VALUES OF BMOA FUNCTIONS ON INTERPOLATING SEQUENCES

Carl Sundberg

1. Introduction and preliminaries. For $0<p\leqslant\infty$ we denote by H^p the usual Hardy class of functions analytic in $\mathbf{D}=\{z:|z|<1\}$. Denote by $\rho(z,w)=|(z-w)/(1-\bar{z}w)|$ the pseudo-hyperbolic distance between $z,w\in\mathbf{D}$. Let $\{z_n\}\subseteq\mathbf{D}$ be an *interpolating sequence,* i.e. a sequence satisfying $\inf_n \prod_{m\neq n}\rho(z_m,z_n)>0$. A theorem of Carleson [1] states that if $\{\alpha_n\}$ is any bounded sequence of numbers then there is an $f\in H^\infty$ such that $f(z_n)=\alpha_n$ for all n. For $0<p<\infty$, a subsequent theorem of Shapiro and Shields [9] characterizes the sequences $\{\alpha_n\}$ for which there exist $f\in H^p$ satisfying $f(z_n)=\alpha_n$ for all n; they are just the sequences for which $\sum_n|\alpha_n|^p(1-|z_n|^2)<\infty$. The purpose of this paper is to extend this characterization to another space of analytic functions closely related to the Hardy spaces, the space BMOA of analytic functions of bounded mean oscillation.

The following definitions and notations will be used in the sequel. If $E\subseteq\partial\mathbf{D}=\{z:|z|=1\}$ we denote by $|E|$ the Lebesgue measure of E. If $f\in L^1(\partial\mathbf{D})$ and $E\subseteq\partial\mathbf{D}$ we write $f_E=(1/|E|)\int_E f$, and

$$\|f\|_*=\sup\left\{\frac{1}{|I|}\int_I|f-f_I|: I \text{ an arc in } \partial\mathbf{D}\right\}.$$

The space BMO is the space of functions f for which $\|f\|_*<\infty$. For $f\in L^1(\partial\mathbf{D})$ and $z\in\mathbf{D}$ we write $f(z)$ for the Poisson extension of f at z:

$$f(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-|z|^2}{|1-e^{-i\theta}z|^2}f(e^{i\theta})\,d\theta.$$

The space BMOA is the set of $f\in$ BMO whose Poisson extensions are analytic in $\mathbf{D}$.

We define a *Carleson measure* to be a positive measure μ on $\mathbf{D}$ for which

$$\|\mu\|_*=\sup_{z_0\in\mathbf{D}}\iint_{\mathbf{D}}\frac{1-|z_0|^2}{|1-\bar{z}_0 z|^2}\,d\mu(z)<\infty.$$

If z_n is an interpolating sequence, then the measure $\sum_n\delta_{z_n}(1-|z_n|^2)$ is a Carleson measure (see Chapter VII, Theorem 1.1 of [5]).

For $a\in\mathbf{D}$ we define $\phi_a:\mathbf{D}\to\mathbf{D}$ by $\phi_a(z)=(z-a)/(1-\bar{a}z)$. It is well known that then $\|f\circ\phi_a\|_*\leqslant C\|f\|_*$, C a universal constant (see Chapter VI, Section 3 of [5]). If $w_n=\phi_a(z_n)$, then it is easy to check that $\|\sum_n\delta_{w_n}(1-|w_n|^2)\|_*=\|\sum_n\delta_{z_n}(1-|z_n|^2)\|$. Finally, we state a theorem of Carleson that we will need (see Chapter II, Theorem 3.9 of [5]):

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If μ is a Carleson measure, $1<p<\infty$, and $f\in L^p(\partial\mathbf{D})$, then there is a constant depending only on p and $\|\mu\|_*$ such that

$$\iint_{\mathbf{D}} |f(z)|^p \, d\mu(z) \leqslant C\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(e^{i\theta})|^p \, d\theta.$$

2. The main result.

THEOREM 2.1. *Let $\{z_n\}\subseteq\mathbf{D}$ be an interpolating sequence. Then the following statements are equivalent for a sequence of numbers $\{\alpha_n\}$.*

(a) *There is a* BMOA *function f such that $f(z_n)=\alpha_n$ for all n.*

(b) *There is a* BMO *function f such that $f(z_n)=\alpha_n$ for all n.*

(c) *There is a positive number λ and numbers $\{\beta(z)\}_{z\in\mathbf{D}}$ such that*

$$\sup_{z\in\mathbf{D}} \sum_n \exp[\lambda|\alpha_n-\beta(z)|][1-\rho(z,z_n)^2]<\infty.$$

