ASYMPTOTIC ESTIMATES FOR THE PERIODS OF PERIODIC SOLUTIONS OF A DIFFERENTIAL-DELAY EQUATION

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Introduction. A periodic solution $x(t)$ of

$$(0.1) \qquad x'(t) = -\alpha f(x(t-1))$$

will be called a "slowly oscillating periodic solution (of (0.1))" if there exist numbers $q>1$ and $\bar{q}>q+1$ such that $x(t)>0$ for $0<t<q$, $x(t)<0$ for $q<t<\bar{q}$, and $x(t+\bar{q})=x(t)$ for all t. The word "slowly" refers to the fact that the separation of zeros of $x(t)$ is greater than the time lag, which is 1.

If $f(x)$ is odd (the case considered in this paper), it is useful to consider a subclass of the slowly oscillating periodic solutions of (0.1). A slowly oscillating periodic solution of (0.1) is called an *S-solution* (in the notation of D. Saupe [7, 8]) if $\bar{q}=2q$ and $x(t+q)=-x(t)$ for all t. Actually, it will be useful to be more pedantic and call an S-solution of (0.1) a pair (α, x) such that $x(t)$ is a periodic solution of $x'(t)=-\alpha f(x(t-1))$, $x(t)$ is positive on an interval $(0,q)$ where $q>1$, and $x(t+q)=-x(t)$ for all t. This paper will treat properties of S-solutions of (0.1) and in particular properties of the maps $(\alpha,x)\to q=q(\alpha,x)$ for α large and (α,x) an S-solution.

The problem of the existence and qualitative properties of slowly oscillating periodic solutions of (0.1) has been studied by several authors, and there is ample evidence by now that the qualitative properties of periodic solutions of (0.1) may depend subtly on the function f. Here we shall consider odd functions $f(x)$ which are similar to $f_r(x)$, where

$$f_r(x) \equiv x(1+|x|^{r+1})^{-1}.$$

Equation (0.1) with such an f was suggested by J. Yorke (in a private communication to the second author) as a model for somewhat more complicated-looking equations like

$$(0.2) \qquad x'(t) = -Ax(t) + Bf(x(t-1)), \quad A,B>0,$$

which had been proposed by Mackey and Glass [2, 3] in connection with physiological control theory.

D. Saupe [7, 8] has carried out a careful numerical study of equation (0.1) for $f(x)=f_r(x)$. Saupe's results suggest that (0.1) displays very complex dynamical behaviour, but little has been proved. It has, however, been proved that if $r>2$ and α is sufficiently large, then equation (0.1) has at least three S-solutions (Saupe's numerical studies actually suggest the existence of at least seven

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S-solutions for large α). In particular, if $f(x)$ is sufficiently like $f_r(x)$ and $r>2$, Nussbaum [5, 6] has proved that, for all α sufficiently large, equation (0.1) has an S-solution (α, x_α) such that $x_\alpha(t)$ is positive on an interval $(0, q_\alpha)$, $x_\alpha(t+q_\alpha)=-x_\alpha(t)$ for all t, and $\lim_{\alpha\to\infty} q_\alpha=\infty$. In general we shall use the notation $2q(\alpha, x)$ to denote the minimal period of an S-solution (α, x) of (0.1), so $q_\alpha \equiv q(\alpha, x_\alpha)$.

D. Saupe has observed [7, 8] (see Proposition 1.2 below) that if f is odd and (α, x) is an S-solution, then one can obtain by a simple transformation another S-solution $(\tilde{\alpha}, \tilde{x})$. If this trick is applied to the family $\{(\alpha, x_\alpha)\}$ of S-solutions described above, one obtains a new family of S-solutions $\{(\tilde{\alpha}, \tilde{x}_\alpha)\}$, and we shall see that

$$\lim_{\tilde{\alpha}\to\infty} q(\tilde{\alpha}, \tilde{x}_\alpha)=1.$$

The purpose of this paper is to give sharp asymptotic estimates on the minimal periods of the periodic solutions $x_\alpha(t)$ and $\tilde{x}_\alpha(t)$. In fact we shall prove that if $r>2$ and $f(x)$ is sufficiently like $f_r(x)$ then there are positive constants c_1 and c_2 (independent of α) such that, for all large α,

$$c_1\alpha^{r-2} \leqslant q(\alpha, x_\alpha) \leqslant c_2\alpha^{r-2}. \tag{0.3}$$

Furthermore, with the aid of Saupe's transformation, we shall prove that there exists $\beta_0>0$ such that, for each $\beta\geqslant\beta_0$, equation (0.1) has an S-solution (β, z_β) with

$$1+c_3\beta^\tau \leqslant q(\beta, z_\beta) \leqslant 1+c_4\beta^\tau, \tag{0.4}$$

where c_3 and c_4 are constants independent of β and $\tau=-1+(r-1)^{-1}$ (so $-1<\tau<0$).

Our direct motivation for proving such results comes from a theorem which plays a central role in recent work by Mallet-Paret and Nussbaum [4]. Mallet-Paret and Nussbaum study equations of the form

$$x'(t)=-\alpha[\gamma x(t)+f(x(t-1))]. \tag{0.5}$$

Assume that $f: \mathbf{R}\to\mathbf{R}$ is C^1 on a neighborhood of zero and continuous and bounded on $\mathbf{R}$, and in addition satisfies the properties that $xf(x)>0$ for all x and that $f'(0)>\gamma>0$ (where γ is as in (0.5)). It is proved in [4] that if ν_0 is the unique solution in $[\pi/2, \pi]$ of

$$\cos\nu_0=-\left(\frac{\gamma}{f'(0)}\right),$$

then (0.5) has a slowly oscillating periodic solution (which can be taken to be an S-solution if f is odd) for each $\alpha>(\nu_0/\sqrt{(f'(0))^2-\gamma^2})$. Furthermore, for γ and f fixed, there exists a positive constant c_5 (independent of $\alpha>0$) such that if $x_\alpha(t)$ is any slowly oscillating periodic solution of (0.3) of period p_α, then

$$p_\alpha-2\leqslant c_5\alpha^{-1}. \tag{0.6}$$

{Thus even though equations (0.1) and (0.5) both have (for f sufficiently like f_r and $r>2$) S-solutions whose minimal periods approach 2 as α approaches ∞, we see from equations (0.4) and (0.6) that the rate of convergence is different for $\gamma>0$ than for $\gamma=0$.}

