K f^{-1}) = F_K^2$. Since the right hand side is outer, each term on the left must be outer. Thus $f$ is $\mathcal{C}$-outer and cyclic by Theorem 4.1. □

The next two corollaries follow immediately from Theorem 4.1 and Corollary 4.1.

COROLLARY 4.2. *If $f$ is non-vanishing and in $N\cap A^2$, then $f$ is cyclic if and only if $f$ is $\mathcal{C}$-outer.*

COROLLARY 4.3. *If $f$ and $f^{-1}$ are in $N\cap A^2$ then both are cyclic.*

We can also give another characterization for a function $f\in N\cap A^2$ to be cyclic.

LEMMA 4.1. *Suppose $f\in N\cap A^2$. Let $f=Bs_1F/s_2$ be the canonical factorization of $f$, where $B$ is a Blaschke product, $F$ is outer, and $s_1$ and $s_2$ are singular. Then $s_2=s_\sigma$ where $\sigma(K)=0$ for all $K\in\mathcal{C}$, that is, $s_2$ is $\mathcal{C}$-outer.*

*Proof.* We know $s_1=s_\mu$ and $s_2=s_\sigma$ where $\mu$ and $\sigma$ live on disjoint sets; this follows from the Hahn decomposition of $\nu$ where

$$\log\left|\frac{s_1(z)}{s_2(z)}\right|=\int_T\frac{1-|z|^2}{|1-\bar{\zeta}z|^2}\,d\nu(\zeta).$$

Suppose $\sigma(K)>0$ for some $K\in\mathcal{C}$. Then, since $f\in A^2$,

$$F_Kf=\frac{Bs_1}{s_2}F\cdot F_K\in H^\infty(\Omega_K).$$

Let $\varphi$ be a Riemann map of $D$ onto $\Omega_K$. Since $B(\varphi)$, $F(\varphi)$, and $F_K(\varphi)$ are outer functions by Theorem 2.2, and since $\sigma(K)>0$ implies that $s_2(\varphi)$ has a singular factor $s_3$, $F_Kf\in H^\infty(\Omega_K)$ implies that $s_3$ divides $s_1(\varphi)$. But by the proof of Theorem 2.1, $s_3=s_{\hat{\sigma}}$, where $\hat{\sigma}(E)\equiv\sigma(\varphi(E))$, and the singular part of $s_1(\varphi)$ is generated by the measure $\hat{\mu}$, where $\hat{\mu}(E)\equiv\mu(\varphi(E))$. Since $s_3$ divides $s_1(\varphi)$, $\hat{\mu}-\hat{\sigma}\geqslant 0$. This contradicts the fact that $\hat{\mu}$ and $\hat{\sigma}$ live on disjoint sets. □

THEOREM 4.2. *Let $f$ be a non-vanishing function in $N\cap A^2$. If $f$ has the canonical factorization $f=(s_1/s_2)F$, then $f$ is cyclic if and only if $s_1$ is cyclic.*

*Proof.* By Lemma 4.1 and Theorem 4.1, if $s_1$ is cyclic then $f$ is $\mathcal{C}$-outer. Since $f\in N$, $f$ is cyclic. Conversely, if $f$ is cyclic then Theorem 4.1 implies that $f$ is $\mathcal{C}$-outer and Lemma 4.1 implies that $s_1$ is $\mathcal{C}$-outer; that is, $s_1$ is cyclic.

REMARK. Observe that Corollary 4.3 may be deduced directly from Lemma 4.1 and Theorem 4.2.

**5. Generalizations to weighted Bergman spaces.** Let $q\geqslant 0$. Define $A^{2,q}$ to be all functions $f$ analytic on $D$ for which

$$\|f\|^2_{A^{2,q}}\equiv\int_0^{2\pi}\int_0^1|f(re^{i\theta})|^2(1-r)^q r\,dr\,d\theta<\infty.$$

Then $A^2=A^{2,0}\subseteq A^{2,q}\subseteq A^{2,q'}$ if $q<q'$.

The theorems of Korenblum [10], Shapiro [13], and Roberts [14] say that all the $A^{2,q}$ spaces have the same cyclic inner functions.

THEOREM II. *The function $s_\sigma$ is cyclic for $A^{2,q}$ if and only if $\sigma(K)=0$ for all BCH-sets $K$.*

It is a simple matter to extend our results to the spaces $A^{2,q}$. We sketch the details.

