DECOMPOSITIONS AND APPROXIMATE FIBRATIONS

R. J. Daverman and L. S. Husch

1. Introduction. In this paper we investigate upper semi-continuous (= u.s.c.) decompositions $\mathcal{G}$ of manifolds M (without boundary) into continua having the shape of closed manifolds of some fixed dimension $k>0$. The fundamental problem considered is the extent to which the decomposition map $p: M \to M/\mathcal{G}$ is an approximate fibration. Coram and Duvall [3] initiated this type of investigation when they considered decompositions of the 3-sphere into 1-spheres satisfying several additional restrictions, including that the decomposition space be the 2-sphere, and they showed that the decomposition map was an approximate fibration over the complement of at most two points. Our main result in §2 (Theorem 2.10) is that if $\mathcal{G}$ is an u.s.c. decomposition of M as above and if the decomposition space is finite dimensional, then there exists a dense open subset $U \subseteq M/\mathcal{G}$ such that the restriction $p \mid p^{-1}(U): p^{-1}(U) \to U$ is an approximate fibration. From §3, it follows that U is a generalized manifold.

Daverman [5] showed that if the dimension of M is $k+1$ then $M/\mathcal{G}$ is a 1-dimensional manifold, and if each element of $\mathcal{G}$ is a locally flat submanifold of M then p is an approximate fibration, provided $M/\mathcal{G}$ has empty boundary. He also constructed examples to show that either local flatness or some condition on the relationship between the fundamental groups of M and the elements of $\mathcal{G}$ is needed in order to show that p is an approximate fibration. Supposing that each element of $\mathcal{G}$ has the shape of a closed k-dimensional manifold, and that $M/\mathcal{G}$ is, homeomorphic to $\mathbf{R}^1$, then we show (Theorem 5.15) that the inclusion of each element of $\mathcal{G}$ into M is a homology equivalence; this generalizes Lemma 6.2 of [5]. Furthermore, if the inclusion-induced $\check{\pi}_1(g) \to \pi_1(M)$ is an isomorphism for each $g \in \mathcal{G}$ and the integral group ring of $\pi_1(M)$ is Noetherian, then we show (Theorem 5.16) that p is an approximate fibration.

The value of knowing that p is an approximate fibration, for example in the latter case, is that p can be approximated by locally trivial bundle maps and hence M can be expressed as a product $N \times \mathbf{R}$, where N is a closed k-manifold which has the shape of the elements of $\mathcal{G}$.

We refer the reader to [2] for the definition of approximate fibrations and their properties. We use the Mardešić–Segal approach to shape theory [11], although we use Borsuk's terminology of FANR (= fundamental absolute neighborhood retract) rather than Mardešić–Segal's term ASNR (= absolute shape neighborhood retract). A fundamental property of FANR's which we often employ in this paper is that if $\{U_i\}_{i=1}^{\infty}$ is a nested sequence of neighborhoods of an FANR X in an ANR such that $\bigcap_{i=1}^{\infty} U_i = X$, then the induced inverse systems of homology and homotopy groups, $\{H_*(U_i)\}_{i=1}^{\infty}$ and $\{\pi_*(U_i)\}_{i=1}^{\infty}$, are stable; that

---

Received September 23, 1983. Revision received February 6, 1984.
Research of the first author was supported in part by NSF Grant MCS 81-20741.
Michigan Math. J. 31 (1984).

is, there exist subsequences—say, $\{H_j(U_{i(n)})\}$ of $\{H_j(U_i)\}$, with bonding maps $\tau_n: H_j(U_{i(n+1)}) \to H_j(U_{i(n)})$—such that $\tau_n \mid \text{image}\, \tau_{n+1}: \text{image}\, \tau_{n+1} \to \text{image}\, \tau_n$, and the natural homomorphisms $\varprojlim_i \{H_j(U_i)\} \to \text{image}\, \tau_n$ are isomorphisms for each n. This constitutes a practical working definition; it is a direct consequence of the more usual, equivalent formulation, in which a progroup G is said to be *stable* if G is isomorphic in the category of progroups to a group.

The following was shown in [8, Theorem 6].

PROPOSITION 1.1. *Suppose that we have a commutative diagram of groups and homomorphisms*

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\cdots \longrightarrow & & A_{i,j} & \longrightarrow & B_{i,j} & \longrightarrow & C_{i,j} & \longrightarrow & A_{i-1,j} & \longrightarrow \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\cdots \longrightarrow & & A_{i,j+1} & \longrightarrow & B_{i,j+1} & \longrightarrow & C_{i,j+1} & \longrightarrow & A_{i-1,j+1} & \longrightarrow \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
 & & \vdots & & \vdots & & \vdots & & \vdots &
\end{array}$$

such that for $i, j \leqslant -1$, $A_{i,j} = B_{i,j} = C_{i,j} =$ trivial group, the rows are exact and the columns corresponding to the $A_{i,j}$'s and the $C_{i,j}$'s are stable. Then the columns corresponding to the $B_{i,j}$'s are also stable and the induced sequence

$$\cdots \to \varprojlim_j \{A_{i,j}\} \to \varprojlim_j \{B_{i,j}\} \to \varprojlim_j \{C_{i,j}\} \to \varprojlim_j \{A_{i-1,j}\} \to \cdots$$

is exact.

$H_*(X; \mathcal{B})$ will denote singular homology with local coefficients [15, p. 223; 16]. If X and $\{U_i\}_{i=1}^{\infty}$ are as above, then $\check{H}_*(X; \mathcal{B}) = \varprojlim_i \{H_*(U_i; \mathcal{B})\}$ where $\mathcal{B}$ denotes, for each i, the restriction of $\mathcal{B}$ to U_i. If $x \in X$, then $\check{\pi}_1(X, x) = \varprojlim_i \{\pi_1(U_i, x)\}$.

2. Approximate fibrations. Let M be a connected n-dimensional manifold without boundary and let $\mathcal{G}$ be a u.s.c. decomposition of M into continua which have the shape of closed k-dimensional manifolds. Let $p: M \to M/\mathcal{G}$ be the natural decomposition map. Given $\mathcal{H} \subseteq \mathcal{G}$, we use the notation $p(\mathcal{H})$ for $\{p(h) \mid h \in \mathcal{H}\}$. Let $\mathcal{B}$ be a bundle of local coefficients on M.

For each $g \in \mathcal{G}$ let $N(g)$ be a closed connected k-dimensional manifold which has the shape of g. Since $N(g)$ is an ANR, g is an FANR [11]. Hence, there exists a sequence of open connected saturated neighborhoods of g in M, $\{U(g, i)\}$, and continuous maps $\alpha_i^g: N(g) \to U(g, i)$ and $\beta_i^g: U(g, i) \to N(g)$ such that

(i) $U(g, i) \supseteq U(g, i+1)$ for each i,

(ii) $\bigcap_{i=1}^{\infty} U(g, i) = g$,

(iii) the inclusion $\mu_i^g: U(g, i+1) \to U(g, i)$ is homotopic to $\alpha_i^g \beta_{i+1}^g$, and

(iv) $\beta_i^g \alpha_i^g$ is homotopic to the identity on $N(g)$.

Let $\mu_{i*}^g : H_j(U(g,i+1);\mathcal{B}) \to H_j(U(g,i);\mathcal{B})$ denote the homomorphism induced by μ_i^g. To simplify notation, we have used $\mathcal{B}$, rather than $\mathcal{B} \mid U(g,i+1)$, to denote the restriction of $\mathcal{B}$ to $U(g,i+1)$. As mentioned in the introduction, we have the following.

PROPOSITION 2.1. *The inverse system* $\{H_j(U(g,i);\mathcal{B})\}$ *is stable and the natural homomorphism* $\kappa_i^g : \check{H}_j(g;\mathcal{B}) = \varprojlim H_j(U(g,i);\mathcal{B})\} \to H_j(U(g,i);\mathcal{B})$ *is an isomorphism onto* image μ_{i*}^g. *Furthermore,* $\alpha_{i*}^g : H_j(N(g);\, \alpha_i^{g*}\mathcal{B}) \to H_j(U(g,i);\mathcal{B})$ *is also an isomorphism onto* image μ_{i*}^g.

If we choose base points in g and $N(g)$, then we may assume that all the above maps and homotopies are base point preserving [10]. If we let $\mu_{i*}^g : \pi_j(U(g,i+1)) \to \pi_j(U(g,i))$ denote the homomorphism induced by μ_i^g, then we have the following.

PROPOSITION 2.2. *Proposition* 2.1 *remains valid if we replace homology groups by homotopy groups.*

Let $h \in \mathcal{G}$ such that $h \subseteq U(g,r)$ for some $r > 1$ and choose s such that $U(h,s) \subseteq U(g,r)$. Let i be the latter inclusion and let $\tau(g,h) = \beta_r^g i \alpha_s^h : N(h) \to N(g)$. Note that $\tau(g,h)$ is independent of r and s; that is, if $\bar{i}$: $U(h,u) \subseteq U(g,v)$, then $\beta_r^g i \alpha_s^h$ is homotopic to $\beta_v^g \bar{i} \alpha_u^h$.