1. Existence of periodic solutions. Recall that an "S-solution" of equation (0.1) is a pair (α,x) such that $x(t)$ is a periodic solution of (0.1) for which there exists $q>1$ such that $x(t)>0$ for $0<t<q$ and $x(t+q)=-x(t)$ for all t. It is proved in [6] that for functions like $f_r(x)\equiv x(1+|x|^{r+1})^{-1}$, $r>2$, and for each α sufficiently large, equation (0.1) has an S-solution of minimal period $2q_\alpha$, where $q_\alpha\to\infty$ as $\alpha\to\infty$.

We list here the assumptions on f that will be used in the sequel.

H1. $f:\mathbf{R}\to\mathbf{R}$ is an odd ($f(-x)=-f(x)$ for all x), continuous map. There exists a number $x_*>0$ such that $f|[0,x_*]$ is nondecreasing and $f|[x_*,\infty)$ is nonincreasing. There exist positive constants a, d, r and σ and a constant x_0 such that, for $x\geqslant x_0$,

(1.1) $$(a-dx^{-\sigma})x^{-r}\leqslant f(x)\leqslant(a+dx^{-\sigma})x^{-r}.$$

By replacing $f(x)$ by $(1/a)f(x)$ and suitably modifying α in equation (0.1), we can assume in the following work (if it is convenient) that $a=1$ in H1.

The following result follows from Theorem 1.1 of [6].

THEOREM 1.1 [6]. *Assume that f satisfies* H1 *and that $r>2$ and $\sigma>r/(r-1)$ (r and σ as in* H1). *Then there exists $\alpha_0>0$ such that*

(i) *equation* (0.1) *has a periodic solution $x_\alpha(t)$ for $\alpha\geqslant\alpha_0$,*
(ii) *$x_\alpha(t)>0$ on an interval $(0,q_\alpha)$,*
(iii) *$x_\alpha(t+q_\alpha)=-x_\alpha(t)$ for all t, and*
(iv) $\lim_{\alpha\to\infty}q_\alpha=\infty$.

REMARK 1.1. It is easy to check that if $\beta>0$ and $r>2$, the function

$$f(x)=(\operatorname{sgn}(x))|x|^\beta(1+|x|^{\beta+r})^{-1}$$

satisfies H1 and gives a class of examples for Theorem 1.1.

Assume that f satisfies the hypotheses of Theorem 1.1 and select positive constants a_1 and a_2 such that

(1.2) $$a_1x^{-r}\leqslant f(x)\leqslant a_2x^{-r}\quad\text{for } x\geqslant x_*,$$

where x_* and r are as in H1. It is proved in Section 1 of [6] that for α sufficiently large equation (0.1) possesses an S-solution (α,x_α) such that

(1.3) $$x_\alpha(2)\geqslant k\alpha^\epsilon,\quad \epsilon\equiv(r+1)^{-1},$$

where

(1.4) $$k=2a_2^\epsilon.$$

It is then proved that such an S-solution has the property that $\lim_{\alpha\to\infty}q_\alpha=\infty$ (where $2q_\alpha$ is the minimal period of x_α). A variety of asymptotic estimates on $x_\alpha(t)$ are also obtained, but all of these depend only on knowing that (α,x_α) is an S-solution satisfying (1.3).

In order to make use of the asymptotic estimates obtained in [6], we must prove that if $x_\alpha(t)$, $\alpha \geqslant \alpha_0$, are S-solutions as in Theorem 1.1, then equation (1.3) is automatically satisfied for α large enough. This technical point is handled by the following proposition.

PROPOSITION 1.1. *Assume that f satisfies* H1 *and let r be as in* H1 *and* $\epsilon = (r+1)^{-1}$. *If c is a positive constant, there exists $Q = Q(c)$ such that if (α, x) is an S-solution of*

$$x'(t) = -\alpha f(x(t-1)) \tag{1.5}$$

for some $\alpha > 0$ and

$$x(2) \leqslant c\alpha^{\epsilon}, \tag{1.6}$$

then

$$q \leqslant Q$$

where $2q$ is the minimal period of $x(t)$.

Proof. Define $Q(c) = 5 + a_1^{-1}c^{r+1}$ (a_1 as in equation (1.2)); more precise estimates of Q can be given. We assume that (α, x) is an S-solution of (1.5) and satisfies (1.6), but that $q > Q$, and we obtain a contradiction. Notice that (1.5) implies that x is increasing on $[0,1]$ and decreasing on $[1, q+1]$, so that (defining $\|x\| = \sup_t |x(t)|$)

$$\|x\| = x(1) = |x(q+1)|.$$

Case 1. Assume that $x(2) \leqslant x_*$ (x_* as in H1). Because $x(t)$ is decreasing on $[1, q+1]$ and $f(x)$ is increasing on $[0, x_*]$, equation (1.5) implies that $x(t)$ is concave up ($x'(t)$ is increasing) for $3 \leqslant t \leqslant q+1$. It follows that for $3 \leqslant t \leqslant q+1$ the graph of $x(t)$ lies above the graph of the straight line with slope $x'(q)$ passing through the point $(q, 0)$. Concavity implies that

$$|x'(q)| \leqslant \frac{x(3)}{q-3},$$

so we obtain that

$$|x(q+1)| = \|x\| \leqslant \left(\frac{x(3)}{q-3}\right) < \left(\frac{\|x\|}{q-3}\right). \tag{1.7}$$

We obtain a contradiction from (1.7), because we are assuming that $q \geqslant 4$.

