THE σ-REGULAR REPRESENTATION OF $\mathbf{Z}\times\mathbf{Z}$

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Let $\mathbf{Z}$ denote the group of integers. There exist multipliers σ on $\mathbf{Z}\times\mathbf{Z}$ such that the group extension of $\mathbf{Z}\times\mathbf{Z}$ by σ is a non-Type I group. In fact, the σ-regular representation of such a lattice group is a Type II_1 factor; the consequences of this fact were investigated by Pukanszky in [5]. The main result of this paper is the existence of decompositions of the σ-regular representation of $\mathbf{Z}\times\mathbf{Z}$ with respect to an infinite family of mutually disjoint measures. The integrands in the decompositions are induced irreducibles; furthermore, they can be canonically chosen so that the restrictions to two given normal subgroups are associated with Lebesgue measure quasi-orbits on tori with arbitrary finite relatively prime multiplicities.

Let G be a locally compact group, T the circle group. A *multiplier* (or cocycle) on G is a Borel function $\sigma: G\times G\to T$ satisfying $\sigma(a,b)\sigma(ab,c)=\sigma(a,bc)\sigma(b,c)$ and $\sigma(a,e)=\sigma(e,a)=1$ for all $a,b,c\in G$, where e is the identity of G. Two multipliers σ and σ' are *similar* if there is a Borel $\beta: G\to T$ such that $\sigma'(a,b)=\beta(a)\beta(b)\beta(ab)^{-1}\sigma(a,b)$ for all a,b. A multiplier similar to unity is called a coboundary. For $G=\mathbf{Z}\times\mathbf{Z}$, we find that every multiplier is similar to one of the form $\exp(iB)$, where B is a real bilinear form on $G\times G$. This follows from [4, Theorem 9.6] and the fact that every multiplier on a cyclic group is a coboundary. For convenience, we will adopt the following conventions. Elements of G will be denoted either by n or by (p,q), with subscripts as needed. We regard T as $R/2\pi\mathbf{Z}$, elements typically denoted by u,w. We view elements of T^2 as vectors V or (V_1,V_2), with group action written additively. Finally, let e_1,e_2 be the usual basis vectors in the real plane, $e_3=e_1+e_2$; $\langle\,,\rangle$ will denote the usual inner product.

For a given multiplier σ on $G=\mathbf{Z}\times\mathbf{Z}$, define the *$\sigma$-regular representation* R^σ by the formula $(R^\sigma_g f)(g')=\sigma(g',g)f(g'g)$ for $f\in L^2(G)$. If $F: L^2(G)\to L^2(T^2)$ is the Fourier transform, define $\hat{R}^\sigma=FR^\sigma F^{-1}$. We wish next to define an action on $\hat{R}^\sigma$ by certain homomorphisms of T^2. To this end, let $M=\left(\begin{smallmatrix}a_1 & a_2\\ a_3 & a_4\end{smallmatrix}\right)\in SL(2,\mathbf{Z})$. It is well known that M induces a measure-preserving homomorphism of T^2, and hence a unitary operator V_M on $L^2(T^2)$, given by $(V_M\phi)(v)=\phi(Mv)$. We will say M acts on $\hat{R}^\sigma$ by $M\cdot\hat{R}^\sigma=V_M\hat{R}^\sigma V_M^{-1}$. To compute the effect of this action, fix a real matrix $A=\left(\begin{smallmatrix}t_1 & t_2\\ t_3 & t_4\end{smallmatrix}\right)$, and let σ be given by $\sigma(n_1,n_2)=\exp(i\langle n_1,An_2\rangle)$. Then, for all $\phi\in L^2(T^2)$,

$$\hat{R}^\sigma_n\phi(v)=c(n)\exp(-i\langle v,n\rangle)\phi(v+An)$$

and

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$$M\cdot\hat{R}_n^\sigma\phi(v)=c(n)\exp(-i\langle Mv,n\rangle)\phi(v+M^{-1}An)$$

where $\overline{c(n)}=\sigma(n,n)$.

Note that if χ is a character of G, $\chi\hat{R}^\sigma$ is unitarily equivalent to $\hat{R}^\sigma$; denote the representation $M\cdot(\chi\hat{R}^\sigma)$ by U. We have $U=\chi(M\cdot\hat{R}^\sigma)$. The character χ will play a crucial role in determining equivalence relations among the irreducible components of disjoint direct integral decompositions, as will the matrix $M^{-1}A$, which we denote by $\begin{pmatrix} s_1 & s_2 \\ s_3 & s_4 \end{pmatrix}$.

We now take a first look at decompositions. For each $r,s,t\in T$, define a representation $V=V(r,s,t)$ of G, acting in $L^2(T)$, by

$$(*)\qquad V_nf(w)=\chi(n)c'(n)\exp(-ir\langle Me_2,n\rangle)\exp(-iw\langle Me_3,n\rangle)f(w+sp+tq),$$

where $\overline{c'(n)}=\exp(i\langle n,A'n\rangle)$ and $A'=M\begin{pmatrix} s & t \\ s & t \end{pmatrix}$.

