DERIVATIONS FROM SUBALGEBRAS OF SEPARABLE C^*-ALGEBRAS

A. J. Lazar, S.-K. Tsui, and S. Wright

1. Introduction. Let A be a C^*-algebra, $M(A)$ its multiplier algebra, B a C^*-subalgebra of A. Suppose $\delta: B \to A$ is a *derivation* of B into A, i.e., a linear map for which $\delta(ab) = a\delta(b) + \delta(a)b$, for all $a, b \in B$. In many important applications, one wishes to know if δ is *inner in* $M(A)$, i.e., if there is an element m of $M(A)$ for which $\delta(b) = mb - bm$, for all $b \in B$. Akemann and Johnson [1] have pointed out in particular the importance of investigating those pairs (B, A) as above for which every derivation of B into A is inner in this sense. A first step in such an investigation would consist of studying the C^*-algebras A for which *all* such pairs (B, A) have this property. We formalize this by saying that a C^*-algebra A is *hereditarily cohomologically trivial* (HCT for short) if for each C^*-subalgebra B of A and each derivation $\delta: B \to A$, there is a multiplier m of A for which $\delta(b) = mb - bm$, for all $b \in B$.

In [6], the authors determined the structure of the HCT C^*-algebras with continuous trace. The only other class of HCT algebras known to us are the finite von Neumann algebras, a result due to Erik Christensen [3, §5] (it is of course an outstanding open problem whether the algebra $B(H)$ of all bounded linear operators on a Hilbert space H is HCT). The HCT algebras are evidently contained in the class of C^*-algebras for which every derivation $\delta: A \to A$ is inner in $M(A)$, and Elliott [5] and Akemann and Pedersen [2] determined the structure of the separable C^*-algebras with this latter property. In the paper before the reader, we will determine the structure of the separable C^*-algebras which are HCT. It turns out that the separable HCT algebras form a rather restricted class; in fact the only simple, separable HCT algebras are the algebras of compact operators on a separable Hilbert space, usually referred to as the *elementary* C^*-algebras. More precisely, we will prove:

THEOREM 1.1. *Let A be a separable C^*-algebra. Then A is HCT if and only if A has a direct sum decomposition of the form $A_1 \oplus A_2$, where A_1 is a commutative algebra and A_2 is the restricted direct sum of a (possibly finite) sequence of separable elementary C^*-algebras.*

2. Proof of Theorem 1.1. We begin with a lemma which is no doubt well-known to the experts, but for which, in the interest of clarity and completeness, we provide a proof (it is stated without proof in the argument of Lemma 3.1 of [7]).

LEMMA 2.1. *Let A be a C^*-algebra, p a closed, central projection in the enveloping von Neumann algebra A^{**} of A. Then pA^{**} is naturally isomorphic to the enveloping von Neumann algebra $(pA)^{**}$ of pA.*

Received June 1, 1983.
Michigan Math. J. 31 (1984).

Proof. Note that pA^{**} is $\sigma(A^{**}, A^*)$-closed in A^{**}, and is hence a W^*-algebra. Let $\pi: pA \to B(H)$ be a representation of pA. We must prove that π extends to a normal representation of pA^{**} into $B(H)$.

Define a representation $\tilde{\pi}$ of A into $B(H)$ by $\tilde{\pi}(a) = \pi(pa)$, $a \in A$. Let $\tilde{\pi}^{**}$ be the canonical extension of $\tilde{\pi}$ to a normal representation of A^{**}. We assert that $\tilde{\pi}^{**}(pa) = \pi(pa)$, for all $a \in A$. If this is so, then $\tilde{\pi}^{**}|_{pA^{**}}$ will be the extension of π that we seek.