PROPOSITION 2.3. *If* $g,h,k \in \mathcal{G}$ *such that* $U(k,t) \subseteq U(h,s+1) \subseteq U(h,s) \subseteq U(g,r)$, *then* $\tau(g,k)$ *is homotopic to the composition* $\tau(g,h) \circ \tau(h,k)$.

Proof. Consider the homotopy commutative diagram

$$\begin{array}{ccccccc}
U(g,r) & \supseteq & U(h,s) & \supseteq & U(h,s+1) & \supseteq & U(k,t) \\
\downarrow \beta_r^g & & \nwarrow \alpha_s^h & & \swarrow \beta_{s+1}^h & & \uparrow \alpha_t^k \\
N(g) & \xleftarrow{\tau(g,h)} & & N(h) & & \xleftarrow{\tau(h,k)} & N(k).
\end{array}$$

□

PROPOSITION 2.4. *For all* $g \in \mathcal{G}$, $\tau(g,g)$ *is homotopic to the identity.*

Let $\bar{\tau}(g,h)_* : \check{H}_j(h;\mathcal{B}) \to \check{H}_j(g;\mathcal{B})$ be the composition

$$\check{H}_j(h;\mathcal{B}) \xrightarrow{\kappa_s^h} H_j(U(h,s);\mathcal{B}) \xrightarrow{i_*} H_j(U(g,r);\mathcal{B})$$
$$\xrightarrow{\mu_{r-1*}^g} \text{image } \mu_{r-1*}^g \xrightarrow{(\kappa_{r-1}^g)^{-1}} \check{H}_j(g;\mathcal{B}).$$

PROPOSITION 2.5. *The following diagram is commutative:*

$$\begin{array}{ccc}
\check{H}_j(h;\mathcal{B}) & \xrightarrow{\bar{\tau}(g,h)_*} & \check{H}_j(g;\mathcal{B}) \\
(\alpha_s^h)_*^{-1} \circ \kappa_s^h \downarrow & & \downarrow (\alpha_{r-1}^g)_*^{-1} \circ \kappa_{r-1}^g \\
H_j(N(h);\alpha_s^{h*}\mathcal{B}) & \xrightarrow{\tau(g,h)_*} & H_j(N(g);\alpha_{r-1}^{g*}\mathcal{B}).
\end{array}$$

Let $\mathcal{S} = \{g \in \mathcal{G} \mid$ for each neighborhood U of g in M there exists $h \in \mathcal{G}$ such that $h \subseteq U$ and $\tau(g,h)$ is not a homotopy equivalence$\}$.

PROPOSITION 2.6. *$p(\mathcal{S})$ is a closed nowhere dense subset of $M/\mathcal{G}$.*

Proof. Let $g\in\mathcal{G}$ such that $p(g)\in \mathrm{cl}(p(\mathcal{S}))$, but suppose that $g\notin\mathcal{S}$. Hence there exists a neighborhood U of g such that for all $h\in\mathcal{G}$ and $h\subseteq U$, $\tau(g,h)$ is a homotopy equivalence. Choose r such that $U(g,r)\subseteq U$, and choose $h\in\mathcal{S}$ such that, for some s, $U(h,s)\subseteq U(g,r)$. By definition, there exists $k\in\mathcal{G}$ such that $k\subseteq U(h,s+1)$ but $\tau(h,k)$ is not a homotopy equivalence. By construction, $\tau(g,h)$ and $\tau(g,k)$ are homotopy equivalences. It follows from Proposition 2.3 that $\tau(h,k)$ is a homotopy equivalence. This contradiction implies that $p(\mathcal{S})$ is closed in $M/\mathcal{G}$.

Now suppose that $U=\mathrm{int}\,p(\mathcal{S})\neq\varnothing$. Let $g_1\in\mathcal{G}$ such that $g_1\subseteq p^{-1}(U)$. In the intersection $p^{-1}(U)\cap U(g_1,2)$ find $g_2\in\mathcal{G}$ such that $U(g_2,r_2)\subseteq p^{-1}(U)\cap U(g,2)$ and $\tau(g_1,g_2)$ is not a homotopy equivalence. Inductively, find $g_{i+1}\in\mathcal{G}$ such that $U(g_{i+1},r_{i+1})\subseteq U(g_i,r_i+1)$ and $\tau(g_i,g_{i+1})$ is not a homotopy equivalence. Without loss of generality we may suppose that $\bigcap_{i=1}^{\infty} U(g_i,r_i)=g\in\mathcal{G}$.

Choose i and s such that $U(g,2)\supseteq U(g_i,r_i)$ and $U(g_{i+1},r_{i+1}+1)\supseteq U(g,s)$. By Propositions 2.3 and 2.4, the composition

$$\tau=\tau(g,g_i)\circ\tau(g_i,g_{i+1})\circ\tau(g_{i+1},g)$$

is homotopic to the identity on $N(g)$.

If $N(g)$ is orientable, then $H_k(N(g))=\mathbf{Z}$ using integer coefficients; it follows that $H_k(N(g_i))$ and $H_k(N(g_{i+1}))$ are non-trivial. Hence $N(g_i)$ and $N(g_{i+1})$ are also orientable. Since $\tau(g,g_i)\circ\tau(g_i,g_{i+1})\circ\tau(g_{i+1},g)$ induces the identity on $H_k(N(g))$, each of $\tau(g,g_i)$, $\tau(g_i,g_{i+1})$ and $\tau(g_{i+1},g)$ induces isomorphisms on the kth homology.

Now assume that $N(g)$ is non-orientable and let $\phi:\widetilde{N(g)}\to N(g)$ be the orientable double covering. If $N(g_i)$ were orientable, then $\tau(g,g_i)$ could be lifted to $\widetilde{N(g)}$ and the homomorphism of fundamental groups induced by $\tau(g,g_i)$ would not be onto, which would give us a contradiction. Similarly, $N(g_{i+1})$ is non-orientable.

We next claim that the pull-back of ϕ by $\tau(g,g_i)$, $\phi_i:\widetilde{N(g_i)}\to N(g_i)$, is the orientation double covering of $N(g_i)$. Consider the various pull-backs

$$\begin{array}{ccccccc}
\widetilde{N(g)} & \xleftarrow{\tau_1} & \widetilde{N(g_i)} & \xleftarrow{\tau_2} & \widetilde{N(g_{i+1})} & \xleftarrow{\tau_3} & \widetilde{N(g)} \\
\phi\downarrow & & \phi_1\downarrow & & \phi_2\downarrow & & \phi_3\downarrow \\
N(g) & \longleftarrow & N(g_i) & \longleftarrow & N(g_{i+1}) & \longleftarrow & N(g).
\end{array}$$

Since $\tau_3\circ\tau_2\circ\tau_1$ is homotopic to the identity on $\widetilde{N(g)}$, we can use the argument from above to show that $\widetilde{N(g_i)}$ is orientable. Hence, ϕ_1 is the orientation double covering. Similarly, $\phi_2:\widetilde{N(g_{i+1})}\to N(g_{i+1})$ is the orientation double covering. Note that we have used the fact that since τ is homotopic to the identity, ϕ_3 is equivalent to ϕ.

Let $\mathbf{Z}^t$ be the coefficient bundle of twisted integers on $N(g)$. From above, it follows that the pull-backs $\tau(g,g_i)*\mathbf{Z}^t$ and $(\tau(g,g_i)\circ\tau(g_i,g_{i+1}))*\mathbf{Z}^t$ are the coefficient bundles of twisted integers on $N(g_i)$ and $N(g_{i+1})$ respectively. Hence, if we use these coefficients, $H_k(N(g))=\mathbf{Z}$ and each of $\tau(g,g_i)$, $\tau(g_i,g_{i+1})$, and $\tau(g_{i+1},g)$ induces isomorphisms on the kth homology.

Therefore, in either case ($N(g)$ orientable or non-orientable), each of $\tau(g,g_i)$, $\tau(g_i,g_{i+1})$, and $\tau(g_{i+1},g)$ is a degree one map.

Since $\tau(g_{i+1},g)$ is degree one, the induced homomorphism of fundamental groups $\pi_1(N(g))\to\pi_1(N(g_{i+1}))$ is onto, and therefore, since τ is homotopic to the identity, is an isomorphism. It then easily follows that both $\tau(g_i,g_{i+1})$ and $\tau(g,g_i)$ also induce isomorphisms on fundamental groups.

