THE RANGE OF THE RESIDUE FUNCTIONAL FOR THE CLASS S_p

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Let U denote the unit disk $\{z: |z|<1\}$. For $0<p<1$, the class S_p will consist of all functions $g(z)$ which are meromorphic and univalent in U and in addition are normalized so that $g(0)=0$, $g'(0)=1$ and $g(p)=\infty$. Define the set

$$\Omega_p=\{a: a=\mathrm{Res}_{z=p}\, g(z),\ g\in S_p\}.$$

In this note we prove the following:

THEOREM. $\Omega_p=\{-p^2(1-p^2)^\epsilon: |\epsilon|\leqslant 1\}$.

Proof. The proof consists of a mutual inclusion argument.

Suppose that $a\in\Omega_p$. Then $a=\mathrm{Res}_{z=p}\, g(z)$ for some $g(z)\in S_p$. Let S denote the class of all functions $f(z)$ which are analytic and univalent in U and are normalized so that $f(0)=0$ and $f'(0)=1$. Then a short argument shows that the function

$$f_c(z)=\frac{cg(z)}{c+g(z)} \quad (-c\notin g(U))$$

belongs to S and that $a=-f_c^2(p)/f_c'(p)$. We shall apply the Golusin Inequalities [1, p. 898] to the function $f_c(z)$. For each $f\in S$, we have

$$\left|\sum_{n=1}^{N}\sum_{k=1}^{N}\lambda_n\bar{\lambda}_k \log\left(\frac{f(z_n)-f(z_k)}{z_n-z_k}\,\frac{z_n z_k}{f(z_n)f(z_k)}\right)\right|\leqslant\sum_{n=1}^{N}\sum_{k=1}^{N}\lambda_n\bar{\lambda}_k\log\left(\frac{1}{1-z_n\bar{z}_k}\right),$$

where the z_n $(0<|z_n|<1)$ are distinct and the λ_n are arbitrary complex numbers. For $z_k=z_n$, the quotient is interpreted as a derivative. We apply these inequalities with $k=n=N=1$, $\lambda_1=1$ and $z_1=p$ to obtain the inequality

$$\left|\log\frac{p^2 f'(p)}{f^2(p)}\right|\leqslant\log\frac{1}{1-p^2}. \tag{1}$$

This inequality was originally discovered by Grunsky [2]. Setting $f(z)=f_c(z)$ in (1), we obtain

$$|\log(-a)-\log p^2|\leqslant\log\frac{1}{1-p^2}.$$

It follows that

$$\log(-a)=\log p^2+\epsilon\log(1-p^2)$$

where $|\epsilon|\leqslant 1$. Exponentiating and multiplying by -1, we obtain $a=-p^2(1-p^2)^\epsilon$, which was what we wanted.

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In order to establish inclusion in the other direction, we shall consider the functions

$$g_\epsilon(z)=\frac{pz}{p-z}(1-pz)^\epsilon,\quad |\epsilon|\leqslant 1, \tag{2}$$

which are clearly meromorphic in U. Also, $g_\epsilon(0)=0$, $g_\epsilon'(0)=1$, $g_\epsilon(p)=\infty$ and $\text{Res}_{z=p}\, g_\epsilon(z)=-p^2(1-p^2)^\epsilon$. If $|\epsilon|=1$, then g_ϵ is univalent and thus $g_\epsilon \in S_p$. Furthermore, each such function maps U onto the plane less a certain logarithmic slit, so that the area of $\hat{C}-g_\epsilon(U)$ is zero. See [3, p. 279–280] for details.

To complete the proof, we must show that $g_\epsilon(z)$ is univalent for $|\epsilon|<1$. To do this, we shall make use of some results for the class Σ. Recall that Σ consists of all functions $h(z)$ which are analytic and univalent on the set $\Delta=\{z: |z|>1\}$ and satisfy the normalizations $h(\infty)=\infty$, and $h'(\infty)=1$. The following facts about Σ are well-known and may be found, for instance, in [4, p. 58–60]. If $h(z)$ and $\lambda_1,\ldots,\lambda_m$ are arbitrary complex numbers not all zero, then the Grunsky Inequalities assert that

$$\sum_{k=1}^{\infty} k\left|\sum_{n=1}^{m} b_{kn}\lambda_n\right|^2 \leqslant \sum_{k=1}^{m} |\lambda_k|^2/k,\quad m=1,2,\ldots, \tag{3}$$

where the numbers b_{kn} are the Grunsky coefficients of $h(z)$ defined by the relation

$$\log\frac{h(z)-h(\zeta)}{z-\zeta}=-\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} b_{kn}z^{-k}\zeta^{-n} \tag{4}$$

where $|z|, |\zeta|>1$. Equality holds in (3) if and only if the area of the complement $\hat{C}-h(\Delta)$ is zero. Also, if $h(z)=z+b_0+\cdots$ is analytic in $1<|z|<\infty$ and if (3) holds for all $\lambda_1,\ldots,\lambda_m$ and $m=1,2,\ldots$, then $h(z)$ is univalent in Δ and therefore belongs to Σ.

We introduce the auxiliary function $h_\epsilon(z)=1/g_\epsilon(1/z)$. Now $h_\epsilon(z)$ is analytic in z and ϵ, $|z|>1$, $|\epsilon|\leqslant 1$. The earlier remarks about $g_\epsilon(z)$ imply that if $|\epsilon|=1$ then $h_\epsilon(z)\in\Sigma$, and that the area of $\hat{C}-h_\epsilon(\Delta)$ is equal to zero.