LEMMA 5.1. *Let $f\in A^{2,q}$. Then there is a constant $A_q$ depending only on $q$ such that*

$$|f(z)|\leqslant A_q(1-|z|)^{-A_q}\|f\|_{A^{2,q}}$$

*for all $z\in D$.*

*Proof.* This fact is well known and we omit the proof.

Let $K$ be a BCH-set and let $G$ be the function defined in §3 associated with $K$.

LEMMA 5.2. *There is an $N$ depending only on $q$ such that if $F_{K,q}\equiv G^N$ then $|F_{K,q}(z)|\leqslant(1-|z|)^{A_q}$ for $z\in\Gamma_K$, where $\Gamma_K$ is as in* §3.

LEMMA 5.3. *Let $f\in A^{2,q}$. Then $F_{K,q}f\in H^\infty(\Omega_K)$ where $\Omega_K$ is as in* §3. *Furthermore,*

$$\|F_{K,q}f\|_{\infty,\Omega_K}\leqslant c\|f\|_{A^{2,q}}.$$

DEFINITION. Let $f\in A^{2,q}$. Then $f$ is called $\mathcal{C}$-outer if $F_{K,q}$ is outer in $H^\infty(\Omega_K)$ for all $K\in\mathcal{C}$.

Putting the three lemmas together yields the following theorem.

THEOREM 5.1. *Let $f$ be cyclic in $A^{2,q}$. Then $f$ is $\mathcal{C}$-outer.*

To obtain the partial converse, we note that since $A^{2,q}\supseteq A^2$, $(A^{2,q})^*\subseteq(A^2)^*$. Using the duality described in §4, we see that $(A^{2,q})^*\subseteq H^2$ and that the proof of Theorem 4.1 (with $F_{K,q}$ used in place of $F_K$) applies, yielding the next theorem.

THEOREM 5.2. *Let $f$ be non-vanishing in $A^{2,q}$ and suppose that $[f]\cap N$ contains a non-vanishing function. Then $f$ is cyclic if and only if $f$ is $\mathcal{C}$-outer.*

We get the next corollary, generalizing Corollary 4.1.

COROLLARY 5.1. *Let $f\in A^{2,q}$ and suppose that $f^{-1}\in A^{2,q'}$ for some $q$ and $q'$. Then if $[f]\cap N$ contains a non-vanishing function, $f$ is cyclic.*

*Proof.* Let $K\in\mathcal{C}$. Then $(F_{K,q}f)(F_{K,q'}f^{-1})=F_{K,q}F_{K,q'}$. Since the right hand side is outer, both factors on the left must be outer. Thus $f$ is $\mathcal{C}$-outer and hence cyclic. □

Corollary 5.1 applies immediately to functions $f\in A^{2,q}\cap N$ as does Theorem 5.2.

COROLLARY 5.2. *Let $f$ be non-vanishing and in $A^{2,q}\cap N$. Then $f$ is cyclic if and only if $f$ is $\mathcal{C}$-outer.*

COROLLARY 5.3. *Let $f\in A^{2,q}\cap N$ and suppose that $f^{-1}\in A^{2,q'}$ for some $q$ and $q'$. Then $f$ is cyclic.*

We can also give a more precise description of the factorization of $f\in A^{2,q}\cap N$.

THEOREM 5.3. *Let* $f \in A^{2,q} \cap N$ *and suppose* $f = B(s_1/s_2)F$ *is the canonical factorization of* $f$. *Then* (i) $s_2$ *is cyclic, and* (ii) $f$ *is cyclic if and only if* $B \equiv 1$ *and* $s_1$ *is cyclic.*

*Proof.* This theorem may be proved by the arguments in Lemma 4.1 and Theorem 4.2, provided that $F_{K,q}$ is used in place of $F_K$. □

CONCLUDING REMARKS. We have raised two natural questions. (1) If $f \in A^{2,p}$ and is non-vanishing, must $[f]_{A^{2,p}} \cap N$ contain a non-vanishing function? (2) If $f \in A^{2,p}$ and is $\mathcal{C}$-outer, is $f$ cyclic?

An affirmative answer to (1) would imply an affirmative answer to (2). An affirmative answer to (2) would show that if $f \in A^{2,p}$ and $f^{-1} \in A^{2,q}$ then both are cyclic.

ADDED IN PROOF. 1. It has been brought to our attention that Corollary 4.3 and Lemma 4.1 can be proved using the theory of pre-measures as developed by B. I. Korenblum in [8] and [9].

2. In a paper which will appear in Proc. Amer. Math. Soc. by Paul Bourdon, "Cyclic Nevanlinna Class Functions in Bergman Spaces", another proof of Corollary 4.3 is given.

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