Case 2. Assume that $x(2) > x_*$. Let $\tau \geqslant 2$ be the first time $t > 2$ such that $x(t) = x_*$. The same argument as in Case 1 shows that $x(t)$ is concave up on $[\tau+1,\ q+1]$ and that (assuming $q > \tau + 1$)

$$|x(q+1)| = \|x\| \leqslant \left(\frac{x(\tau)}{q-\tau-1}\right) < \left(\frac{\|x\|}{q-\tau-1}\right). \tag{1.8}$$

We conclude from (1.8) that

$$q < \tau + 2. \tag{1.9}$$

Because $x(t)$ is decreasing for $1 \leqslant t \leqslant q+1$ and $f(x)$ is decreasing for $x \geqslant x_*$, we obtain from (1.5) that

$$|x'(t)| \geqslant \alpha f(x(2)) \geqslant \alpha (c\alpha^{\epsilon})^{-r} a_1 = a_1 c^{-r} \alpha^{\epsilon} \tag{1.10}$$

for $3 \leqslant t \leqslant \tau+1$. If $\tau \geqslant 3$, equation (1.10) implies

$$c\alpha^{\epsilon} \geqslant x(3) - x(\tau) \geqslant (a_1 c^{-r} \alpha^{\epsilon})(\tau - 3) \tag{1.11}$$

or

$$\tau \leqslant c^{r+1} a_1^{-1} + 3. \tag{1.12}$$

Of course equation (1.12) follows automatically if $2 < \tau < 3$. Equations (1.9) and (1.12) contradict the assumption that $q > Q$. □

Proposition 1.1 allows us to use all the asymptotic estimates on $x_\alpha(t)$ obtained in Section 1 of [6]. We also obtain as a bonus a slight sharpening of the main theorem of Section 2 of [6].

COROLLARY 1.1. *Assume that f satisfies* H1 *and that* $0 < r < 2$ *and* $\sigma > \max(r-1, 0)$ *(r and σ as in* H1*). Then the minimal period* $2q$ *of any S-solution of equation* (1.5) *is bounded above by a constant independent of* $\alpha > 0$.

Proof. Is is proved in Lemma 2.3 of [6] that there exist $\alpha_0 > 0$ and $c > 0$ such that if (α, x) is an S-solution of equation (1.5) for some $\alpha \geqslant \alpha_0$, then

$$x(2) \leqslant c\alpha^{\epsilon}, \quad \epsilon = (r+1)^{-1}. \tag{1.13}$$

Proposition 1.1 then implies that there exists $Q_1 > 0$ such that the minimal period $2q$ of any S-solution of equation (1.5) for some $\alpha \geqslant \alpha_0$ satisfies

$$q \leqslant Q_1. \tag{1.14}$$

On the other hand, if $x(t)$ is an S-solution of (1.5) for some α with $0 < \alpha \leqslant \alpha_0$, we have

$$x(2) \leqslant \|x\| = x(1) = -\alpha \int_{-1}^{0} f(x(s))\, ds \leqslant \alpha f(x_*). \tag{1.15}$$

Equation (1.15) implies that

$$x(2) \leqslant d\alpha^{\epsilon}, \tag{1.16}$$

where

$$d = f(x_*)\, \alpha_0^{1-\epsilon}. \tag{1.17}$$

Proposition 1.1 then implies that there exists $Q_2 > 0$ such that the minimal period $2q$ of any S-solution (α, x) with $0 < \alpha \leqslant \alpha_0$ satisfies

$$q \leqslant Q_2. \tag{1.18}$$

Finally, we obtain the desired conclusion from (1.14) and (1.18). □

REMARK 1.2. In Theorem 2.1 of [6] it is also assumed that there exist positive constants c_1 and c_2 such that

$$c_1 x \leqslant f(x) \leqslant c_2 x \quad \text{for } 0 \leqslant x \leqslant x_*. \tag{1.19}$$

Corollary 1.1 shows (1.19) is unnecessary in discussing S-solutions.

One can check that Corollary 1.1 is applicable to

$$f(x) = (\operatorname{sgn}(x))|x|^{\beta}(1+|x|^{r+\beta})^{-1}$$

if $\beta > 0$ and $0 < r < 2$. However, equation (1.19) will only hold if $\beta = 1$, so Theorem 2.1 of [6] only applies for $\beta = 1$.

If f satisfies H1 with $r > 2$ and $\sigma > (r/(r-1))$, then the existence of an S-solution of equation (0.1) of minimal period $2\tilde{q}_\alpha$ (where $\tilde{q}_\alpha \to 1$ from above as $\alpha \to \infty$) now follows directly from Theorem 1.1 and the following simple but ingenious and useful observation.

PROPOSITION 1.2 (D. Saupe [7, 8]). *Let $f: \mathbf{R} \to \mathbf{R}$ be a continuous, odd function. Assume that for some $\alpha \neq 0$, $x(t)$ is a periodic solution of $x'(t) = -\alpha f(x(t-1))$ of minimal period $2q > 0$ and $x(t+q) = -x(t)$ for all t. If $q \neq 1$ (q is necessarily unequal to 1 if f is locally Lipschitzian) and if we define $\tilde{q}$ and $\tilde{x}(t)$ by $\tilde{q} = (q/(q-1))$ and $\tilde{x}(t) = -x(-(q-1)t)$, then $\tilde{x}(t)$ is a periodic solution of $\tilde{x}'(t) = -\tilde{\alpha} f(\tilde{x}(t-1))$, where $\tilde{\alpha} = (q-1)\alpha$. Furthermore, $2|\tilde{q}|$ is the minimal period of $\tilde{x}(t)$ ($\tilde{x}(t+\tilde{q}) = -\tilde{x}(t)$ for all t) and if $x(t) > 0$ for $t \in (0,q)$, then $\tilde{x}(t) > 0$ for $t \in (0,\tilde{q})$.*