Let Λ be the integral group ring of these groups. If $N(g)$ is orientable let us now use Λ as a coefficient bundle for homology, and if $N(g)$ is non-orientable let us use $\Lambda\otimes\mathbf{Z}^t$ as a coefficient bundle [15, p. 223]. By [15, Lemma 2.1], the degree one maps $\tau(g,g_i)$, $\tau(g_i,g_{i+1})$, and $\tau(g_{i+1},g)$ induce epimorphisms in all homology groups. Again, using the fact that τ is homotopic to the identity, one can show that $\tau(g_{i+1},g)$, $\tau(g_i,g_{i+1})$, and then $\tau(g,g_i)$ induce isomorphisms of all homology groups. With our choice of coefficients, the Whitehead theorem [17, Theorem 1] attests that the induced maps on the universal coverings of $N(g)$, $N(g_i)$, and $N(g_{i+1})$ are homotopy equivalences. It then follows that in particular $\tau(g_i,g_{i+1})$ is a homotopy equivalence, and we have a contradiction. Therefore $p(\mathcal{S})$ is nowhere dense in $M/\mathcal{G}$, and the proof of Proposition 2.6 is completed. □

Dydak and Segal [6] have introduced the following definition. A map $f: X\to Y$ is *homology r-stable* if, given $y\in Y$ and a neighborhood U of y in Y, there exists a neighborhood V of y contained in U such that for all $z\in V$:

(a) the natural homomorphism $\check{H}_i(f^{-1}(z))\to H_i(f^{-1}(V))$ is a monomorphism for all $i<r$, and

(b) the image of $H_i(f^{-1}(V))\to H_i(f^{-1}(U))$ is equal to the image of $\check{H}_i(f^{-1}(z))\to H_i(f^{-1}(U))$ for all $i\leqslant r$.

Here all homology groups have integral coefficients and $\check{H}_i(f^{-1}(z))=\varprojlim\{H_i(f^{-1}(W))\mid W$ is a neighborhood of z in Y and the bonding maps are induced by inclusions$\}$. We have the following result from Dydak and Segal [6, cf. Corollary 4.9].

THEOREM 2.7. *Let $f: X\to Y$ be a closed surjection of metrizable spaces such that $X\in LC^{r+1}$, Y is complete, f is homology r-stable, and for each $y\in Y$, $f^{-1}(y)$ is an* FANR. *Then $Y\in LC^{r+1}$.*

PROPOSITION 2.8. $p(\mathcal{G}-\mathcal{S})\in LC^r$ *for each r.*

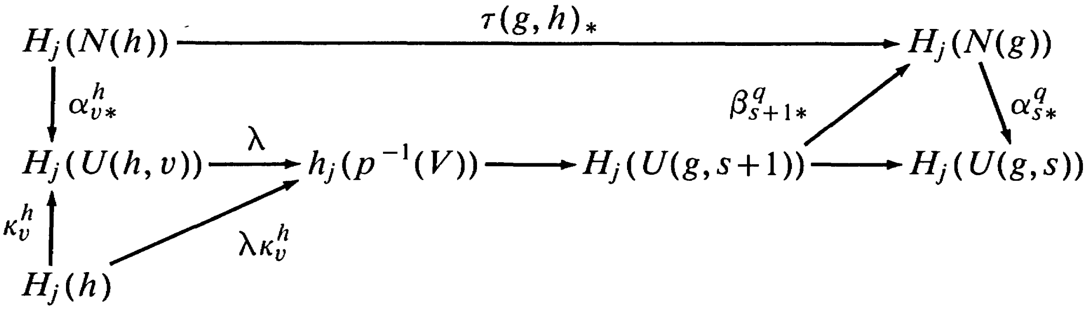
Proof. By the previous theorem, it suffices to show that $p\mid p^{-1}(p(\mathcal{G}-\mathcal{S}))$ is homology r-stable for each r. This follows directly from the definitions; however, for completeness we include the proof. Suppose U is a neighborhood of $p(g)$ in $p(\mathcal{G}-\mathcal{S})$ for some $g\in\mathcal{G}-\mathcal{S}$. Let V be a neighborhood of $p(g)$ such that $p^{-1}(V)\subseteq U(g,s+1)\subseteq U(g,s)\subseteq p^{-1}(U)$, for some s for which all $h\in\mathcal{G}$ with $h\subseteq U(g,s)$ give rise to a homotopy equivalence $\tau(g,h)$. Choose t such that $U(g,t)\subseteq p^{-1}(V)$.

From Proposition 2.1,

$$\mu^g_{s*}\,\mu^g_{s-1*}\cdots\mu^g_{t*}\mid \text{image}\,\mu^g_{t+1*}:\text{image}\,\mu^g_{t+1*}\to\text{image}\,\mu^g_{s*}$$

is an isomorphism. It follows that the image $H_j(p^{-1}(V))\to H_j(U(g,s))$ is precisely image μ^g_{s*}. By Proposition 2.1, the natural homomorphism $\check{H}_j(g)\to H_j(U(g,s))$ and $\alpha^g_{s*}:H_j(N(g))\to H_j(U(g,s))$ are isomorphisms onto image μ^g_{s*}.

Let $h\in\mathcal{G}$ such that $h\subseteq p^{-1}(V)$; choose v such that $U(h,v)\subseteq p^{-1}(V)$. Consider the commutative diagram



where all unnamed homomorphisms and λ are induced by inclusion. By Proposition 2.1, κ^h_v and α^h_{v*} are isomorphisms onto image u^h_{v*}. Since $\tau(g,h)_*$ is an isomorphism, $\lambda\mid\text{image}\,\alpha^h_{v*}$ is one-to-one; hence $\lambda\kappa^h_v$ is one-to-one and we have shown (a) of the definition of homology r-stable.

To see (b), note that by using the above commutative diagram, the image of $\check{H}_j(h)$ in $H_j(U(g,s))$ is image$(\alpha^g_{s*}\circ\tau(g,h))=\text{image}\,\alpha^g_{s*}$ which (from above) is the image of $H_j(p^{-1}(V))$ in $H_j(U(g,s))$. □

Note that if, in the definition of homology r-stable, we replace the homology groups by homotopy groups, then we have the definition of an *r-movable map* given by Coram and Duvall in [4]. Using an argument similar to that given in the proof of Proposition 2.8, we have the following.

PROPOSITION 2.9. *$p\mid p^{-1}(p(\mathcal{G}-\mathcal{S}))$ is an r-movable map for each r.*

THEOREM 2.10. *Suppose M is a connected n-manifold without boundary, $\mathcal{G}$ is a* u.s.c. *decomposition of M into continua having the shape of closed connected k-manifolds such that $M/\mathcal{G}$ is finite dimensional, and $p:M\to M/\mathcal{G}$ is the natural map. Then there exists a dense open subset $U\subseteq M/\mathcal{G}$ such that $p\mid p^{-1}(U):p^{-1}(U)\to U$ is an approximate fibration.*

Proof. Let $U=p(\mathcal{G}-\mathcal{S})$. Since $M/\mathcal{G}$ is finite dimensional, it follows from Proposition 2.8 and [1, p. 124] that U is an ANR. By Proposition 2.9, $p\mid p^{-1}(U)$ is an r-movable map for all r and the conclusion follows from [4, Corollary 3.4]. □

As mentioned in the introduction, examples of such decompositions $\mathcal{G}$, for which $p:M\to M/\mathcal{G}$ is not an approximate fibration over all of $M/\mathcal{G}$, can be found in [5, §5]. Coram and Duvall [4, p. 240] have given examples with the same effect, involving decompositions $\mathcal{G}$ of $M=S^{2k+1}$ into k-spheres such that $M/\mathcal{G}\approx S^{k+1}$. We want to exhibit an elementary example: if N is any closed, connected k-manifold not homotopy equivalent to S^k, and $\mathcal{G}$ is the decomposition of $M=N\times\mathbf{R}^{k+1}$ into $N\times\{0\}$ and the spheres $\{q\}\times rS^k$, $q\in N$, and $r>0$, then $p: M\to M/\mathcal{G}$ cannot be an approximate fibration everywhere, because the fibers do not all have the same shape.

3. Approximate fibrations and generalized manifolds. The main result of this section is the following theorem.

THEOREM 3.1. *Let $p: M\to B$ be a proper map which is an approximate fibration of the connected m-manifold (without boundary) onto an* ANR *B. Then B is a k-dimensional generalized manifold over $\mathbf{Z}$; moreover, if M is orientable, then the fiber of p has the shape of a Poincaré duality space of formal dimension $m-k$.*

Our proof is motivated by the corresponding result for Hurewicz fibrations proved by Raymond [14] (see also [13]). A version of the above theorem was stated without proof by Quinn in [12].

We refer the reader to [13] for a definition of generalized manifold. We shall actually show that B is a singular homology k-manifold over $\mathbf{Z}$ and then apply Proposition 3.4 of [14] to deduce that B is a generalized manifold.

A space B is a singular homology k-manifold over $\mathbf{Z}$ if B is a Hausdorff space such that

(A) $$H_r(B,B-\{y\})=\begin{cases}\mathbf{Z} & r=k\\ 0 & r\neq k\end{cases}\quad\text{for all } y\in B;$$

(B) there exists a covering of B by open sets $\{U_\alpha\}$ such that the inclusion-induced homomorphism $H_k(B,B-\mathrm{cl}(U))\to H_k(B,B-\{y\})$ is an isomorphism for all $y\in U_\alpha$; and

(C) B has finite cohomological dimension over $\mathbf{Z}$; that is, there exists an integer l such that $H_c^j(U)=0$ for all open subsets U of B and all $j\geqslant l$.