(d) *There is a positive number λ such that*

$$\sup_{z\in\mathbf{D}} \sum_n \exp[\lambda|\alpha_n-\alpha(z)|][1-\rho(z,z_n)^2]<\infty,$$

where the average $\alpha(z)$ is defined as

$$\alpha(z)=\frac{\sum_n \alpha_n[1-\rho(z,z_n)^2]}{\sum_n [1-\rho(z,z_n)^2]}.$$

Proof. The implications from (a) to (b) and from (d) to (c) are of course immediate. Those from (b) to (c), (c) to (d), and (b) to (a) are not difficult and are done below. The hard part of the proof is the implication from (c) to (b); this will be done using the result of a construction that will be given in Section 3.

(b)$\Rightarrow$(c) By the John–Nirenberg Theorem [7] there are positive numbers λ_0, M such that if $h\in$ BMO with $\|h\|_*\leqslant 1$, then $(1/2\pi)\int_{-\pi}^{\pi}\exp(\lambda_0|h(e^{i\theta})-h(0)|)\,d\theta\leqslant M$. Now say $f\in$ BMO satisfies $f(z_n)=\alpha_n$ for all n. As was mentioned above, there is a constant C such that for any $z\in\mathbf{D}$, $\|f\circ\phi_{-z}\|_*\leqslant C\|f\|_*$. Pick $z\in\mathbf{D}$ and set $g=f\circ\phi_{-z}$, $w_n=\phi_z(z_n)$. Writing $\lambda=\lambda_0/C\|f\|_*$, we then have that $(1/2\pi)\int_{-\pi}^{\pi}\exp(\lambda|g(e^{i\theta})-g(0)|)\,d\theta\leqslant M$. Let G be the Poisson extension of $\exp[(\lambda/2)|g(e^{i\theta})-g(0)|]$. Since $\exp[(\lambda/2)|g(w)-g(0)|]$ is subharmonic as a function of $w\in\mathbf{D}$, we have that $G(w)\geqslant\exp[(\lambda/2)|g(w)-g(0)|]$ for all $w\in\mathbf{D}$. By the result of Carleson mentioned at the end of the Introduction, there is a constant C_1 depending only on $\|\delta_{w_n}(1-|w_n|^2)\|_*=\|\delta_{z_n}(1-|z_n|^2)\|_*$ such that

$$\begin{aligned}\sum_n \exp(\lambda|g(w_n)-g(0)|)(1-|w_n|^2) &\leqslant \sum_n G(w_n)^2(1-|w_n|^2)\\ &\leqslant C_1\frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{i\theta})^2\,d\theta\\ &= C_1\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp(\lambda|g(e^{i\theta})-g(0)|)\,d\theta\\ &\leqslant C_1 M.\end{aligned}$$

Using the definitions of g and w_n, we see that this is equivalent to

$$\sum_n \exp(\lambda|\alpha_n - f(z)|)[1-\rho(z,z_n)^2] \leqslant C_1 M,$$

which yields (c) with $\beta(z)=f(z)$.

(c)$\Rightarrow$(d) The proof of this implication is due to Wolff; it is exactly the same as the proof of the parallel point in [10]. Say $\lambda>0$, $\{\beta(z)\}_{z\in\mathbf{D}}$ and $M<\infty$ are such that $\sum_n \exp[\lambda|\alpha_n-\beta(z)|][1-\rho(z,z_n)^2] \leqslant M$, all $z\in\mathbf{D}$. Then for $z\in\mathbf{D}$ fixed and $t>0$,

$$\sum_{\{n:|\alpha_n-\beta(z)|>t\}} [1-\rho(z,z_n)^2] \leqslant \min\left\{\sum_n [1-\rho(z,z_n)^2], Me^{-\lambda t}\right\}.$$

Hence (using, e.g., Lemma 4.1 from Chapter I of [5]),

$$\begin{aligned}
|\alpha(z)-\beta(z)| &\leqslant \frac{1}{\sum_n [1-\rho(z,z_n)^2]} \sum_n |\alpha_n-\beta(z)|[1-\rho(z,z_n)^2] \\
&\leqslant \frac{1}{\sum_n [1-\rho(z,z_n)^2]} \int_0^\infty \min\left\{\sum_n [1-\rho(z,z_n)^2], Me^{-\lambda t}\right\} dt \\
&= \frac{1}{\lambda} \log \frac{M}{\sum_n [1-\rho(z,z_n)^2]} + \frac{1}{\lambda},
\end{aligned}$$

so by Hölder's inequality

$$\begin{aligned}
&\sum_n \exp[(\lambda/2)|\alpha_n-\alpha(z)|][1-\rho(z,z_n)^2] \\
&\quad\leqslant \left\{\sum_n \exp[\lambda|\alpha_n-\beta(z)|][1-\rho(z,z_n)^2]\right\}^{1/2} \\
&\quad\quad\times \exp[(\lambda/2)|\alpha(z)-\beta(z)|]\left\{\sum_n [1-\rho(z,z_n)^2]\right\}^{1/2} \\
&\quad\leqslant e^{1/2}M,
\end{aligned}$$

proving (d) with λ replaced by $\lambda/2$ and M replaced by $e^{1/2}M$.