Proof. Lemma 4.1 of [1] implies that equation (1.5) has no nonconstant periodic solution of period 2 if f is locally Lipschitzian. Thus in our case $q \neq 1$ if f is locally Lipschitzian; otherwise, assume $q \neq 1$. The rest of the lemma follows by direct calculation, using the fact that f is odd and $x(t+q) = -x(t)$ for all t. □

In Theorem 1.1 we have not used the full strength of the results in Section 1 of [6], and we shall need a slightly stronger version of Theorem 1.1 later in this paper. To state this stronger version we first need a definition. Let k and ϵ be constants as in equations (1.3) and (1.4) and define a set Γ of S-solutions by

$$\Gamma = \{(\alpha, x) : (\alpha, x) \text{ is an } S\text{-solution of equation (1.5) and } x(2) \geqslant k\alpha^{\epsilon}\}. \tag{1.20}$$

Recall that $q(\alpha, x)$ is the minimal period of an S-solution (α, x).

THEOREM 1.3. *Assume that f satisfies* H1 *and that $r > 2$ and $\sigma > (r/(r-1))$. There exist constants $\alpha_0 > 0$ and $c > 0$ such that* (1) *for each $\alpha \geqslant \alpha_0$ there exists $(\alpha, x) \in \Gamma$ (the α-slice of Γ is nonempty), and* (2) *if $(\alpha, x) \in \Gamma$ and $\alpha \geqslant \alpha_0$, then*

$$x(2) \geqslant c\alpha^{(r-1)\epsilon}, \quad \epsilon = (r+1)^{-1}. \tag{1.21}$$

There is a function $\rho(\alpha)$ such that $\lim_{\alpha \to +\infty} \rho(\alpha) = \infty$ and such that if $(\beta, x) \in \Gamma$ for some $\beta \geqslant \alpha$, then

$$q(\beta, x) \geqslant \rho(\alpha). \tag{1.22}$$

Finally, there exists $p_0 > 0$ such that for every $p \geqslant p_0$ there exists $(\alpha, x) \in \Gamma$ with $\alpha \geqslant \alpha_0$ and

$$\alpha(q(\alpha,x)-1)=p. \tag{1.23}$$

Proof. The fact that the α-slice of Γ is nonempty for all large α is proved in Section 1 of [6]. Equation (1.21) is a consequence (for large α) of equation (1.68) in [6]. Equation (1.22) follows from the fact that equation (1.4) in [6] is valid for any (α,x) in Γ (or one can derive a cruder estimate for $\rho(\alpha)$ by using Proposition 1.1 and equation (1.21)).

It remains to prove equation (1.23). This follows by exactly the connectivity argument used to prove Corollary 3.1 in [5]; the necessary estimates are given in Section 1 of [6]. We leave the details to the reader. □

Although Theorem 1.3 will be adequate for our applications, it is a special case of a more general and natural theorem which we shall state but (for reasons of length) shall not prove. Recall that if (α,x) is an S-solution, one can identify (α,x) with (α,ϕ), where $\phi=x\,|\,[0,1]$ (the map $(\alpha,x)\to(\alpha,\phi)$ is 1-1). In this way the set of S-solutions inherits a metric as a subset of $(0,\infty)\times C[0,1]$. For each $\alpha_0>0$ define a subset Γ_{α_0} of Γ by

$$\Gamma_{\alpha_0}=\{(\alpha,x)\in\Gamma:\alpha\geqslant\alpha_0\}. \tag{1.24}$$

Then we have the following.

THEOREM 1.4. *Assume that f satisfies* H1 *and that $r>2$ and $\sigma>(r/(r-1))$. Then there exists $\alpha_0>0$ such that Γ_{α_0} has an unbounded connected component $G_{\alpha_0}\subset\Gamma_{\alpha_0}$ and G_{α_0} has nonempty intersection with $\{(\alpha_0,x)\in\Gamma\}$.*

Note that if J is any compact interval of reals, $J\subset(0,\infty)$, one can easily prove that there exists a constant M such that if $x(t)$ is any S-solution of equation (0.1) for some $\alpha\in J$, then $\|x\|\leqslant M$. Using this fact and the connectedness and unboundedness of G_{α_0} (Theorem 1.4), one obtains that for each $\alpha\geqslant\alpha_0$ there exists an S-solution $(\alpha,x)\in G_{\alpha_0}\subset\Gamma_{\alpha_0}$.

The proof of Theorem 1.4 involves only a slight extension of the arguments of Corollary 3.1 and Lemma 3.4 of [5]. Theorem 1.3 follows immediately from Theorem 1.4 by connectivity arguments and the estimates of Section 1 of [6]. For example, it is proved in [6] that

$$\lim_{\alpha\to\infty}\,[\inf\{q(\beta,x):(\beta,x)\in\Gamma\text{ and }\beta\geqslant\alpha\}]=\infty. \tag{1.25}$$

We have just remarked that the α-slice of G_{α_0} is nonempty for each $\alpha\geqslant\alpha_0$. If we define a continuous function $h: G_{\alpha_0}\to\mathbf{R}$ by $h(\alpha,x)=\alpha[q(\alpha,x)-1]$, then equation (1.25) implies that the supremum of h on G_{α_0} is ∞. If the infimum of h on G_{α_0} is β_0 (the inf is achieved), connectivity of G_{α_0} implies that the range of h is $[\beta_0,\infty)$, which is the conclusion of Theorem 1.3.

2. Asymptotic estimates of the periods of periodic solutions. We proceed now to obtain asymptotic estimates for the minimal periods of the periodic solutions described in the previous section. Our main result is the following.