Let us now assume that M is orientable and that the fibers of p have the shape of a compact connected polyhedron F. Let $x\in B$ and let U_0 be a neighborhood of x. As in §2, there exist a sequence of neighborhoods $U(p^{-1}(x),i)$ and maps $\alpha_i: U(p^{-1}(x),i)\to F$ and $\beta_i: F\to U(p^{-1}(x),i)$ having similar properties to those in §2. Suppose that $U(p^{-1}(x),1)\subseteq p^{-1}(U_0)$ and choose a neighborhood U of x such that $p^{-1}(U)\subseteq U(p^{-1}(x),2)$. By local contractibility, choose a connected neighborhood V of x such that V contracts to x in U. Let $\alpha: p^{-1}(V)\to F$ be the restriction of α_2. The result stated below is well known [9].

PROPOSITION 3.2. *The map $p\times\alpha: p^{-1}(V)\to V\times F$ induces isomorphisms of homology groups $H_*(p^{-1}(V),p^{-1}(V-\{x\}))\to H_*(V\times F,(V-\{x\})\times F)$.*

By duality,

$$\mathbf{Z}=H^0(p^{-1}(x))=H_m(p^{-1}(V),p^{-1}(V-\{x\}))=H_m(V\times F,(V-\{x\})\times F)$$
$$=\left[\bigoplus_{i=1}^{m}H_{m-i}(V,V-\{x\})\otimes H_i(F)\right]\oplus[\text{torsion}].$$

Hence, there exists k such that

$$H_i(V, V-\{x\}) \otimes H_{m-i}(F) = \begin{cases} \mathbf{Z} & i=k \\ 0 & i \neq k. \end{cases}$$

Let γ and β be generators of $H_k(V, V-\{x\})$ and $H_{m-k}(F)$, respectively.

Letting $V_1 = V$, choose a nested sequence of open neighborhoods $\{V_i\}_{i=1}^{\infty}$ of x in B such that $\bigcap_{i=1}^{\infty} V_i = \{x\}$ and V_{i+1} is contractible in V_i for each i. Let γ_i be the generator of $H_k(V_i, V_i-\{x\})$ which is sent by the inclusion map to $\gamma_1 = \gamma$.

Consider the following

$$\begin{array}{ccccc} H^j(F) & \xrightarrow{\pi_{i+1}^*} & H^j(V_{i+1} \times F) & \xrightarrow{\cap(\gamma_{i+1} \times \beta)} & H_{m-j}(V_{i+1} \times F, (V_{i+1}-\{x\}) \times F) \\ \| & \nwarrow \delta_i^* & \uparrow \mu_i^* & & \downarrow \mu_{i*} \\ H^j(F) & \xrightarrow{\pi_i^*} & H^j(V_i \times F) & \xrightarrow{\cap(\gamma_i \times \beta)} & H_{m-j}(V_i \times F, (V_i-\{x\}) \times F) \end{array}$$

where μ_i and δ_i are inclusions, π_i is projection, and $\cap(\gamma_i \times \beta)$ is the homomorphism induced by cap products. Note that using properties of cap products, one can show that the above diagram commutes up to sign; that is, $\mu_* \circ (\cap(\gamma_{i+1} \times \beta)) \circ \mu^* = \pm(\cap(\gamma_i \times \beta))$. Therefore, if we are careful in our choice of the γ_i's, then we may assume that the above diagram commutes. Taking limits, we have an induced homomorphism

$$D: H^j(F) \to \varprojlim_i \{H_{m-j}(V_i \times F, (V_i-\{x\}) \times F)\}.$$

Since each μ_{i*} is an isomorphism, we have a natural isomorphism

$$\mu_*: \varprojlim_i \{H_{m-j}(V_i \times F, (V_i-\{x\}) \times F)\} \to H_{m-j}(V \times F, (V-\{x\}) \times F)$$

and it is straightforward to check that $\mu_* D(w) = \gamma \times (w \cap \beta)$.

If $y \in p^{-1}(x)$, note that the inclusion maps induce isomorphisms

$$H_m(M, M-\{y\}) \cong H_m(M, M-p^{-1}(x)) \cong H_m(p^{-1}(V_i), p^{-1}(V_i-\{x\}));$$

let $\xi_i \in H_m(p^{-1}(V_i), p^{-1}(V_i-x))$ correspond to the orientation class of M. We have a commutative diagram

$$\begin{array}{ccc} H^j(p^{-1}(V_{i+1})) & \xrightarrow{\cap \xi_{i+1}} & H_{m-j}(p^{-1}(V_{i+1}), p^{-1}(V_{i+1}-\{x\})) \\ \uparrow \bar{\mu}_i^* & & \downarrow \bar{\mu}_{i*} \\ H^j(p^{-1}(V_i)) & \xrightarrow{\cap \xi_i} & H_{m-j}(p^{-1}(V_i), p^{-1}(V_i-\{x\})) \end{array}$$

which, when we take limits, gives us an isomorphism

$$\check{H}^j(p^{-1}(x)) \to \varprojlim_i H_{m-j}(p^{-1}(V_i), p^{-1}(V_i-\{x\})) \cong H_{m-j}(p^{-1}(V), p^{-1}(V-\{x\}))$$

which, of course, is Alexander duality. By using the map

$$p \times \alpha: (p^{-1}(V), p^{-1}(V-\{x\})) = (V \times F, (V-\{x\}) \times F)$$

and Proposition 3.2, one can show that the fact that the above limit homomorphism is bijective implies that D is an isomorphism.

From the Kunneth theorem we then obtain the facts that

$$\cap\beta: H^j(F)\to H_{m-k-j}(F)$$

is an isomorphism for each i, and that

$$\left[\bigoplus_{\substack{l=0\\ l\neq k}}^{m-j} H_{m-j-l}(V,V-\{x\})\otimes H_l(F)\right]\oplus[\text{torsion}]=0.$$

It follows that $H_j(V,V-\{x\})=0$ unless $j=k$, and we have shown condition (A). The proof of condition (B) is very similar to that given by Raymond for Hurewicz fibrations [14]. Condition (C) is a well-known consequence of (A) and the fact that B is an ANR. Therefore, B is a k-dimensional generalized manifold.

If M is non-orientable, then consider the composition $\tilde{M}\xrightarrow{\lambda}M\xrightarrow{p}B$, where λ is the orientation double covering. Note that $p\circ\lambda$ is also an approximate fibration.

If the fiber of p does not have the shape of a compact polyhedron, then consider the composition $M\times S^1\xrightarrow{\pi}M\xrightarrow{p}B$ where π is projection. The fiber of $p\circ\pi$ now has the shape of a compact polyhedron [7] and $p\circ\pi$ is still an approximate fibration.

If the fiber of p is not connected, consider the monotone-light factorization of p, $M\xrightarrow{\check{p}}\check{B}\xrightarrow{q}B$. It is easily seen that p is an approximate fibration and q is a covering map.

4. An exact sequence. In this section we construct an exact sequence, analogous to the Thom-Gysin sequence, for a subcontinuum of M having the shape of a manifold. (We emphasize the non-orientable case because it is more complicated than the orientable one.) This exact sequence will be used in §5 (Proposition 5.13) as a key part of our analysis of the homological equivalence in codimension one decompositions.

Let M be an open connected non-orientable n-manifold and let $g\subseteq M$ be a continuum with the shape of a closed non-orientable r-dimensional manifold N. Furthermore, suppose that if $\lambda:\tilde{M}\to M$ is the orientation double covering of M, then $\lambda^{-1}(g)$ is connected. Let $\{U_i\}_{i=1}^{\infty}$ be a sequence of compact connected n-dimensional submanifolds of M such that $U_{i+1}\subseteq$ interior U_i for each i, boundary U_i is locally flat in M, and $\bigcap_{i=1}^{\infty}U_i=g$. Note that for each i, $\lambda^{-1}(U_i)$ is connected and hence U_i is non-orientable. Let $\mathbf{Z}^t$ be the bundle of twisted integers on M; then the restriction of $\mathbf{Z}^t$ to U_i is the bundle of twisted integers on U_i for each i.