(b)$\Rightarrow$(a) It clearly will suffice to consider the case where the α_n's are real. Let u be a real-valued BMO function such that $u(z_n)=\alpha_n$ for all n. By the Duality Theorem of C. Fefferman ([3], [4]) we can find real-valued $g, h\in L^\infty$ such that $u=g+\tilde{h}$ and $h(0)=0$—as usual $\tilde{h}$ denotes the Hilbert transform of h. Then $\tilde{u}=\tilde{g}-h$. Using Carleson's Interpolation Theorem we can find $F\in H^\infty$ such that $F(z_n)=g(z_n)+ih(z_n)$. Define $f=u+i\tilde{u}-(g+i\tilde{g})+F$. Clearly $f\in$ BMOA, and it is easy to check that $f(z_n)=\alpha_n$ for all n.

(c)$\Rightarrow$(b) Let λ, $\{\beta(z)\}_{z\in\mathbf{D}}$, $\{\alpha_n\}$ be as in (c). We again may assume the α_n's are real. Proposition 3.3 in the following section will yield $u\in$ BMO such that $\{\alpha_n-u(z_n)\}$ is bounded. By Carleson's Interpolation Theorem there is $g\in H^\infty$ such that $g(z_n)=\alpha_n-u(z_n)$ for all n. Then $f=g+u$ is in BMO and satisfies $f(z_n)=\alpha_n$, all n.

3. A construction. The main result of this section, Proposition 3.3, completes the proof of Theorem 2.1. We will first need to discuss some definitions and results. In this section all functions are real-valued.

DEFINITIONS. A *dyadic arc* is an arc $I \subseteq \partial\mathbf{D}$ of the form

$$I=\left\{e^{i\theta}: 2\pi\frac{l}{2^k} \leqslant \theta \leqslant 2\pi\frac{l+1}{2^k}\right\}$$

for some nonnegative integers k, l. We will denote the set of dyadic arcs by $\mathfrak{D}$, and if $J\in\mathfrak{D}$ we define $\mathfrak{D}_J=\{I\in\mathfrak{D}: I\subseteq J\}$.

Let $J\in\mathfrak{D}$. We define $\mathrm{BMO}_d(J)$ to be the set of those $f\in L^1(J)$ for which

$$\|f\|_{\mathrm{BMO}_d(J)}=\sup_{I\in\mathfrak{D}_J}\frac{1}{|I|}\int_I|f-f_I|<\infty.$$

We will write BMO_d for $\mathrm{BMO}(\partial\mathbf{D})$ and refer to this space as *dyadic BMO*.

A theorem of Davis (Theorem 3.1 of [2]) easily implies the following result, first stated and exploited by Garnett and Jones in [6].

THEOREM 3.1 (Davis-Garnett-Jones). *Let $\tau\rightarrow u^{(\tau)}$ be a measurable mapping from $[0,2\pi)$ to BMO_d. Define u by*

$$u(e^{i\theta})=\frac{1}{2\pi}\int u^{(\tau)}(e^{i(\theta+\tau)})\,d\tau.$$

Then there is a universal constant C such that $\|u\|_*\leqslant C\sup_\tau\|u^{(\tau)}\|_{\mathrm{BMO}_d}$.

The construction needed to prove Proposition 3.3 is based on the dyadic version of a theorem of Wolff [10].

THEOREM 3.2 (Wolff). *Let $J\in\mathfrak{D}$ and let $\Omega\subseteq J$ be measurable with $|\Omega|>0$. Let f be a function on Ω and suppose there exist numbers $\lambda>0, M$, and for each $I\in\mathfrak{D}_J$ a number a_I such that*

$$\frac{1}{|I|}\int_{\Omega\cap I}\exp(\lambda|f-a_I|)\leqslant M.$$

Then f is the restriction to Ω of a function $F\in\mathrm{BMO}_d(J)$ with $\|F\|_{\mathrm{BMO}_d(J)}$ bounded by a constant depending only on λ and M. Moreover we can choose F so that $\int_J|F-a_J|\leqslant C\int_\Omega|f-a_J|$, where C depends only on λ and M.