THEOREM 1. *Let $s_1=s_3=s$ and $s_2=s_4=t$. Then U is unitarily equivalent to the direct integral $\int_T V(r,s,t)\,dr$.*

Proof. Define $W\colon L^2(T,L^2(T))\to L^2(T^2)$ by

$$(Wf)(v_1,v_2)=f(v_2-v_1)(v_1)\quad\text{for all }(v_1,v_2)\in T^2,\ f\in L^2(T,L^2(T)).$$

Then

$$\begin{aligned}\left(W\left(\int V\,dr\right)_n W^{-1}\phi\right)(v_1,v_2)&=\left(\left(\int V\,dr\right)_n W^{-1}\phi\right)(v_2-v_1)(v_1)\\ &=V_n(v_2-v_1,s,t)(W^{-1}\phi)(v_2-v_1)(v_1)\\ &=\chi(n)c'(n)\exp(-i(v_2-v_1)\langle Me_2,n\rangle)\\ &\quad\times\exp(-iv_1\langle Me_3,n\rangle)\phi(v_1+sp+tq,v_2+sp+tq)\\ &=U_n\phi(v_1,v_2)\end{aligned}$$ □

THEOREM 2. *Let V be as above. Then, if σ^k is not a coboundary for $k\neq 0$, V is irreducible.*

Proof. Keeping M as before, let $d_1=a_1+a_2$, $d_2=a_3+a_4$. Regard each V as a unitary representation of the appropriate group extension G^σ. Let

$$S=\{(a,b)\mid(a,b)=(1,0),(0,1),\text{ or }a\text{ and }b\text{ are relatively prime}\}.$$

For $(a,b)\in S$, let $N_{(a,b)}=\{(w,ak,bk)\mid k\in\mathbf{Z},w\in T\}$. Clearly, each $N_{(a,b)}$ is a normal abelian subgroup, and the condition on σ implies that orbits of non-unity characters in $\hat{N}_{(a,b)}$ are countable dense subsets of a torus. By direct calculation, the restriction of V to $N_{(-d_2,d_1)}$ is a product of a character on T with a direct sum of characters over an orbit in $\hat{\mathbf{Z}}$. Hence, we may apply Theorem 8.1 of [4], since the stability subgroup of $\chi\in\hat{N}_{(a,b)}$, $\chi\not\equiv 1$, is $N_{(a,b)}$. □

Let us write $\mathcal{R}$ for the regular representation of $\mathbf{Z}$, and 1 for the identity character of T.

LEMMA. *The representations V satisfy:*

(i) $V|_{N_{(1,0)}} \sim 1\cdot|d_1|\mathfrak{R}$,

(ii) $V|_{N_{(0,1)}} \sim 1\cdot|d_2|\mathfrak{R}$,

where ~ denotes unitary equivalence and d_1, d_2 are as in the proof of Theorem 2.

Proof. (i) For each $m=0,1,\ldots,|d_1|-1$, let

$$H_m = \{f\in L^2(T) \mid f(w)=\Sigma c_k \exp(iw(d_1 k+m))\}.$$

Each H_m is invariant under each $V_{(p,0)}$. We identify H_m with l^2 and find that the restriction of $V_{(p,0)}$ has the form $V_{(p,0)}\hat{f}(k)=Q(p,k)\hat{f}(k-p)$, $f\in H_m$, where $|Q|=1$ and $Q(p,k)Q(q,k-p)=Q(p+q,k)$. By a slight modification of Lemma 3.7 of [2], $V|_{N_{(1,0)}}|_{H_m}\sim 1\cdot\mathfrak{R}$. Hence $V|_{N_{(1,0)}}\sim 1\cdot|d_1|\mathfrak{R}$. □

COROLLARY. *$V|_{N_{(1,0)}}$ is associated with the product of a point-mass on $\hat{T}$ and Lebesgue measure quasi-orbit on T with multiplicity $|d_1|$.*

We come now to the question of equivalence relations. The multiplier for each V given by (*) is similar to one of the form $\sigma(n_1,n_2)=\exp(i(d_1 t-d_2 s)p_1 q_2)$, where $n_i=(p_i,q_i)$, $i=1,2$. Further, let $\bar{\chi}$ denote the "character part" of V; that is, $\bar{\chi}(n)=\chi(n)\exp(ir\langle Me_2,n\rangle)$. Then V depends on the parameters M, s, t, and $\bar{\chi}$. Let V and V' be two such representations, with primes on the parameters of V'; finally, let $d_1=a_1+a_2$, $d_2=a_3+a_4$, $d_1'=a_1'+a_2'$, $d_2'=a_3'+a_4'$.