By Poincaré duality [15, p. 224], there exists a class $[U_i]\in H_n(U_i,\text{bdry }U_i;\mathbf{Z}^t)$ such that cap product with $[U_i]$ induces isomorphisms

$$\phi_i: H^{n-s}(U_i;\mathbf{Z})\to H_s(U_i,\text{bdry }U_i;\mathbf{Z}^t).$$

Consider

$$\begin{array}{ccccc}
H_s(M,M-U_i;\mathbf{Z}^t) & \xleftarrow{\alpha_i} & H_s(U_i,\text{bdry }U_i;\mathbf{Z}^t) & \xleftarrow{\phi_i} & H^{n-s}(U_i;\mathbf{Z})\\
\Big\downarrow{\scriptstyle\beta_i} & & & & \Big\downarrow{\scriptstyle\gamma_i}\\
H_s(M,M-U_{i+1};\mathbf{Z}^t) & \xleftarrow[\alpha_{i+1}]{} & H_s(U_{i+1},\text{bdry }U_{i+1};\mathbf{Z}^t) & \xleftarrow[\phi_{i+1}]{} & H^{n-s}(U_{i+1};\mathbf{Z})
\end{array}$$

where all homomorphisms except ϕ_i and ϕ_{i+1} are induced by inclusions. By excision, α_i and α_{i+1} are isomorphisms. If $[U_i]$ is chosen such that $\beta_i\alpha_i([U_i]) = \alpha_{i+1}[U_{i+1}]$, then it follows from properties of cap product that the above diagram commutes. [In order to see this, consider the following diagram:

$$\begin{array}{ccc} H_s(U_i, \text{bdry } U_i; \mathbf{Z}^t) & \xleftarrow{\phi_i} & H^{n-s}(U_i; \mathbf{Z}) \\ \downarrow \tau_1 & & \| \\ H_s(U_i, W_i; \mathbf{Z}^t) & \xleftarrow{\phi_i'} & H^{n-s}(U_i; \mathbf{Z}) \\ \downarrow \tau_2 & & \downarrow \gamma_i \\ H_s(U_{i+1}, \text{bdry } U_{i+1}; \mathbf{Z}^t) & \xleftarrow{\phi_{i+1}} & H^{n-s}(U_{i+1}; \mathbf{Z}) \end{array}$$

where $W_i = \text{cl}(U_i - U_{i+1})$, τ_1 and τ_2 are induced by inclusions, and ϕ_i' is cap product with $\tau_1([U_1])$]. Passing to direct limits, we obtain an isomorphism

$$\begin{aligned} \check{H}^{n-s}(g; \mathbf{Z}) = \varinjlim H^{n-s}(U_i; \mathbf{Z}) \to \varinjlim H_s(M, M - U_i; \mathbf{Z}^t) \\ = H_s(M, M-g; \mathbf{Z}^t). \end{aligned}$$

Since g has the shape of a non-orientable r-manifold, $\check{H}^{n-s}(g; \mathbf{Z})$ is isomorphic to $\check{H}_{r-n+s}(g; \mathbf{Z}^t)$. By excision, we have the following.

PROPOSITION 4.1. *For each neighborhood U of g in M, $H_s(U, U-g; \mathbf{Z}^t)$ is isomorphic to $\check{H}_{r-n+s}(g; \mathbf{Z}^t)$.*

Let $U \supseteq V$ be neighborhoods of g in M and consider the following commutative diagram:

$$\begin{array}{ccccccccc} \cdots \to & H_s(U-g; \mathbf{Z}^t) & \to & H_s(U; \mathbf{Z}^t) & \to & H_s(U, U-g; \mathbf{Z}^t) & \to & H_{s-1}(U-g; \mathbf{Z}^t) & \to \cdots \\ & \uparrow & & \uparrow & & \uparrow \alpha & & \uparrow & \\ \cdots \to & H_s(V-g; \mathbf{Z}^t) & \to & H_s(V; \mathbf{Z}^t) & \to & H_s(V, V-g; \mathbf{Z}^t) & \to & H_{s-1}(V-g; \mathbf{Z}^t) & \to \cdots. \end{array}$$

Note that α is an isomorphism by excision. Hence, if we take inverse limits using the neighborhood system of g in M, then $\varprojlim_U H_s(U, U-g; \mathbf{Z}^t) \cong \check{H}_{r-n+s}(g; \mathbf{Z}^t)$ and $\varprojlim_U H_s(U; \mathbf{Z}^t) \cong \check{H}_s(g; \mathbf{Z}^t)$. Therefore, $\{H_s(U-g; \mathbf{Z}^t)\}$ is stable and we have an exact sequence

$$\begin{aligned} \cdots \to \varprojlim \{H_s(U-g; \mathbf{Z}^t)\} \to H_s(g; \mathbf{Z}^t) \\ \to H_{r-n+s}(g; \mathbf{Z}^t) \to \varprojlim \{H_{s-1}(U-g; \mathbf{Z}^t)\} \to \cdots. \end{aligned}$$

We write $H_s(\text{end}_M(g); \mathbf{Z}^t)$ for $\varprojlim \{H_s(U-g; \mathbf{Z}^t)\}$, where we think of $\text{end}_M(g)$ as the end of M determined by g.

This establishes the non-orientable case in the following theorem. The orientable case is easier; Proposition 4.1 is well-known duality, and the rest of the argument carries over directly.

THEOREM 4.2. *Let M be an open connected n-dimensional manifold and let $g \subseteq M$ be a continuum such that g has the shape of a closed r-dimensional* *manifold N. In case M is orientable, suppose N is orientable and $\mathcal{B}$ is a product coefficient bundle over M; in case M is non-orientable, suppose N is non-orientable, the preimage of g in the orientation double covering of M is connected, and $\mathcal{B}$ is the twisted integers coefficient bundle over M. Then there exists an exact sequence*

$$\cdots \to H_s(\mathrm{end}_M(g);\mathcal{B}) \to \check{H}_s(g;\mathcal{B}) \to \check{H}_{r-n+s}(g;\mathcal{B}) \to H_{s-1}(\mathrm{end}_M(g);\mathcal{B}) \to \cdots.$$

5. Codimension one decompositions. In this section we focus attention on an u.s.c. decomposition $\mathcal{G}$ of an n-manifold M into continua having the shape of $(n-1)$-manifolds such that $M/\mathcal{G}$ is $\mathbf{R}^1$. The goal is to prove that the inclusion $g \to M$ induces homology isomorphisms for all $g \in \mathcal{G}$.

PROPOSITION 5.1. *If M is orientable, then each $g \in \mathcal{G}$ has the shape of an orientable manifold.*

Proof. By duality, $H_1(M, M-g)$ is isomorphic to $\check{H}^{n-1}(g)$ (using $\mathbf{Z}$-coefficients). If g had the shape of a non-orientable manifold, then $\check{H}^{n-1}(g)$ would be cyclic of order two. Hence, from the long exact sequence of the pair $(M, M-g)$, we would see that $H_0(M-g)$ is isomorphic to $H_0(M)$, which is isomorphic to $\mathbf{Z}$. Therefore g would not separate M, contradicting the fact that $p(g)$ separates $\mathbf{R}^1$. □

Our next aim is to prove the following non-orientable analogue of Proposition 5.1.

PROPOSITION 5.2. *If M is non-orientable, then each $g \in \mathcal{G}$ has the shape of a non-orientable manifold.*

In order to show this, we need to derive some other results. Let M be non-orientable and let $\lambda: \tilde{M} \to M$ be the orientable double covering of M. Let $\tilde{\mathcal{G}}$ be the decomposition of $\tilde{M}$ whose elements are the components of $\lambda^{-1}(g)$ for each $g \in \mathcal{G}$. Hence, λ induces an onto mapping $\bar{\lambda}: \tilde{M}/\tilde{\mathcal{G}} \to M/\mathcal{G}$. We claim that $\bar{\lambda}$ is one-to-one. Note that for $x \in M/\mathcal{G}$, $\bar{\lambda}^{-1}(x)$ contains either one or two points. By [5, Theorem 3.3′], $\tilde{M}/\tilde{\mathcal{G}}$ is a 1-manifold (possibly with boundary); hence, let $<$ designate order relations on $\tilde{M}/\tilde{\mathcal{G}}$ and $M/\mathcal{G}$ which are induced from $\mathbf{R}^1$. Now suppose that there exists $x \in M/\mathcal{G}$ such that $\lambda^{-1}(x) = \{y, z\}$, $y < z$. Choose $w \in \tilde{M}/\tilde{\mathcal{G}}$ such that $y < w < z$ and, without loss of generality, suppose that $\lambda(w) > x$. It follows that the image of $\bar{\lambda}$ is bounded above and we contradict the fact that λ is onto. Therefore $\bar{\lambda}$ is a homeomorphism. We have the following.