THEOREM 2.1. *Assume that f satisfies* H1 *and that $r>2$ and $\sigma>(r/(r-1))$ (r and σ as in* H1*). For each α sufficiently large let (α,x_α) be an S-solution of*

$$x'(t) = -\alpha f(x(t-1)) \tag{2.1}$$

such that $2q_\alpha$, *the minimal period of* $x_\alpha(t)$, *satisfies*

$$\lim_{\alpha\to\infty} q_\alpha = \infty. \tag{2.2}$$

(The existence of $x_\alpha(t)$ *is ensured by Theorem* 1.1.) *Then there exist* $\alpha_0 > 0$ *and constants* k_1 *and* k_2 *such that for all* $\alpha \geqslant \alpha_0$,

$$k_1 \alpha^{r-2} \leqslant q_\alpha \leqslant k_2 \alpha^{r-2}. \tag{2.3}$$

Furthermore, there exists $\alpha_1 > 0$ *such that*

$$k_1 \alpha^{r-2} \leqslant q(\alpha, x) \leqslant k_2 \alpha^{r-2} \tag{2.4}$$

for all $(\alpha, x) \in \Gamma$ *with* $\alpha \geqslant \alpha_1$ *(where* Γ *is as in equation* (1.20)).

The proof of Theorem 2.1 is based on the following lemma, whose proof we defer to an appendix.

LEMMA 2.1. *Assume that* f *satisfies* H1 *and that* $r > 2$ *and* $\sigma > (r/(r-1))$ (r *and* σ *as in* H1). *Let* Γ *be as in equation* (1.20). *Define a function* $m = m(\alpha, x)$ *for* $(\alpha, x) \in \Gamma$ *by*

$$m(\alpha, x) = x(q-1), \quad q = q(\alpha, x).$$

Then there exist positive constants d_1 *and* d_2 *(independent of* α*) such that for all* α *sufficiently large and* $(\alpha, x) \in \Gamma$,

$$d_1 \alpha^\epsilon \leqslant m(\alpha, x) \leqslant d_2 \alpha^\epsilon, \quad \epsilon = (r+1)^{-1}. \tag{2.5}$$

Furthermore, there exist positive constants c_1, c_2, c_3 *and* c_4 *(independent of* α*) such that for all pairs* $(\alpha, x) \in \Gamma$ *with* α *sufficiently large,*

$$c_1 m^r \leqslant x(1) \leqslant c_2 m^r \tag{2.6}$$

and

$$c_3 m^{r-1} \leqslant x(2) \leqslant c_4 m^{r-1}, \tag{2.7}$$

where $m = m(\alpha, x)$ *in equations* (2.6) *and* (2.7).

Proof of Theorem 2.1. Proposition 1.1 implies that the solutions (α, x_α) must lie in Γ for α sufficiently large, so it suffices to prove equation (2.4). By using equation (2.5) we can assume that $m(\alpha, x) \geqslant x_*$ for $\alpha \geqslant \alpha_1$, and it then follows from (2.1) that if $(\alpha, x) \in \Gamma$ and $\alpha \geqslant \alpha_1$, $x(t)$ is decreasing and concave down on $[2, q]$, where $q = q(\alpha, x)$. If we assume that equations (2.6) and (2.7) hold for $\alpha \geqslant \alpha_1$ and $(\alpha, x) \in \Gamma$, we see that (possibly increasing α_1) $x(2) \geqslant x_*$ for any $(\alpha, x) \in \Gamma$ with $\alpha \geqslant \alpha_1$, and we obtain from (2.7) and obvious estimates that for $(\alpha, x) \in \Gamma$ and $\alpha \geqslant \alpha_1$

$$c_3 m^{r-1} - \alpha(c_3 m^{r-1})^{-r} a_2 \leqslant x(3) \leqslant c_4 m^{r-1}, \tag{2.8}$$

where a_2 is as in equation (1.2). It follows that there exists $c_5 > 0$ such that for $(\alpha, x) \in \Gamma$ and α sufficiently large (say, $\alpha \geqslant \alpha_1$),

$$c_5 m^{r-1} \leqslant x(3) \leqslant c_4 m^{r-1}, \quad m = m(\alpha, x). \tag{2.9}$$

By using the fact that $f(u)$ is decreasing for $u \geqslant x_*$ we find that if $\alpha \geqslant \alpha_1$ and $(\alpha, x) \in \Gamma$, then $y(t) \equiv x(t-1) - x(t)$ is increasing from $3 \leqslant t \leqslant q$, $q = q(\alpha, x)$. Concavity of $x(t)$ on $[3, q]$ also implies that $x(q-2) - x(q-1) \leqslant m = x(q-1)$, and we obtain from the two previous equations

$$x(t-1) - x(t) \leqslant x(q-2) - x(q-1) \leqslant m \leqslant x(t), \quad 3 \leqslant t \leqslant q-1. \tag{2.10}$$

It follows that if $(\alpha, x) \in \Gamma$ and $\alpha \geqslant \alpha_1$ we have

$$x(t) \leqslant x(t-1) \leqslant 2x(t), \quad 3 \leqslant t \leqslant q(\alpha, x) - 1. \tag{2.11}$$

Because f is decreasing on $[x_*, \infty)$ we obtain from (2.11) that

$$\alpha f(2x(t)) \leqslant -x'(t) \leqslant \alpha f(x(t)), \quad 3 \leqslant t \leqslant q(\alpha, x) - 1. \tag{2.12}$$

If we use (1.2) we derive from (2.12) that

$$(2^{-r} a_1) \alpha (x(t))^{-r} \leqslant -x'(t) \leqslant a_2 \alpha (x(t))^{-r}, \quad 3 \leqslant t \leqslant q-1. \tag{2.13}$$

If we multiply both sides of (2.13) by $(x(t))^r$ and integrate from $t=3$ to $t=q-1$, we obtain

$$(2^{-r} a_1) \alpha (q-4) \leqslant (r+1)^{-1} [x(3)^{r+1} - m^{r+1}] \leqslant a_2 \alpha [q-4]. \tag{2.14}$$

If we use the estimates on m given by (2.5) and those on $x(3)$ in (2.9), inequality (2.14) gives estimates on q in terms of powers of α and yields inequality (2.4). □

With the aid of Theorem 2.1 we can give precise asymptotic estimates for periodic solutions of (2.5) whose minimal periods approach 2 as $\alpha \to \infty$.