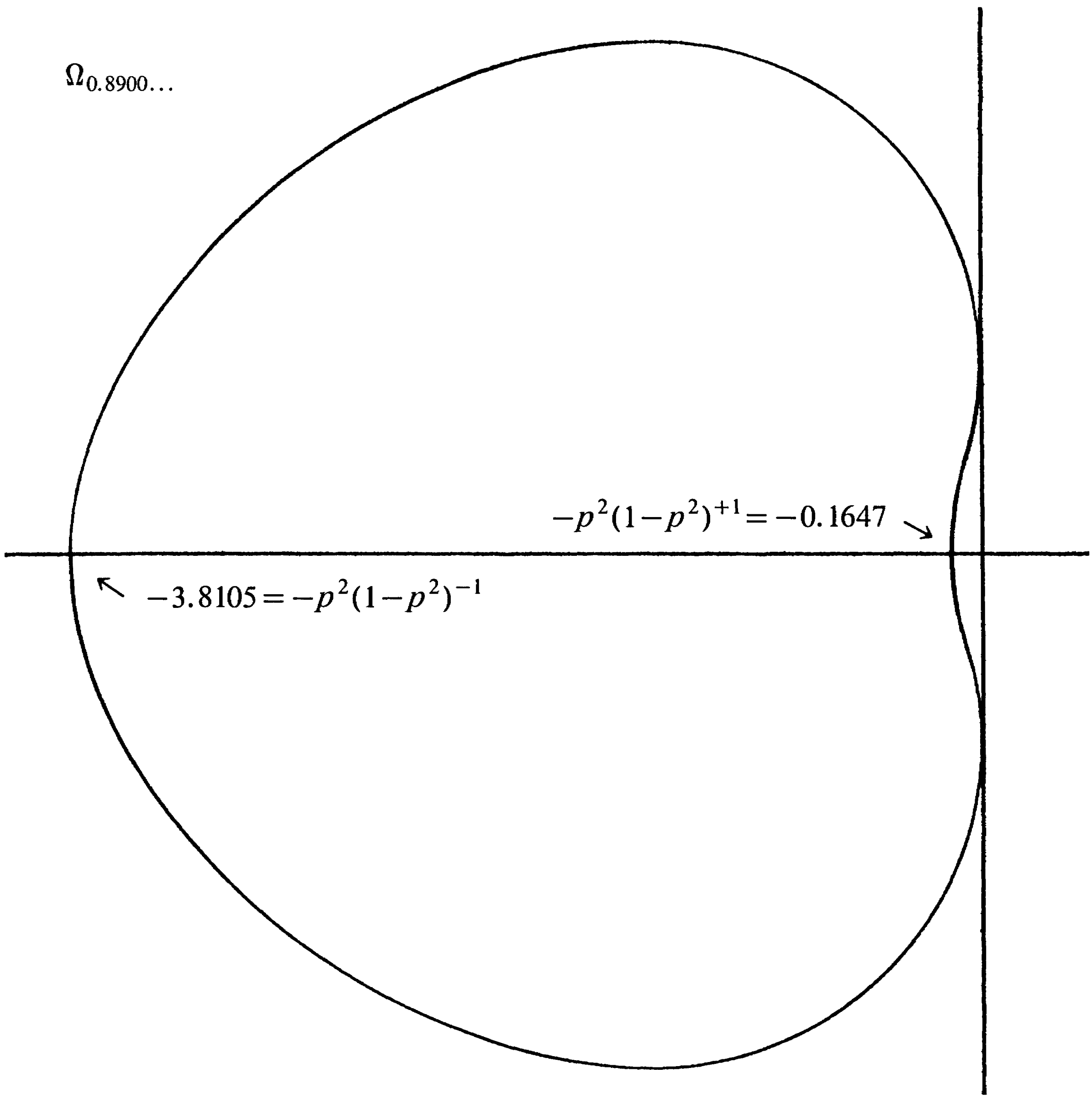
Let $\{b_{kn}(\epsilon)\}$ $(k,n=1,2,\ldots)$ denote the Grunsky coefficients of $h_\epsilon(z)$. Then integration of (4) above yields

$$b_{kn}(\epsilon)=-\iint\limits_{\substack{|z|=R\\|\zeta|=R}}\log\frac{h_\epsilon(z)-h_\epsilon(\zeta)}{z-\zeta}\,z^{k-1}\zeta^{n-1}\,dz\,d\zeta\quad (R>1).$$

Clearly, $b_{kn}(\epsilon)$ is an analytic function of ϵ. It follows that

$$B(\epsilon)=\sum_{k=1}^{\infty} k\left|\sum_{n=1}^{m} b_{kn}(\epsilon)\lambda_n\right|^2$$

is a subharmonic function of ϵ in the closed unit disk for every $m=1,2,\ldots$ By the maximum principle for subharmonic functions, B must take its greatest value when $|\epsilon|=1$. Indeed, for $|\epsilon|=1$, the equality



$$B(\epsilon)=\sum_{k=1}^{m}|\lambda_k|^2/k$$

must hold since $h_\epsilon(z)\in\Sigma$ and the area of $\hat{C}-h_\epsilon(\Delta)$ is equal to zero.

Consequently, for $|\epsilon|<1$, we have the strict inequality

$$B(\epsilon)<\sum_{k=1}^{m}|\lambda_k|^2/k.$$

By a prior remark, this implies that $h_\epsilon(z)$ and hence that $g_\epsilon(z)$ is univalent. Thus, $g_\epsilon(z)\in S_p$ for $|\epsilon|\leqslant 1$, and the proof is complete. □

If p is small, the set Ω_p is a small oval-shaped disk which covers the point $w=-p^2$. The region $\Omega_{0.8900\ldots}$ is sketched above. As p tends to one, the two bulges tangent to the imaginary axis advance toward one another and then overlap infinitely often as they circle around the origin in opposite directions. The union $\bigcup \Omega_p$ covers the entire complex plane, punctured at the origin.

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