LEMMA 5.3. *For each $g \in \mathcal{G}$, $\lambda^{-1}(g)$ is connected.*

LEMMA 5.4. *For each $g \in \mathcal{G}$ and each i, $U(g,i)$ is non-orientable.*

Proof. Since $U(g,i)$ is connected, then $\lambda^{-1}(U(g,i))$ is connected. Hence, there is a loop l in $U(g,i)$ which is not homotopic to a loop from the image of $\lambda^{-1}(U(g,i))$. By definition of orientation double covering, l must be orientation-reversing. □

LEMMA 5.5. *Let N be a locally flat connected closed $(n-1)$-dimensional submanifold of M such that N separates M and $\lambda^{-1}(N)$ is connected. Then N is non-orientable.*

Proof. As in the proof of the previous proposition, there exists a loop l in N which is orientation-reversing in M. Since N separates M, N is bicollared in M; it follows that l is orientation-reversing in N. □

COROLLARY 5.6. *Let $\mathbf{Z}^t$ be the twisted integers coefficient bundle over M; then the pull-back of $\mathbf{Z}^t$ by the inclusion map $N \subseteq M$ is the twisted integers coefficient bundle over N.*

Proof. Since $\lambda^{-1}(N)$ is connected and $\lambda^{-1}(N)$ separates $\tilde{M}$, the argument in Proposition 5.1 shows that $\lambda^{-1}(N)$ is orientable. Hence $\lambda \mid \lambda^{-1}(N)$ is the orientation double covering of N, and the conclusion of Corollary 5.6 follows. □

LEMMA 5.7. *Let U be an open neighborhood of N in M such that no component of $U-N$ has compact closure. Then the inclusion-induced homomorphism $i_*: H_{n-1}(N;\mathbf{Z}^t \mid N) \to H_{n-1}(U;\mathbf{Z}^t \mid U)$ is a monomorphism.*

Proof. Since the pair (U,N) is homotopy equivalent to a pair (K,N) where K is an $(n-1)$-dimensional complex, $H_n(U,N;\mathbf{Z}^t \mid U)=0$. Now consider the long exact homology sequence of the pair (U,N).

LEMMA 5.8. *Given $g \in \mathcal{G}$ and r, there exists a connected locally flat $(n-1)$-dimensional submanifold $N \subseteq U(g,r)$ which separates the two ends of $U(g,r)$, $N \cap g = \varnothing$, and such that $\lambda^{-1}(N)$ is connected.*

Proof. Let $N_1 \subseteq U(g,r)$ be a locally flat submanifold which separates the two ends of $U(g,r)$ and $N_1 \cap g = \varnothing$. There exists a component W of $U(g,r)-(N_1 \cup g)$ such that $\text{cl}(W)$ is compact. Since each $\lambda^{-1}(g^1)$ is connected, it is easily seen that $\lambda^{-1}(\text{cl}(W))$ is also connected. Suppose that $\lambda^{-1}(N_1)$ is not connected and let α be a locally flat arc in $\text{cl}(W)-g$ such that $\text{bdry}\,\alpha = N_1 \cap \alpha$ and such that a component of $\lambda^{-1}(\alpha)$ meets both components of $\lambda^{-1}(N_1)$. If we attach to N_2 a 1-handle whose core is α, and remove its interior, we obtain the required N. □

Proof of Proposition 5.2. Suppose that for some $g \in \mathcal{G}$, g has the shape of an orientable manifold $N(g)$. Find a locally flat closed connected $(n-1)$-manifold $N_1 \subseteq U(g,1)$ such that N_1 separates the ends of M, $N_1 \cap g = \varnothing$, and $\lambda^{-1}(N_1)$ is connected. Find a locally flat closed connected $(n-1)$-manifold $N_2 \subseteq U(g,3)$ such that N_2 separates the ends of M, $N_1 \cap N_2 = \varnothing$, and $\lambda^{-1}(N_2)$ is connected. By Lemma 5.5, N_1 and N_2 are non-orientable, and (by Corollary 5.6 and [15, p. 224]) $H_{n-1}(N_1;\mathbf{Z}^t \mid N_1)$ and $H_{n-1}(N_2;\mathbf{Z}^t \mid N_2)$ are isomorphic to the integers. If N_1 and N_2 are chosen sufficiently close to g, we may assume that $N_1 \cup N_2$ is the boundary of a compact submanifold of M. If we let $i_j: N_j \to U(g,1)$ denote inclusion, $j=1,2$, then $i_{1*}(H_{n-1}(N_1;\mathbf{Z}^t \mid N_1)) = i_{2*}(H_{n-1}(N_2;\mathbf{Z}^t \mid N_2))$ in $H_{n-1}(U(g,1);\mathbf{Z}^t)$. By Lemma 5.7, these subgroups are isomorphic to $\mathbf{Z}$.

Recall the homotopy commutative diagram

$$\begin{array}{ccccc} U(g,1) & \xleftarrow{\mu_1} & U(g,2) & \xleftarrow{\mu_2} & U(g,3) \\ \uparrow{\scriptstyle \alpha_1} & \swarrow{\scriptstyle \beta_2} & \uparrow{\scriptstyle \alpha_2} & \swarrow{\scriptstyle \beta_3} & \uparrow{\scriptstyle \alpha_3} \\ N(g) & = & N(g) & = & N(g). \end{array}$$

Since α_1 is homotopic to $\mu_1\alpha_2$, the bundles $\alpha_1^*(\mathbf{Z}^t \mid U(g,1))$ and $\alpha_2^*(\mathbf{Z}^t \mid U(g,2))$ are isomorphic. Since $N(g)$ is orientable (by Poincaré duality [15, p. 223]), $H_{n-1}(N(g);\alpha_i^*(\mathbf{Z}^t))$ is isomorphic to $H^0(N(g);\alpha_i^*(\mathbf{Z}^t))$.

Let $\widetilde{N(g)}$ be the double cover of $N(g)$ which is induced by the covering $\lambda\colon \tilde{M}\to M$. Since $\lambda^{-1}(g)$ is connected, $\widetilde{N(g)}$ is connected. Fix $y_0\in N(g)$ and consider the action of $\pi_1(N(g),y_0)$ on the bundle $\alpha_i^*(\mathbf{Z}^t)$ restricted to y_0, which we can identify with $\mathbf{Z}$. Since $N(g)$ is connected, any loop not in the image of $\pi_1(\widetilde{N(g)})$ in $\pi_1(N(g))$ will induce the automorphism of $\mathbf{Z}$ given by $x\to -x$. By [16, p. 275], $H^0(N(g);\alpha_i^*(\mathbf{Z}^t))$ is the trivial group.

Recall that by Proposition 2.1 $\mu_{1*}\mid\mu_{2*}(H_{n-1}(U(g,3);\mathbf{Z}^t))$ is an isomorphism onto $\mu_{1*}(H_{n-1}(U(g,2);\mathbf{Z}^t)$ and that

$$\alpha_{2*}\colon H_{n-1}(N(g);\alpha_2^*(\mathbf{Z}^t))\to\mu_{2*}(H_{n-1}(U(g,3);\mathbf{Z}^t))$$

is an isomorphism. In particular, μ_{1*} is the trivial homomorphism. Since $i_{2*}\colon H_{n-1}(N_2;\mathbf{Z}^t)\to H_{n-1}(U(g,1);\mathbf{Z}^t)$ factors through μ_{1*}, we have a contradiction. The proof of Proposition 5.2 is completed. □

Note that if $i\colon N_2\to U(g,2)$ denotes inclusion, then $i_*(H_{n-1}(N_2;\mathbf{Z}^t))\subseteq\alpha_{2*}(H_{n-1}(M(g);\alpha_2^*(\mathbf{Z}^t))$. Consider the exact homology sequence of the pair $(U(g,2),N_2)$:

$$0\to H_{n-1}(N_2;\mathbf{Z}^t)\to H_{n-1}(U(g,2);\mathbf{Z}^t)\to H_{n-1}(U(g,2),N_2;\mathbf{Z}^t)\to\cdots.$$

As in the proof of Lemma 5.7, there exists an $(n-1)$-dimensional complex K containing N_2 such that the pairs $(U(g,2),N_2)$ and (K,N_2) are homotopy equivalent. Hence, $H_{n-1}(U(g,2),N_2;\mathbf{Z}^t)$ has no torsion. It follows that $i_*(H_{n-1}(N_2;\mathbf{Z}^t))=\alpha_{2*}(H_{n-1}(M(g);\alpha_2^*(\mathbf{Z}^t))$. More generally one can show Proposition 5.9 below.

If M is orientable let $\mathcal{B}$ denote the product $\mathbf{Z}$-coefficient bundle over M; if M is non-orientable let $\mathcal{B}$ denote the twisted integers coefficient bundle over M. In order to simplify the notation we shall also use $\mathcal{B}$ to denote the restriction, $\mathcal{B}\mid X$, of $\mathcal{B}$ over a subset X of M.