Since p is closed, there is an increasing net $\{a_\alpha\}$ of positive elements of A with $a_\alpha \nearrow (1-p)$. Hence $pa_\alpha^{1/2} = 0$, for all α, and so for all $a \in A$ and α,

$$\tilde{\pi}(a^* a_\alpha a) = \tilde{\pi}(a^* a_\alpha^{1/2})\, \tilde{\pi}(a_\alpha^{1/2} a) = 0.$$

Hence,

$$\begin{aligned} \tilde{\pi}^{**}((1-p)a^*a) &= \tilde{\pi}^{**}(a^*(1-p)a) \\ &= \lim_\alpha \tilde{\pi}(a^* a_\alpha a) = 0, \quad \text{for all } a \in A. \end{aligned}$$

It follows from the norm identity in a C^*-algebra that $\tilde{\pi}^{**}((1-p)a) = 0$, $a \in A$, and our assertion hence obtains. □

We recall for use in the next lemma that if n is a positive integer, a family $\{e(i,j): i,j = 1, \ldots, n\}$ of elements of a C^*-algebra is an $(n \times n)$ *system of matrix units* if

(a) $e(i,j)e(k,l) = \delta_{jk} e(i,l)$; $i,j,k,l = 1, \ldots, n$, where δ_{jk} denotes the Kronecker delta;

(b) $e(i,j)^* = e(j,i)$; $i,j = 1, \ldots, n$.

We also recall that every C^*-algebra A has a unique maximal, liminary, closed, two-sided ideal [4, Proposition 4.2.6], and that A is said to be *antiliminary* if its maximal liminary ideal is the zero ideal. The next lemma is the key ingredient in the proof of Theorem 1.1.

LEMMA 2.2. *Let A be a separable, antiliminary C^*-algebra. Then A is not HCT.*

Proof. By Lemma 6.7.2 of [8], A contains a quasi-matrix system of rank $\{2,2,2,\ldots\}$, i.e., sequences $\{e_n\}_{n \geq 1}$ and $\{v_n\}_{n \geq 1}$ satisfying the following conditions:

(1) $$e_n \geq 0, \quad \|e_n\| = \|v_n\| = 1, \quad \text{for all } n;$$

(2) $$v_n^* v_n e_n = e_n, \quad \text{for all } n;$$

(3) $$v_n^2 = 0, \quad \text{for all } n;$$

(4) $$e_n e_{n+1} = e_{n+1}, \quad e_n v_{n+1} = v_{n+1}, \quad e_n v_{n+1}^* = v_{n+1}^*, \quad \text{for all } n;$$

(5) $$e_m v_n = 0, \quad m \geq n;$$

(6) $$v_m v_n = 0, \quad m > n;$$

(7) $$v_m^* v_n = 0, \quad m \neq n.$$

By Proposition 6.6.5 of [8], there is a nonzero projection q in A^{**} such that q commutes with $\{e_n\}_{n \geq 1} \cup \{v_n\}_{n \geq 1}$ and $\{qe_n\}_{n \geq 1} \cup \{qv_n\}_{n \geq 1}$ is a matrix system of rank $\{2,2,2,\ldots\}$, i.e., (1) through (7) hold for these elements, as well as

(8) $$qe_n = (qv_n^*)(qv_n), \quad \text{for all } n;$$

(9) $$qe_n = qe_{n+1} + (qv_{n+1})(qv_{n+1}^*), \quad \text{for all } n.$$

The matrix system

$$\{e_n q\}_{n\geqslant 1} \cup \{v_n q\}_{n\geqslant 1}$$

defines systems of matrix units $\{e_n(i,j): i,j=1,\ldots,2^n\}_{n\geqslant 1}$ in qAq such that if A_n = linear span of $\{e_n(i,j): i,j=1,\ldots,2^n\}$, then A_n is a C^*-subalgebra of qAq isomorphic to the algebra of complex $2^n \times 2^n$ matrices, all A_n's have the same unit, $A_n \subseteq A_{n+1}$, $n \geqslant 1$, and the embedding of A_n into A_{n+1} is given by

$$e_n(i,j) = e_{n+1}(i,j) + e_{n+1}(i+2^n, j+2^n); \quad i,j=1,\ldots,2^n$$

(cf. the discussion following Definition 6.6.1 of [8]).

We need to determine the formula for $e_n(i,1)$, $i=1,\ldots,2^n$, in terms of q, the e_k's, and the v_k's. We claim that for $n \geqslant 2$,

(i) $e_n(1,1) = qe_n$;

(ii) if $2 \leqslant k \leqslant 2^{n-1}$, there is an increasing set of indices $1 \leqslant i_1 < i_2 < \cdots < i_p \leqslant n-1$ such that $e_n(k,1) = qv_{i_1}\ldots v_{i_p} e_n$;

(iii) $e_n(1+2^{n-1},1) = qv_n$;

(iv) if $2+2^{n-1} \leqslant k \leqslant 2^n$, there is a set of indices as in (ii) such that $e_n(k,1) = qv_{i_1}\ldots v_{i_p} v_n$.