THEOREM 3. *$V\sim V'$ if and only if $d_1=d_1'$, $d_2=d_2'$,*

$$d_1 t'-d_2 s' = d_1 t - d_2 s \bmod 2\pi,$$

and $\bar{\chi}((d_2,-d_1))=\bar{\chi}'((d_2,-d_1))\exp(ij(d_1 t-d_2 s))$ for some integer j.

Proof. If all of the above equalities hold, then V and V' are both concentrated on the same (discrete) orbit in $\hat{N}_{(-d_2,d_1)}$; hence $V\sim V'$. Conversely, if $V\sim V'$, then they have the same multiplier, and their restrictions to any normal subgroup are equivalent. Hence $d_1=d_1'$ and $d_2=d_2'$ by the lemma; negative signs cannot occur since the restrictions to, say, $N_{(-d_2,d_1)}$ must both be direct sums of characters. This also yields the last assertion of the theorem. □

Thus, the decomposition of $M\cdot\hat{R}^\sigma$ depends upon an a priori specialized choice of parameters. We will show next that this may always be done, without changing the cohomology class of the representation.

In the matrix A used to define σ, the diagonal entries produce coboundaries (e.g., $\exp(it_1 p_1 p_2)$). Thus, they may be changed as convenient. Suppose M is chosen such that $d_1=a_1+a_2\neq 0$, $d_2=a_3+a_4\neq 0$. Let us then choose $t_1=t_3 d_1/d_2$, $t_4=t_2 d_2/d_1$. We may readily check that the entries of $M^{-1}A$ satisfy the condition of Theorem 1.

THEOREM 4. *Let σ be a multiplier on G such that σ^k is not a coboundary for $k\neq 0$. Then R^σ is cohomologous to a direct integral of induced irreducibles. Furthermore, if d_1 and d_2 are relatively prime positive integers, there exists a decomposition of R^σ such that each irreducible in the decomposition restricts on*

$N_{(1,0)}$ (respectively, $N_{(0,1)}$) to a product of a point mass and Lebesgue measure quasi-orbit with multiplicity d_1 (respectively, d_2).

We note only that, if M is as above, the sums of the rows are relatively prime integers; further, given d_1 and d_2, choose h, l such that $d_1 h + d_2 l = 1$. Then take $M = \begin{pmatrix} l & d_1 - l \\ -h & d_2 + h \end{pmatrix}$.

Finally, let us investigate disjointness of these decompositions. Let

$$C = \{M \in SL(2, \mathbf{Z}) \mid \langle Me_3, e_i \rangle \neq 0, i = 1, 2\}.$$

If $M_1, M_2 \in C$, define $M_1 \sim M_2$ if and only if $|\langle M_1 e_3, e_i \rangle| = |\langle M_2 e_3, e_i \rangle|$, $i = 1, 2$.

PROPOSITION. *Suppose $M_1, M_2 \in C$, $M_1 \not\sim M_2$. Then the decompositions of $M_1 \cdot \hat{R}^\sigma$ and $M_2 \cdot \hat{R}^\sigma$ are disjoint.*

Proof. Let $M_1 \cdot \hat{R}^\sigma \sim \int V(r)\, dr$, $M_2 \cdot \hat{R}^\sigma \sim \int V(r')\, dr'$, $M_1 = (a_i)$, $M_2 = (b_i)$. Since $M_1 \not\sim M_2$, at least one of the following holds:

$$|a_1 + a_2| \neq |b_1 + b_2| \quad \text{or} \quad |a_3 + a_4| \neq |b_3 + b_4|.$$

Comparing the restrictions of $V(r)$ and $V(r')$ to $N_{(1,0)}$ or $N_{(0,1)}$, we see that no $V(r)$ could be equivalent to any $V(r')$. □

COROLLARY. *There are infinitely many disjoint decompositions of R^σ into induced irreducibles.*

The above work was a result of investigations of certain "twisted" tensor products of irreducible multiplier representations of $\mathbf{Z} \times \mathbf{Z}$; see [1]. It turned out that all such products formed from induced irreducibles could be written as direct sums of some R^σ (except in certain degenerate cases).

The representations V discussed in Theorem 1, which depend essentially upon a relatively prime pair of integers (d_1, d_2), a character (ζ, η) of G, and translation parameters s, t, can be used to yield an infinite family of irreducible cocycle representations, of arbitrary finite dimension, of the simplest virtual group, $T \times \mathbf{Z}$. We shall discuss this in more detail in a forthcoming paper.

REFERENCES

1. L. Baggett, *Multiplier extensions other than the Mackey extension,* Proc. Amer. Math. Soc. 56 (1976), 351–356.
2. ———, *Representations of the Mautner group* I, Pacific J. Math. 77 (1978), 7–22.
3. G. W. Mackey, *Induced representations of locally compact groups* I, Ann. of Math. (2) 55 (1952), 101–139.
4. ———, *Unitary representations of group extensions* I, Acta Math. 99 (1958), 265–311.
5. L. Pukanszky, *Representations of solvable Lie groups,* Ann. Sci. École Norm. Sup. (4) 4 (1971), 457–608.

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