PROPOSITION 5.9. *Let $g\in\mathcal{G}$ and let $N\subseteq U(g,r+1)$ be a locally flat closed connected $(n-1)$-dimensional submanifold such that N separates the ends of M, $N\cap g=\varnothing$, and N is non-orientable when M is non-orientable. Then $i_*(H_{n-1}(N;\mathcal{B}))=\alpha_{r*}(H_{n-1}(N(g);\alpha_r^*(\mathcal{B})))$ where $i\colon N\to U(g,r)$ denotes inclusion and $r\geqslant 2$. Furthermore, both i_* and α_{r*} are one-to-one.*

Let $g\neq h\in\mathcal{G}$ such that $h\subseteq U(g,3)$ and choose $r\geqslant 3$ such that $U(h,r)\subseteq U(g,3)$. Let N be a locally flat closed connected $(n-1)$-dimensional submanifold of $U(h,r+1)$ such that N separates g from h and such that N is non-orientable when M is non-orientable. Let $i: N \to U(h,r)$ and $j: U(h,r) \to U(g,2)$ denote the inclusion maps. By Proposition 5.9, $\alpha_{2*}(H_{n-1}(N(g);\alpha_2^*(\mathcal{B}))) = (ji)_*(H_{n-1}(N;\mathcal{B}))$ and $\alpha_{r*}(H_{n-1}(N(h);\alpha_r^*(\mathcal{B}))) = i_*(H_{n-1}(N;\mathcal{B}))$. Furthermore, $(ji)_*$ and i_* are one-to-one. It follows that

$$\tau(g,h)_*: H_{n-1}(N(h);\alpha_r^*(\mathcal{B})) \to H_{n-1}(N(g);\alpha_2^*(\mathcal{B}))$$

is an isomorphism. Therefore, we have the following.

PROPOSITION 5.10. *$\tau(g,h)$ is a degree one map from $N(h)$ to $N(g)$.*

The next result is a well-known consequence of Proposition 5.10 (see, for example, the proof of Lemma 2.1 in [15]).

PROPOSITION 5.11. *For each i, $\tau(g,h)_*: H_i(N(h);\alpha_r^*(\mathcal{B})) \to H_i(N(g);\alpha_2^*(\mathcal{B}))$ is onto and splits.*

COROLLARY 5.12. *For each i, $\bar{\tau}(g,h)_*: \check{H}_i(h;\mathcal{B}) \to \check{H}_i(g;\mathcal{B})$ is onto and splits.*

In order to simplify notation, we shall omit the mention of $\mathcal{B}$ in what follows.

Let V_i and W_i denote the components of $U(g,i) - g$ so that $V_{i+1} \subseteq V_i$ for all i. Let $g_i \in \mathcal{G}$ such that $g_i \subseteq V_i$. Consider the following commutative diagram (see §2):

$$\begin{array}{ccccccc}
\multicolumn{7}{c}{\overset{\bar{\tau}(g,g_i)_*}{\longrightarrow}} \\
\check{H}_j(g_i) & \xrightarrow{\sigma_{i*}} & H_j(V_i) & \xrightarrow{\xi_{i*}} & H_j(U(g,i)) & \xrightarrow{\kappa_{i*}} & \check{H}_j(g) \\
 & & \uparrow{\scriptstyle \gamma_{i*}} & & \uparrow{\scriptstyle \mu_{i*}} & & \| \\
\check{H}_j(g_{i+1}) & \xrightarrow{\sigma_{i+1*}} & H_j(V_{i+1}) & \xrightarrow{\xi_{i+1*}} & H_j(U(g,i+1)) & \xrightarrow{\kappa_{i+1*}} & \check{H}_j(g) \\
\multicolumn{7}{c}{\underset{\bar{\tau}(g,g_{i+1})_*}{\longrightarrow}}
\end{array}$$

where all the homomorphisms except the $\bar{\tau}(g,g_i)_*$'s are induced by inclusion maps. It follows from results in §3 that $\{H_j(V_i)\}$ is essentially constant; hence, without loss of generality, we may assume that for each i,

$$\gamma_{i*} \mid \text{image } \gamma_{i+1*}: \text{image } \gamma_{i+1*} \to \text{image } \gamma_{i*}$$

is an isomorphism.

By Corollary 5.12, $\bar{\tau}(g,g_i)_*$ is onto and splits; therefore,

$$\xi_{i*}\sigma_{i*}: \check{H}_j(g_i) \to \text{image } \mu_{i*}$$

is onto and splits. Now note that when we pass to inverse limits, the induced homomorphism $\varprojlim_i\{H_j(V_i)\} \to \varprojlim_i\{H_j(U(g,i))\} = \check{H}_j(g)$ is onto and splits. Let $\check{H}_j(g) = \varprojlim\{H_j(V_i)\} \oplus K_j^+$ where K_j^+ is the kernel of the latter homomorphism.

Similarly, the induced homomorphism $\varprojlim_i\{H_j(W_i)\} \to \varprojlim_i\{H_j(U(g,i))\} = \check{H}_j(g)$ is onto and splits; again, write $H_j(g) = \varprojlim\{H_j(W_i)\} \oplus K_j^-$.

By Theorem 4.2, there exists a long exact sequence

$$\cdots \to \check{H}_i(g) \to H_i(\mathrm{end}_M(g)) \to \check{H}_i(g) \to \check{H}_{i-1}(g) \to \cdots.$$

Note that $H_j(\mathrm{end}_M(g))$ is naturally isomorphic to $\varprojlim_i\{H_j(W_i)\} \oplus \varprojlim_i\{H_j(V_i)\}$ and, from the results above, this long exact sequence consists of short split exact sequences $0 \to \check{H}_j(g) \to H_j(\mathrm{end}_M(g)) \to \check{H}_j(g) \to 0$. Hence

$$\begin{aligned} H_j(g) \oplus H_j(g) = H_j(\mathrm{end}_M(g)) &= \varprojlim_i\{H_j(W_i)\} \oplus \varprojlim_i\{H_j(V_i)\} \\ &= (\check{H}_j(g) \oplus K_j^-) \oplus (\check{H}_j(g) \oplus K_j^+). \end{aligned}$$

Since all these groups are finitely generated, K_j^- and K_j^+ are the trivial groups. Hence we have the following.

PROPOSITION 5.13. *The inclusion induced homomorphisms*

$$\varprojlim_i\{H_j(V_i)\} \to \varprojlim_i\{H_j(U(g,i))\} \quad \textit{and} \quad \varprojlim_i\{H_j(W_i)\} \to \varprojlim_i\{H_j(U(g,i))\}$$

are isomorphisms.

PROPOSITION 5.14. *Let Y be a connected open subset of $M/\mathcal{G}$; then the inclusion-induced $H_*(p^{-1}(Y)) \to H_*(p^{-1}(\mathrm{cl}(Y)))$ is an isomorphism.*

Proof. Let $g \in \mathcal{G}$ such that $p(g) \in \mathrm{cl}(Y) - Y$. Continuing with the notation from above, we may assume that $p^{-1}(Y) \cap U(g,i) = V_i$. Let $Y_i^+ = p^{-1}(Y) \cup U(g,i)$. Consider the commutative diagram:

$$\begin{array}{ccccccccc} \cdots \to & H_{j+1}(Y_i^+) & \to & H_j(V_i) & \to & H_j(p^{-1}(Y)) \oplus H_j(U(g,i)) & \to & H_j(Y_i^+) & \to \cdots \\ & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\ \cdots \to & H_{j+1}(Y_{i+1}^+) & \to & H_j(V_{i+1}) & \to & H_j(p^{-1}(Y)) \oplus H_j(U(g,i+1)) & \to & H_j(Y_{i+1}^+) & \to \cdots \end{array}$$

where rows are from the Mayer–Vietoris sequence and where vertical homomorphisms are induced by inclusions. Passing to inverse limits and using Proposition 5.13, we obtain short exact sequences

$$0 \to \varprojlim_i\{H_j(V_i)\} \xrightarrow{\delta_* \oplus \xi_*} H_j(p^{-1}(Y)) \oplus H_j(g) \xrightarrow{\epsilon_{1*} - \epsilon_{2*}} H_j(p^{-1}(Y) \cup g) \to 0$$

where all indicated homomorphisms are induced by inclusion maps. Using the fact that ξ_* is an isomorphism, one can easily show that

$$\epsilon_{1*}: H_j(p^{-1}(Y)) \to H_j(p^{-1}(Y) \cup g)$$

is an isomorphism.

Similarly, if $g' \in \mathcal{G}$ such that $p(g') \in \mathrm{cl}(Y) - Y - p(g)$, then

$$H_*(p^{-1}(Y)) \to H_*(p^{-1}(Y) \cup g')$$

is an isomorphism. Finally, we can repeat this argument to obtain Proposition 5.14. □

THEOREM 5.15. *Let $\mathcal{G}$ be an* u.s.c. *decomposition of an n-manifold M into continua which have the shape of closed $(n-1)$-manifolds such that $M/\mathcal{G}$ is $\mathbf{R}^1$. Then, for each $g \in \mathcal{G}$, the inclusion induced homomorphism $\check{H}_*(g) \to H_*(M)$ is an isomorphism.*

Proof. The proof is similar to the proof of Lemma 6.2 in [5]. Let us identify $M/\mathcal{G}$ with $\mathbf{R}$ and let $p(g)=0$. Let $S=\{s\in(0,+\infty)\,|\,\check{H}_*(g)\to H_*(p^{-1}(-s,s))$ is one-to-one$\}$. If we choose $s>0$ such that $p^{-1}(-s,s)\subseteq U(g,2)$, then $s\in S$ by Proposition 5.14. Note that for each $t\in(0,s)$, if $s\in S$ then $t\in S$. Hence, $S=(0,s]$, $(0,s)$, or $(0,+\infty)$. Since $p^{-1}([-s,s])$ is an FANR, an analogue of Proposition 2.1 implies that there is a neighborhood U of $p^{-1}([-s,s])$ such that $\check{H}_*(p^{-1}([-s,s]))\to H_*(U)$ is one-to-one. Hence, $S\neq(0,s]$. Suppose that $S=(0,s)$ and hence $\check{H}_*(g)\to H_*(p^{-1}(-s,s))$ is not one-to-one. Note that a cycle representing an element of the kernel of the latter homomorphism bounds on a compact subset of $p^{-1}(-s,s)$ and we have a contradiction. It follows that $\check{H}_*(g)\to H_*(M)$ is one-to-one.