THEOREM 2.2. *Assume that f satisfies* H1 *and that $r > 2$ and $\sigma > (r/(r-1))$ (r and σ as in* H1). *Then there exists $\beta_0 > 0$ such that for $\beta \geqslant \beta_0$ the equation*

$$z'(t) = -\beta f(z(t-1)) \tag{2.15}$$

has an S-solution (β, z_β) whose minimal period $2q(\beta, z_\beta)$ satisfies

$$1 + k_3 \beta^\tau \leqslant q(\beta, z_\beta) \leqslant 1 + k_4 \beta^\tau, \tag{2.16}$$

where $\tau = -1 + (r-1)^{-1}$ and k_3 and k_4 are positive constants independent of β.

Proof. The constant α_0 in Theorem 1.3 can be chosen as large as desired, so select $\alpha_0 \geqslant \alpha_1$, where α_1 is as in Theorem 2.1. According to Theorem 1.3 there exists β_0 such that for every $\beta \geqslant \beta_0$ there exists $(\alpha, x) \in \Gamma_{\alpha_0}$ such that

$$\alpha (q(\alpha, x) - 1) = \beta. \tag{2.17}$$

If we define $z_\beta(t)$ by

$$z_\beta(t) = -x(-(q-1)t), \quad q = q(\alpha, x), \tag{2.18}$$

where (α, x) is as in (2.17), then Proposition 1.2 shows that (β, z_β) is an S-solution of (2.15). Furthermore, we have

$$q(\beta, z_\beta) = \frac{q}{q-1}, \quad q = q(\alpha, x). \tag{2.19}$$

Theorem 2.1 implies that there are positive constants b_1 and b_2 (independent of α) such that if β is given by (2.17) and $\alpha_0 \geqslant \alpha_1$ is sufficiently large, then

$$b_1 \beta^{(1/(r-1))} \leqslant \alpha \leqslant b_2 \beta^{(1/(r-1))}. \tag{2.20}$$

Theorem 2.1 also implies that there are positive constants b_3 and b_4 such that if α_0 is sufficiently large,

$$1 + b_3 \alpha^{-r+2} \leqslant q(\beta, z_\beta) \leqslant 1 + b_4 \alpha^{-r+2}. \tag{2.21}$$

By combining (2.20) and (2.21) one obtains Theorem 2.2. □

Appendix. *Proof of Lemma* 2.1. Inequality 2.5 is proved in Lemma 1.2 of [6]. If $(\alpha, x) \in \Gamma$, the lower bounds on $x(1)$ and $x(2)$ follow from Lemmas 1.3 and 1.4 of [6].

Section 2 of [6] is concerned with obtaining upper bounds on $x(1)$ and $x(2)$ when (α, x) is an S-solution of (2.1). However, it is assumed in Section 2 of [6] that $1 < r < 2$ (r as in H1), so we must indicate how the arguments there can be modified to handle the case $r > 2$.

PROPOSITION 3.1. *Assume that f satisfies* H1 *and that $r > 2$ and $\sigma \geqslant 1$ (r and σ as in* H1). *There exists $\alpha_1 > 0$ and a constant c independent of α such that if $(\alpha, x) \in \Gamma$ (Γ as in equation* (1.20)),

$$x(1) \leqslant \left(\frac{1}{f(m)}\right) \int_0^\infty f(x)\, dx + cm^{r-1}, \tag{3.1}$$

where $m \equiv x(q-1)$ and $q \equiv q(\alpha, x)$.

Proof. The upper estimate on $x(1)$ in equation (2.6) follows from (3.1) and the estimates on $x(q-1)$, so it suffices to prove (3.1). For purposes of this proof we shall adopt the notation of [6], although q here corresponds to z_1 in [6]. As remarked before, we can assume that $a = 1$ in equation (1.1).

An examination of the proof of Lemma 2.1 in [6] shows that even though it is assumed there that $1 < r < 2$, the same arguments apply until equation (2.18) in [6] for the case $r > 2$. Thus we obtain from equations (2.4) and (2.18) in [6] that if $(\alpha, x) \in \Gamma$ and α is sufficiently large, then

$$\begin{aligned} x(1) = \alpha \int_0^1 f(\psi_0(s))\, ds \leqslant \frac{1}{f(m+m\delta)} \int_0^{x_*} f(u)\, du \\ + \frac{m^r(1+\delta)^r}{b_1(m)} \int_{x_*}^{\theta_0(0)} f(v)[1-\gamma(v-x_*)]^{-r/(r-1)}\, dv. \end{aligned} \tag{3.2}$$

In equation (3.2), $m = x(q-1)$ (where $q = q(\alpha, x)$) and $\delta > 0$ is the smallest positive number such that $x(q-\delta) = x_*$ (where x_* is as in H1). Furthermore, ψ_0 and $\theta_0(0)$ are defined by

$$\psi_0(s) = x(q-1+s), \quad 0 \leqslant s \leqslant 1$$

and

$$\theta_0(0) = x_* + \frac{\alpha}{m} \int_{m+\delta m}^{2m} f(u)\, du.$$

The function $b_1(m)$ is defined by $b_1(m)=1-dm^{-\sigma}$, where d is as in inequality (1.1), and γ is given by

$$\gamma=\frac{(r-1)m^r(1+\delta)^{r-1}}{\alpha b_1(m+\delta m)}.$$

Lemma 1.2 in [6] implies that there are positive constants d_1 and d_2 such that for α large enough

$$d_1 m^{-1}\leqslant\delta\leqslant d_2 m^{-1}, \tag{3.3}$$

and equation (2.13) in [6] implies that there exist positive constants d_3 and d_4 such that for sufficiently large α

$$d_3 m\leqslant\theta_0(0)\leqslant d_4 m. \tag{3.4}$$

Equation (2.19) in [6] is still valid for $r>2$ and shows that there exists a constant k with $0<k<1$ such that for all sufficiently large α

$$\gamma(\theta_0(0)-x_*)\leqslant k. \tag{3.5}$$

(In proving (3.5) one needs equation (2.12) in [6]; that equation has a typographical error and should read, for $r>1$, $c(r)=(r-1)^{-1}[(1+\delta)^{1-r}-2^{1-r}]$.)