(i) and (iii) follow from (2), (3), (6), (8), and (9). For each k, $2 \leqslant k \leqslant 2^{n-1}$, we have from the embedding of A_{n-1} into A_n that

$$\begin{aligned} e_n(k,1) &= (e_n(k,1) + e_n(k+2^{n-1}, 1+2^{n-1}))e_n(1,1) \\ &= e_{n-1}(k,1)e_n(1,1), \\ e_n(k+2^{n-1},1) &= (e_n(k,1) + e_n(k+2^{n-1}, 1+2^{n-1}))e_n(1+2^{n-1},1) \\ &= e_{n-1}(k,1)e_n(1+2^{n-1},1). \end{aligned}$$

It follows that $e_2(1,1) = qe_2$, $e_2(2,1) = qv_1 e_2$, $e_2(3,1) = qv_2$, and $e_2(4,1) = qv_1 v_2$, which is (ii) and (iv) for $n=2$. Assuming inductively that (ii) and (iv) hold for $n-1$, we compute from (i), (iii), (4) and the above embedding formulae that

$$e_n(k,1) = \begin{cases} qv_{i_1}\ldots v_{i_p} e_{n-1} e_n = qv_{i_1}\ldots v_{i_p} e_n, & 2 \leqslant k \leqslant 2^{n-2} \\ qv_{n-1} e_n, & k = 1+2^{n-2} \\ qv_{j_1}\ldots v_{j_r} v_{n-1} e_n, & 2+2^{n-2} \leqslant k \leqslant 2^{n-1}, \end{cases}$$

$$e_n(k+2^{n-1},1) = \begin{cases} qv_{i_1}\ldots v_{i_p} e_{n-1} v_n = qv_{i_1}\ldots v_{i_p} v_n, & 2 \leqslant k \leqslant 2^{n-2} \\ qv_{n-1} v_n, & k = 1+2^{n-2} \\ qv_{j_1}\ldots v_{j_r} v_{n-1} v_n, & 2+2^{n-2} \leqslant k \leqslant 2^{n-1}, \end{cases}$$

for appropriately chosen indices $1 \leqslant i_1 < \cdots < i_p \leqslant n-2$ and $1 \leqslant j_1 < \cdots < j_r \leqslant n-2$. This is (ii) and (iv) for n.

We claim next that for $n \geqslant 1$,

(v) $e_{n+2}(2^n,1) = qv_1\ldots v_n e_{n+2}$,

(vi) $e_{n+2}(2^{n+1},1) = qv_1\ldots v_{n+1} e_{n+2}$,

(vii) $e_{n+2}(2^{n+2},1) = qv_1\ldots v_{n+1} v_{n+2}$.

This follows straightforwardly from the embedding of A_n into A_{n+1}, (4), and induction.

We can now compute $e_{n+2}(3\cdot 2^n, 2^n)$, $n \geqslant 2$. By the embedding of A_{n+1} into A_{n+2}, (4), (iii), and (v)–(vii),

$$\begin{aligned} e_{n+2}(3\cdot 2^n, 1) &= e_{n+2}(2^n + 2^{n+1}, 1) \\ &= (e_{n+2}(2^n, 1) + e_{n+2}(2^n + 2^{n+1}, 1+2^{n+1}))\, e_{n+2}(1+2^{n+1}, 1) \\ &= e_{n+1}(2^n, 1)\, e_{n+2}(1+2^{n+1}, 1) \\ &= q v_1 \ldots v_n\, e_{n+1}\, v_{n+2} \\ &= q v_1 \ldots v_n\, v_{n+2} \end{aligned}$$

and so

(viii) $$\begin{aligned} e_{n+2}(3\cdot 2^n, 2^n) = e_{n+2}(3\cdot 2^n, 1)\, e_{n+2}(1, 2^n) &= e_{n+2}(3\cdot 2^n, 1)\, e_{n+2}(2^n, 1)^* \\ &= q v_1 \ldots v_n\, v_{n+2}\, e_{n+2}\, v_n^* \ldots v_1^*. \end{aligned}$$

Now, for $n \geqslant 1$, set

$$t_n = \sum_{i=1}^{n} v_1 \ldots v_i\, v_{i+2}\, e_{i+2}\, v_i^* \ldots v_1^*.$$

Claim 1: $\|t_n\| \leqslant 1$, $n \geqslant 1$.