Let

$$T=\{s\in(0,+\infty)\,|\,\mathrm{image}(H_*(p^{-1}(-s,s))\to H_*(M)) = \mathrm{image}(H_*(g)\to H_*(M))\}.$$

An argument similar to the above shows that $T=(0,+\infty)$ when one notices that if X is an FANR, then there are neighborhoods $U\subseteq V$ of X such that $\mathrm{image}(\check{H}_*(X)\to H_*(V))=\mathrm{image}(H_*(U)\to H_*(V))$ (see Proposition 2.1). Again, this implies that $\check{H}_*(g)\to H_*(M)$ is onto. □

COROLLARY 5.16. *Suppose $\mathcal{G}$ is a* u.s.c. *decomposition of an n-manifold M into continua having the shape of $(n-1)$-manifolds such that $M/\mathcal{G}$ is a 1-manifold without boundary. Then all pairs of elements of $\mathcal{G}$ are homologically equivalent.*

Illustrating the sharpness of Theorem 5.15, Example 5.3 of [5] sets forth a decomposition $\mathcal{G}$ of an n-manifold M $(n\geqslant 6)$ into closed $(n-1)$-manifolds such that $M/\mathcal{G}$ is $\mathbf{R}^1$, but some pairs of elements fail to be homotopically equivalent due to differences in π_1.

With controls on π_1 we can obtain stronger results.

THEOREM 5.17. *Let $\mathcal{G}$ be an* u.s.c. *decomposition of an n-manifold M into continua which have the shape of closed $(n-1)$-manifolds such that $M/\mathcal{G}$ is $\mathbf{R}^1$ and such that, for each $g\in\mathcal{G}$, the inclusion induced homomorphism $\check{\pi}_1(g)\to\pi_1(M)$ is an isomorphism. Suppose that the integral group ring $\mathbf{Z}\pi_1(M)$ is Noetherian. Then the natural map $M\to M/\mathcal{G}$ is an approximate fibration.*

COROLLARY 5.18. *For $n\geqslant 6$, there exists a closed $(n-1)$-manifold N such that M is homeomorphic to the product $N\times\mathbf{R}$ and N has the shape of elements of $\mathcal{G}$.*

COROLLARY 5.19. *For $n\geqslant 6$, the natural map $M\to M/\mathcal{G}$ can be approximated by locally trivial bundle maps.*

The changes in the above proof which are needed in order to prove Theorem 5.17 are essentially in the choice of coefficients for the homology groups. We use the bundle of local coefficients as described in [15, p. 223]. If $g, h \in \mathcal{G}$, then the fact that $\tau(g,h)$ is a homology isomorphism now implies that $\tau(g,h)$ is a homotopy equivalence. As in §2 this implies that $M \to M/\mathcal{G}$ is an approximate fibration. Corollaries 5.18 and 5.19 follow from [9]. The collection of groups whose integral group rings are Noetherian include the finite extensions of poly-cyclic groups (in particular, finite groups and finitely generated Abelian groups).

COROLLARY 5.20. *Suppose G is an* u.s.c. *decomposition of an n-manifold M into continua having the shape of closed $(n-1)$-manifolds with Abelian or finite fundamental groups, such that $M/\mathcal{G}$ is a* 1*-manifold without boundary. Then the natural map $M \to M/\mathcal{G}$ is an approximate fibration.*

One can derive the following result as we did Theorem 5.17.

THEOREM 5.21. *Let $\mathcal{G}$ be an* u.s.c. *decomposition of a closed n-manifold M into continua which have the shape of closed $(n-1)$-manifolds such that for each $g \in \mathcal{G}$, the inclusion-induced homomorphism $\check{\pi}_1(g) \to \pi_1(M)$ is an isomorphism onto a fixed subgroup of $\pi_1(M)$. Suppose that the integral group ring of $\check{\pi}_1(g)$ is Noetherian. Then the natural map $M \to M/\mathcal{G}$ is an approximate fibration and, for $n \geqslant 6$, M can be obtained from a near-product h-cobordism W* [5] *by homeomorphically identifying the two components of the boundary of W.*

For $n \geqslant 6$, if a certain obstruction in the Whitehead group of $\mathbf{Z}\check{\pi}_1(g)$ vanishes, then the natural map $M \to M/\mathcal{G}$ can be approximated by locally trivial bundle maps whose fibers have the same shape as elements of $\mathcal{G}$.

THEOREM 5.22. *Let $\mathcal{G}$ be an* u.s.c. *decomposition of an n-manifold M into continua which have the shape of closed $(n-1)$-manifolds such that $M/\mathcal{G}$ is* $[0, +\infty)$. *Then the inclusion of $p^{-1}(0)$ into M is a homology equivalence.*

Proof. Choose $t > 0$ such that $p^{-1}([0,t]) \subseteq U(p^{-1}(0), 2)$. It follows that the image of $\check{H}_*(p^{-1}([0,t]))$ in $H_*(M)$ is the image of $\check{H}_*(p^{-1}([0,t]))$ in $H_*(M)$ and $\check{H}_*(p^{-1}(0)) \to H_*(p^{-1}([0,t]))$ is one-to-one. It follows from Proposition 5.14 that the inclusion-induced $\check{H}_*(p^{-1}(t)) \to \check{H}_*(p^{-1}([t, +\infty)))$ is an isomorphism. By excision, $\check{H}_*(M, p^{-1}([0,t])) \cong \check{H}_*(p^{-1}([t, +\infty)), p^{-1}(t)) \cong 0$ and the theorem follows. □

QUESTION. Suppose that the inclusion induced $\check{\pi}_1(p^{-1}(t)) \to \pi_1(M)$ is an isomorphism for $t = 0$ and is an isomorphism onto a subgroup of index two for $t > 0$. It is true that M is the twisted line bundle over some submanifold which has the shape of $p^{-1}(0)$? If $p^{-1}(0)$ is a manifold, then it is not difficult to see that this is true.

REFERENCES

1. K. Borsuk, *Theory of retracts,* PWN, Warsaw, 1967.
2. D. S. Coram and P. F. Duvall, Jr., *Approximate fibrations,* Rocky Mountain J. Math. 7 (1977), 275–288.

3. ———, *Mappings from S^3 to S^2 whose point inverses have the shape of a circle,* General Topology Appl. 10 (1979), 239–246.
4. ———, *Approximate fibrations and a movability condition for maps,* Pacific J. Math. 72 (1977), 41–56.
5. R. J. Daverman, *Decompositions of manifolds into codimension one submanifolds,* Compositio Math., to appear.
6. J. Dydak and J. Segal, *Local n-connectivity of decomposition spaces,* preprint.
7. D. A. Edwards and R. Geoghegan, *Shapes of complexes, ends of manifolds, homotopy limits and the Wall obstruction,* Ann. of Math. (2) 101 (1975), 521–535; correction, ibid. (2) 104 (1976), 389.
8. L. S. Husch, *Shape fibrations and* FANR's, Glas. Mat. Ser. III 14 (34) (1979), 363–373.
9. ———, *Approximating approximate fibrations by fibrations,* Canad. J. Math. 29 (1977), 897–913.
10. H. M. Hastings and A. Heller, *Splitting homotopy idempotents.* Shape theory and geometric topology (Dubrovnik, 1981), 23–36, Lecture Notes in Math., 870, Springer, Berlin, 1981.
11. S. Mardešić and J. Segal, *Shape theory,* North-Holland, Amsterdam, 1982.
12. F. Quinn, *Ends of maps I,* Ann. of Math. (2) 110 (1979), 275–331.
13. F. Raymond, *Separation and union theorems for generalized manifolds with boundary,* Michigan Math. J. 7 (1960), 7–21.
14. ———, *Local triviality for Hurewicz fiberings of manifolds,* Topology 3 (1965), 43–57.
15. C. T. C. Wall, *Surgery of non-simply-connected manifolds,* Ann. of Math. (2) 84 (1966), 217–276.
16. G. W. Whitehead, *Elements of homotopy theory,* Springer, New York, 1978.
17. J. H. C. Whitehead, *Combinatorial homotopy, I,* Bull. Amer. Math. Soc. 55 (1949), 213–245.

Department of Mathematics
University of Tennessee
Knoxville, Tennessee 37996-1300