By using inequality (1.1) it is easy to see that there exists a constant d_5 such that for all sufficiently large α and $(\alpha,x)\in\Gamma$,

$$\left(\frac{1}{f(m+m\delta)}\right)\int_0^{x_*}f(u)\,du\leqslant\frac{1}{f(m)}\int_0^{x_*}f(u)\,du+d_5 m^{r-1}. \tag{3.6}$$

To estimate the second term on the right of (3.2) we define $\eta=-(r/(r-1))$ and $\mu=\theta_0(0)$, and use the binomial theorem (as is justified by (3.5)). We obtain

$$\begin{aligned}\int_{x_*}^{\mu}f(v)[1-\gamma(v-x_*)]^{\eta}\,dv\leqslant&\int_{x_*}^{\infty}f(v)\,dv-\int_{\mu}^{\infty}f(v)\,dv\\&+\sum_{j=1}^{\infty}\gamma^j\binom{\eta}{j}(-1)^j\int_{x_*}^{\mu}f(v)(v-x_*)^j\,dv.\end{aligned} \tag{3.7}$$

Our previous estimates show that there is a constant d_6 such that

$$\gamma\leqslant d_6 m^{-1}, \tag{3.8}$$

so by replacing k with a slightly larger constant we can assume that for sufficiently large α

$$\gamma\mu\leqslant k<1. \tag{3.9}$$

If we observe that $\binom{\eta}{j}(-1)^j\geqslant 0$ and select a_2 as in equation (1.2), we thus obtain from (3.7) that

$$\int_{x_*}^{\mu}f(v)[1-\gamma(v-x_*)]^{\eta}\,dv\leqslant\int_{x_*}^{\infty}f(v)\,dv+a_2\sum_{j=1}^{\infty}\gamma^j\binom{\eta}{j}(-1)^j\int_{x_*}^{\mu}v^{j-r}\,dv. \tag{3.10}$$

If we denote the summation on the right in (3.10) by $\Sigma_{j=1}^{\infty}$, we have in the obvious notation

$$\sum_{j=1}^{\infty} = \sum_{j-r\leqslant -1} + \sum_{j-r>-1}. \tag{3.11}$$

By using (3.8) we see that there exists d_7 such that for all sufficiently large α

$$\sum_{j-r\leqslant -1} \leqslant d_7 m^{-1}. \tag{3.12}$$

On the other hand, term-by-term integration (using (3.9)) shows that

$$\begin{aligned}\sum_{j-r>-1} &\leqslant \sum_{j-r>-1} \gamma^j(-1)^j \binom{\eta}{j}\left(\frac{1}{j-r+1}\right)\mu^{j-r+1} \\ &\leqslant \mu^{1-r} \sum_{j-r>-1} k^j(-1)^j \binom{\eta}{j}\left(\frac{1}{j-r+1}\right) \\ &\leqslant d_8 m^{1-r},\end{aligned} \tag{3.13}$$

where d_8 is a constant independent of α. (Note that if $1<r<2$, $j-r>-1$ for all $j\geqslant 1$ and d_7 can be taken zero, but that for $r>2$ the term $\sum_{j-r\leqslant -1}$ is always of the order m^{-1}.)

By using (3.12) and (3.13) we see that there exists $d_9>0$ such that for sufficiently large α,

$$\int_{x_*}^{\mu} f(v)[1-\gamma(v-x_*)]^{\eta}\, dv \leqslant \int_{x_*}^{\infty} f(v)\, dv + d_9 m^{-1}. \tag{3.14}$$

Some easy estimates show that for sufficiently large α

$$\frac{m^r(1+\delta)^r}{b_1(m)} \leqslant \frac{1}{f(m)} + d_{10} m^{r-1}, \tag{3.15}$$

where d_{10} is independent of α. Thus there exists a constant d_{11} such that

$$\frac{m^r(1+\delta)^r}{b_1(m)} \int_{x_*}^{\mu} f(v)[1-\gamma(v-x_*)]^{\eta}\, dv \leqslant \frac{1}{f(m)} \int_{x_*}^{\infty} f(v)\, dv + d_{11} m^{r-1}. \tag{3.16}$$

Combining (3.2), (3.6), and (3.16) completes the proof. □

If $(\alpha,x)\in\Gamma$, define $\psi_0(s)=x(q-1+s)$ (where $q=q(\alpha,x)$) and define ψ_1 and ψ_2 by

$$\psi_1(t)=\alpha\int_0^t f(\psi_0(s))\, ds, \qquad \psi_2(t)=\alpha\int_0^t f(\psi_1(s))\, ds \tag{3.17}$$

for $0\leqslant t\leqslant 1$, so

$$x(2)=\psi_1(1)-\psi_2(1). \tag{3.18}$$

To obtain an upper bound on $x(2)$ we need a lower bound on $\psi_2(1)$, and we shall obtain this by modifying the proof of Lemma 2.2 in [6].