To verify this, set $a_i = v_1 \ldots v_i\, v_{i+2}\, e_{i+2}\, v_i^* \ldots v_1^*$, and notice first that by (2) and (4), $v_k^* v_k$ acts as a unit for v_m, $m \geqslant k+1$, so for $i \geqslant j+1$,

$$v_i^* \ldots v_{j+1}^*\, v_j^* \ldots v_1^* v_1 \ldots v_j = v_i^* \ldots v_{j+1}^*,$$

and it therefore follows that for $i \geqslant j+1$, there are elements $b_i, b_j \in A$ for which $a_i a_j^* = b_i\, v_{j+1}^*\, e_{j+2}\, b_j$, whence by (5), $a_i a_j^* = 0$. A similar computation shows that $a_i a_j^* = 0$ for $i < j$. Hence by (2) and (4),

$$\begin{aligned} t_n t_n^* &= \sum_{i=1}^{n} v_1 \ldots v_i\, v_{i+2}\, e_{i+2}\, v_i^* \ldots v_1^* v_1 \ldots v_i\, e_{i+2}\, v_{i+2}^*\, v_i^* \ldots v_1^* \\ &= \sum_{i=1}^{n} v_1 \ldots v_i\, v_{i+2}\, e_{i+2}^2\, v_{i+2}^*\, v_i^* \ldots v_1^*. \end{aligned}$$

If c_i denotes the i-th term of this sum, then $\|c_i\| \leqslant 1$ and $c_i \geqslant 0$. By (2) and (4) there are elements $d_i, d_j \in A$ such that $c_i c_j = d_i\, v_{i+2}^*\, v_{i+1}\, d_j$ for $j \geqslant i+1$, and so by (7), $c_i c_j = 0$. Thus for $n \geqslant 1$, $t_n t_n^*$ is a sum of pairwise orthogonal, self-adjoint elements each of norm not exceeding 1, and so $t_n t_n^*$, and hence t_n, also has norm not exceeding 1. This verifies Claim 1.

Let B = the C^*-subalgebra of A generated by $\{e_n\}_{n\geqslant 1} \cup \{v_n\}_{n \geqslant 1}$, and let C = the C^*-subalgebra of B generated by $\{e_n\}_{n\geqslant 1}$ and all elements of the form $v_{i_1} \ldots v_{i_p}\, e_n^2\, v_{i_p}^* \ldots v_{i_1}^*$, where $1 \leqslant i_1 < i_2 < \cdots < i_p \leqslant n-1$, $n \geqslant 2$. By Claim 1, there is a $\sigma(A^{**}, A^*)$-limit point t of $\{t_n\}$ in A^{**}.

Claim 2: $(adt)(C) \subseteq A$.

Here adt denotes the derivation of A^{**} given by $a \to ta - at$, $a \in A^{**}$. To verify this, fix $n \geqslant 1$ and let $k \geqslant n+1$. By (5), $a_k e_n = e_n a_k = 0$. Let $x = v_{i_1} \ldots v_{i_p}\, e_n^2\, v_{i_p}^* \ldots v_{i_1}^*$ with $1 \leqslant i_1 < i_2 < \cdots < i_p \leqslant n-1$. By (3), (4), and (7) either $a_k x = 0$ or there exist elements $b_k, c_k \in A$ with $a_k x = b_k v_k^* \dots v_j^* e_n c_k$ for some $j \leqslant n$, and so by (5), $a_k x = 0$ also in this instance. Similarly, $x a_k = 0$. Now all words in the generators of C begin and end with either an e_n or a product of v_n's or v_n^*'s, and the preceding computation hence shows that $ad(a_k)$ vanishes on all words from generators of C which are formed from elements of $\{e_1, \dots, e_n\}$, $\{v_1, \dots, v_{n-1}\}$, and $\{v_1^*, \dots, v_{n-1}^*\}$ for $k \geqslant n+1$. If $\{t_{n(\alpha)}\}$ is a net from $\{t_n\}$ with $\sigma(A^{**}, A^*)$-limit t, we hence conclude that $(adt_{n(\alpha)})(c) = \sum_{i=1}^{n(\alpha)} (ada_i)(c)$ is equal to a fixed element of A for all α sufficiently large, c ranging over a norm-dense subset of C. Thus $(adt)(c) \in A$ for c ranging over a norm-dense subset of C, and Claim 2 follows.