PROPOSITION 3.2. *Assume that f satisfies* H1 *and that $r>2$ and $\sigma\geqslant 1$. There exist $\alpha_1>0$ and $c_1>0$ such that if $(\alpha,x)\in\Gamma$ and $\alpha\geqslant\alpha_1$ and $\psi_2(t)$ is defined by equation* (3.17), *then*

(3.19) $$\psi_2(1) \geqslant \frac{1}{f(m)} \int_0^\infty f(v)\,dv - c_1 m^{r-1}.$$

Proof. As before, assume that $a=1$ in inequality (1.1) and define $b_2(x) = 1+dx^{-\sigma}$, where d and σ are as in (1.1). If $(\alpha, x) \in \Gamma$, define $q=q(\alpha,x)$ and $m=x(q-1)$, and define δ_1 to be the smallest $\delta_1>0$ such that $x(q+\delta_1) = -x_*$. It is proved in Lemma 1.2 of [6] that there exist constants c_2 and c_3 such that for sufficiently large α

(3.20) $$c_2 m^{-1} \leqslant \delta_1 \leqslant c_3 m^{-1}.$$

If one uses (3.20) and equation (2.28) of [6], one can easily show that for sufficiently large α

(3.21) $$0 \leqslant x_* - \alpha\delta_1 f(m) \leqslant c_4 m^{-1},$$

where c_4 is independent of α. Inequality (3.21) and inequality (2.27) of [6] imply that there exists a constant c_5 such that for large enough α,

(3.22) $$\alpha \int_0^{\delta_1} f(\psi_1(s))\,ds \geqslant \frac{1}{f(m)} \int_0^{x_*} f(v)\,dv - c_5 m^{r-1}.$$

If $(\alpha, x) \in \Gamma$, define ν, μ, and η by

$$\nu = -\frac{m-x_*}{1-\delta}, \quad \mu = m+\delta_1\nu, \quad \text{and} \quad \eta = -\left(\frac{r}{r-1}\right).$$

Lemma 1.2 of [6] implies that there exists $c_6>0$ such that for all large enough α,

(3.23) $$m \geqslant \mu \geqslant m-c_6 \quad \text{and} \quad m \geqslant -\nu \geqslant m-c_6.$$

In Lemma 2.2 of [6] it is assumed that $1<r<2$, but an examination of the argument giving equation (2.46) of [6] shows that it is valid for $r>2$. Thus there exist positive constants c_7 and k such that for sufficiently large α

(3.24) $$\begin{aligned}\alpha \int_{\delta_1}^{1-\delta} f(\psi_1(t))\,dt &\geqslant \frac{\mu^r}{\beta} \int_{x_*}^{x_*+km} f(w)\,dw \\ &\quad + \frac{\mu^r}{\beta} \sum_{j=1}^{\infty} \binom{\eta}{j} \gamma^j \int_{x_*}^{x_*+km} f(w)(w-x_*)^j\,dw.\end{aligned}$$

In equation (3.24), δ is as in the proof of Proposition 3.1, $\beta \equiv b_2(c_7 m)$, and

(3.25) $$\gamma \equiv \frac{(r-1)|\nu|\mu^{r-1}}{\alpha b_1(c_7 m)},$$

where $b_1(x) = 1-dx^{-\sigma}$ and d is as in equation (1.1). The constant k can also be chosen so that for all sufficiently large α

(3.26) $$\gamma(x_*+km) \leqslant k_1 < 1,$$

where k_1 is independent of α (see equation (2.45) in [6]).

The same sort of argument used in Proposition 3.1 shows that there is a constant c_8 such that

$$\left| \sum_{j=1}^{\infty} \binom{\eta}{j} \gamma^j \int_{x_*}^{x_*+km} f(w)(w-x_*)^j \, dw \right| \leqslant c_8 m^{-1}. \tag{3.27}$$

Notice that it is at this point that the fact that $r>2$ enters: just as in the proof of Proposition 3.1 there will be integers $j \geqslant 1$ such that $j-r \leqslant -1$.

If we use (3.23) and (3.27), we obtain from (3.24) that there exists a positive constant c_9 such that for large α,

$$\alpha \int_{\delta_1}^{1-\delta} f(\psi_1(t)) \, dt \geqslant \frac{\mu^r}{\beta} \int_{x_*}^{\infty} f(w) \, dw - c_9 m^{r-1}. \tag{3.28}$$

It is easy to show (using (3.23) and the fact that $\sigma \geqslant 1$) that there exists a constant c_{10} such that for large enough α

$$\frac{\mu^r}{\beta} \geqslant f(m)^{-1} - c_{10} m^{r-1}, \tag{3.29}$$

and we obtain from (3.28) and (3.29) that there exists a constant c_1 such that

$$\alpha \int_{\delta_1}^{1-\delta} f(\psi_1(t)) \, dt \geqslant (f(m))^{-1} \int_{x_*}^{\infty} f(w) \, dw - c_{11} m^{r-1}. \tag{3.30}$$

Inequalities (3.22) and (3.30) imply that for large enough α

$$\psi_2(1) > \alpha \int_0^{1-\delta} f(\psi_1(t)) \, dt \geqslant \frac{1}{f(m)} \int_0^{\infty} f(w) \, dw - (c_5 + c_{11}) m^{r-1}, \tag{3.31}$$

which is the desired result. □

As an immediate corollary of Propositions 3.1 and 3.2, we obtain the upper bound in equation (2.7).

COROLLARY 3.1. *Assume that f satisfies* H1 *and that $r>2$ and $\sigma \geqslant 1$* (*r and σ as in* H1). *There exists $\alpha_1 > 0$ and a constant $d>0$ such that if* $(\alpha, x) \in \Gamma$ (Γ *as in equation* (1.19)) *and $\alpha \geqslant \alpha_1$, then*

$$x(2) \leqslant d m^{r-1}, \tag{3.32}$$

where $m = x(q-1)$ and $q = q(\alpha, x)$.

Proof. Propositions 3.1 and 3.2 show that there are constants c and c_1 such that for $(\alpha, x) \in \Gamma$ and $\alpha \geqslant \alpha_1$,

$$\begin{aligned} x(2) = \psi_1(1) - \psi_2(1) &\leqslant \left(\frac{1}{f(m)} \int_0^{\infty} f(w) \, dw + c m^{r-1} \right) \\ &\quad - \left(\frac{1}{f(m)} \int_0^{\infty} f(w) \, dw - c_1 m^{r-1} \right) = (c + c_1) m^{r-1}. \end{aligned}$$

which is the desired estimate. □

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