Claim 3: qC contains $e_n(i,i), i = 1, \dots, 2^n - 1,\ n \geqslant 1$. By (i), $e_n(1,1) = qe_n \in qC$, $n \geqslant 1$, and by (ii),

$$e_n(i,i) = e_n(i,1)e_n(i,1)^* = qv_{i_1}\dots v_{i_p} e_n^2 v_{i_p}^* \dots v_{i_1}^* \in qC, \quad 2 \leqslant i \leqslant 2^{n-1}, \quad n \geqslant 2. \tag{10}$$

Thus Claim 3 holds for $n=1$, and assuming inductively that it holds for n we have, for $2 \leqslant i \leqslant 2^n - 1$, that

$$e_{n+1}(i+2^n, i+2^n) = e_n(i,i) - e_{n+1}(i,i),$$

which by (10) and the induction hypothesis is in qC. This and (10) shows that Claim 3 holds for $n+1$.

We now assert that $\delta = adt|_C : C \to A$ is outer in $M(A)$. This will show that A is not HCT and finish the proof.

Suppose to the contrary that there exists $m \in M(A)$ with $\delta = adm|_C$. Then we can find an element y of the commutant C' of C relative to A^{**} such that $m = t + y$. By Theorem 6.7.3 of [8], q can be chosen such that it commutes with B and $qAq = qB$. It hence follows, by the Kaplansky density theorem and the $\sigma(A^{**}, A^*)$-precompactness of bounded sets of A^{**}, that $qB^- = (qB)^- = qA^{**}q$ (here and in what follows, S^- denotes the $\sigma(A^{**}, A^*)$-closure of a subset S of A^{**}). Thus

$$qyq \in qB^- \cap C'. \tag{11}$$

Let $b \in B$. Since $m \in M(A)$, we have $qmqb = qmbq \in qAq = qB$, and similarly $bqmq \in qB$. Hence $qmq \in M(qB)$. But qB is a UHF algebra, and hence has an identity, and so

$$qmq \in qB. \tag{12}$$

Now, let H be a fixed separable Hilbert space with orthonormal basis $\{\xi_n\}_{n \geqslant 1}$. By construction, qB is a UHF algebra with matrix units $\{e_n(i,j) : i, j = 1, \dots, 2^n\}$, $n \geqslant 1$, and we now define a faithful representation π of qB on H as follows. Fix positive integers n and p, and write $p = m \cdot 2^n + r$ in its unique representation with m and r integers, $m \geqslant 0$, $1 \leqslant r \leqslant 2^n$. Then set

$$\pi(e_n(i,j))\xi_p = \begin{cases} 0 & , \quad r \neq j \\ \xi_{m \cdot 2^n + i}, & r = j. \end{cases} \tag{13}$$

By examining the proof of Theorem 6.7.3 of [8], we find that q is central and closed with respect to $B^- = B^{**}$, and so by Lemma 2.1, π has a unique extension to a normal representation of qB^- into $B(H)$, which we also denote by π.

Set

$$T_n = \sum_{i=1}^{n} \pi(e_{i+2}(3\cdot 2^i, 2^i)), \quad n \geqslant 1.$$

Each T_n is a finite sum of partial isometries with orthogonal initial spaces and orthogonal final spaces, and hence has norm 1. We wish to evaluate T_n at ξ_p. If p does *not* have a representation of the form $m\cdot 2^{i+2}+2^i$, then $T_n\xi_p=0$. Otherwise, $T_n\xi_p=\xi_{2^i(4m+3)}$. Thus for $n>p$, $T_n\xi_p$ is constant, and so by the uniform boundedness of $\{T_n\}$, the infinite sum $\sum_{i=1}^{\infty} \pi(e_{i+2}(3\cdot 2^i, 2^i))$ converges in the ultraweak topology on $B(H)$ to an operator T with $\|T\|=1$. Now $qt=\lim_\alpha qt_{n(\alpha)}(\sigma(A^{**}, A^*))$, and by (viii) and the definition of t_n, we have $\pi(qt_{n(\alpha)})=T_{n(\alpha)}$, for each α. Thus by the normality of π, $T=\pi(qt)$.

Let $D=$ the C^*-subalgebra of qB generated by $e_n(i,i)$; $i=1,\ldots,2^n$, $n\geqslant 1$. It follows from Claim 3 that the commutant of qC relative to qB^- equals the commutant of D relative to qB^-, and so by (11), we deduce that

$$\pi(qyq)\in\pi(D)', \tag{14}$$

where the commutant in (14) is taken relative to $B(H)$. Thus by (12) and (14),

$$T=\pi(qt)\in\pi(qB)+\pi(D)'. \tag{15}$$

We will now show that for each $d\in\pi(D)'$, $T-d$ has distance at least 1 from $\pi(qB)$. This contradicts (15), and hence shows that δ is outer in $M(A)$.

In what follows, we identify qB with its image under π, and hence repress π in the notation.

Let $d\in D'$. We claim first that d is diagonal relative to the basis $\{\xi_n\}$. Fix distinct positive integers p and q, and consider $(d\xi_p, \xi_q)$. If n is chosen so that 2^n exceeds both p and q, then by (13), $e_n(q,q)$ fixes ξ_q and maps ξ_p to 0. Thus $(d\xi_p,\xi_q)=(de_n(q,q)\xi_p,\xi_q)=0$, and so d is diagonal. Let (d_n) denote the diagonal of d.

Let k be a fixed positive integer, and select $S\in$ span of $\{e_k(i,j): i,j=1,\ldots,2^k\}$. Let h be a fixed integer exceeding k. Then by (13),

$$\begin{aligned}(T-d)\xi_{2^h} &= -d\xi_{2^h}+\sum_{i=1}^{\infty} e_{i+2}(3\cdot 2^i,2^i)\xi_{2^h}\\ &= -d_{2^h}\cdot\xi_{2^h}+\xi_{3\cdot 2^h}.\end{aligned} \tag{16}$$

On the other hand, there are scalars s_{ij} such that

$$S\xi_{2^h}=\sum_{1\leqslant i,j\leqslant 2^k} s_{ij}e_k(i,j)\xi_{2^h}.$$

From (13) and the fact that $2^h=(2^{h-k}-1)2^k+2^k$ is the unique decomposition of 2^h in the form $m\cdot 2^k+r$, $m\geqslant 0$, $1\leqslant r\leqslant 2^k$, m and r integers, we conclude that

$$e_k(i,j)\xi_{2^h}=\begin{cases}0 &, \quad j\neq 2^k,\\ \xi_{2^h-2^k+i}, & \quad j=2^k,\end{cases}$$

and so

$$S\xi_{2^h} = \sum_{i=1}^{2^k} s_{i,2^k}\xi_{2^h-2^k+i}. \tag{17}$$

Thus by (16) and (17), $\|T-d-S\| \geqslant 1$, and since k and $S \in A_k$ are arbitrary, we hence deduce from the fact that $\bigcup_k A_k$ is norm-dense in qB that $T-d$ has distance at least 1 from qB. □

Proof of Theorem 1.1. ($\Leftarrow$). This is a consequence of Theorem 1.1 of [6].

($\Rightarrow$). If A is a separable HCT algebra, then in particular every derivation of A into itself is inner in $M(A)$, and so by Theorem 3.9 of [2] (see also Theorem 1 of [5]), A has a direct sum decomposition of the form $B \oplus C$, where B has continuous trace and C is the restricted direct sum of a family of simple C^*-algebras. Now it is easy to see that hereditary cohomological triviality passes to ideals, and we hence conclude that B and each direct summand of C are HCT. Thus by the separability of B and Theorem 1.1 of [6], B has a direct-sum decomposition of the form $A_1 \oplus B_1$, where A_1 is commutative and B_1 is the restricted direct sum of a sequence of separable elementary C^*-algebras. On the other hand, each direct summand of C is separable, simple, and HCT, and so by Lemma 2.2, each summand of C is liminary, and hence elementary. Setting $A_2 = B_1 \oplus C$, we thus obtain a direct-sum decomposition of A of the desired type. □

REMARKS. (1) As mentioned in the Introduction, Christensen has shown that infinite-dimensional finite factors are HCT. Since such algebras are antiliminary, it follows that the nonseparable analog of Theorem 1.1 does not hold.

(2) Let A be a C^*-algebra, with B a C^*-subalgebra of A. In the introduction to [1], Akemann and Johnson indicated the importance of studying pairs (B,A) as above for which every derivation of B into A is inner. They in fact showed that if $A = C \otimes B(H)$, where C is a separable, unital C^*-algebra with only inner derivations and H is a separable Hilbert space, then every derivation of $1 \otimes B(H)$ into A is inner. In contrast to this example, the construction used in the proof of Lemma 2.2 can be used to show that if A is any UHF algebra, generated by an ascending sequence $\{A_n\}$ of full matrix algebras, say, and if D is the commutative C^*-subalgebra of A generated by the sequence $\{D_n = \text{diagonal subalgebra of } A_n\}$, then there are outer derivations of D into A. We will indicate briefly how this construction proceeds in the case when A is $U(2^\infty)$, the CAR algebra of mathematical physics.

In this example, A is generated by an ascending sequence of matrix algebras $\{A_n\}$, with A_n isomorphic to the algebra of $2^n \times 2^n$ complex matrices, $n \geqslant 1$, and A_n is embedded in A_{n+1} via the relations

$$e_n(i,j) = e_{n+1}(i,j) + e_{n+1}(i+2^n, j+2^n); \quad i,j = 1,\ldots,2^n,$$

where $\{e_n(i,j) : i,j = 1,\ldots,2^n\}$ is a system of matrix units for A_n, $n \geqslant 1$ (this is of course the algebra qB which appears in the proof of Lemma 2.2). Representing A on a Hilbert space H as in the proof of Lemma 2.2, one now notes that if $T \in B(H)$, then $(adT)(D) \subseteq A$ whenever

$$(*) \qquad e_n(i,i)Te_n(j,j)\in A; \quad i,j=1,\ldots,2^n, \quad i\neq j, \quad n\geq 1.$$

Letting $T_n=\sum_{i=1}^{n} e_{i+2}(3\cdot 2^i, 2^i)$, $n\geq 1$, one then shows that $\{T_n\}$ converges in the strong operator topology to a $T\in B(H)$ and that T satisfies $(*)$. The last part of the proof of Lemma 2.2 then applies to place $T-S$ at a distance at least 1 from A for each operator S in the commutant of D. Restricting adT to D therefore yields an outer derivation of D into A.

REFERENCES

1. C. A. Akemann and B. E. Johnson, *Derivations of nonseparable C*-algebras,* J. Funct. Anal. 33 (1979), 311–331.
2. C. A. Akemann and G. K. Pedersen, *Central sequences and inner derivations of separable C*-algebras,* Amer. J. Math. 101 (1979), 1047–1061.
3. E. Christensen, *Extensions of derivations,* J. Funct. Anal. 27 (1978), 234–247.
4. J. Dixmier, *C*-algebras,* North-Holland, Amsterdam, 1977.
5. G. A. Elliott, *Some C*-algebras with outer derivations,* III, Ann. of Math. (2) 106 (1977), 121–143.
6. A. J. Lazar, S.-K. Tsui, and S. Wright, *Derivations from subalgebras of C*-algebras with continuous trace,* to appear.
7. G. K. Pedersen, *Isomorphisms of* UHF *algebras,* J. Funct. Anal. 30 (1978), 1–16.
8. ———, *C*-algebras and their automorphism groups,* Academic Press, London, 1979.

Department of Mathematics
Oakland University
Rochester, Michigan 48063