REMARK. Wolff does not state his result in this way—in particular he works with BMO instead of dyadic BMO and defines $a_I=(1/|\Omega\cap I|)\int_{\Omega\cap I}f$. However, the above statement follows from his proof.

We are now ready for the main result of this section.

PROPOSITION 3.3. *Let $\{z_n\}\subseteq\mathbf{D}$ be an interpolating sequence. Let $\{\alpha_n\}\subseteq\mathbf{R}$, $\lambda>0$, $M>0$, and $\{\beta(z)\}_{z\in\mathbf{D}}$ be such that*

$$\sum_n \exp[\lambda\,|\,\alpha_n-\beta(z)|][1-\rho(z,z_n)^2]\leq M \quad \text{for all } z\in\mathbf{D}.$$

Then there is a $u\in$ BMO *such that the sequence* $\{\alpha_n-u(z_n)\}$ *is bounded.*

Proof. To get an idea of the proof, let us temporarily define I_n to be the arc with center $z_n/|z_n|$ and length $2\pi(1-|z_n|)$. Suppose that the collection $\{I_n\}$ is pairwise disjoint and define f on $\bigcup I_n$ by setting $f\equiv\alpha_n$ on I_n. If we could extend f to a BMO function F on all of $\partial\mathbf{D}$, then clearly $F_{I_n}=\alpha_n$, and so by a well-known property of BMO functions the sequence $\{\alpha_n-F(z_n)\}$ would be bounded. This approach will not work in general, however, since the arcs $\{I_n\}$ of course need not be pairwise disjoint and also since the function f defined above might not be extendable to a BMO function. These difficulties are taken care of by using instead of the arcs $\{I_n\}$ a certain closely related collection of dyadic arcs, and by building the function F in nested stages using the above idea. The resulting function is not in BMO but is in BMO_d, and Theorem 3.1 is then used to finish the proof.

In this proof the letter C will stand for various constants depending at most on λ and M, and not necessarily the same at each occurrence. Clearly we may assume the numbers $\beta(z)$ are real.

We first thin the sequence $\{z_n\}$ out somewhat; find a subsequence $\{z_{n_j}\}$ such that (i) $\rho(z_{n_j},z_{n_k})>99/100$ if $j\neq k$ and (ii) $\forall n\,\exists j$ such that $\rho(z_{n_j},z_n)<999/1000$. Clearly the hypothesis implies that $\exp(\lambda|\alpha_n-\beta(z_n)|)\leq M$, hence $|\alpha_n-\beta(z_n)|\leq(1/\lambda)\log M$. If $\rho(z_m,z_n)<999/1000$ then by the hypothesis $\exp(\lambda|\alpha_m-\beta(z_n)|)<1000M$, so $|\alpha_m-\beta(z_n)|<(1/\lambda)\log(1000M)$. This shows that if $\rho(z_m,z_n)<999/1000$, then $|\alpha_n-\alpha_m|<(1/\lambda)[\log(1000M)+\log M]$. It is a standard fact that there is a constant C such that if $\rho(z,w)<999/1000$ and $f\in$ BMO, then $|f(z)-f(w)|<C\|f\|_*$. Say now that $f\in$ BMO, $\|f\|_*$ is bounded by a constant depending only on λ and M, and $|f(z_{n_j})-\alpha_{n_j}|<C$ for all j. For a given n find by (ii) a j such that $\rho(z_n,z_{n_j})<999/1000$. Then

$$|f(z_n)-\alpha_n|\leq|f(z_n)-f(z_{n_j})|+|f(z_{n_j})-\alpha_{n_j}|+|\alpha_{n_j}-\alpha_n|<C$$

by the above observations. These remarks show that we may as well assume from the outset that $\rho(z_m,z_n)>99/100$ if $m\neq n$. Since we may conformally translate the points $\{z_n\}$ without altering the hypothesis or the conclusion we may assume $z_0=0$. Finally we may subtract α_0 from each term of the sequence $\{\alpha_n\}$ without affecting hypothesis or conclusion, so we may assume that $\alpha_0=0$.

To a dyadic arc

$$I=\left\{e^{i\theta}: 2\pi\frac{l}{2^k}\leq\theta\leq 2\pi\frac{l+1}{2^k}\right\}$$

we associate the "dyadic square"

$$S_I=\left\{re^{i\theta}: 2\pi\frac{l}{2^k}\leq\theta<2\pi\frac{l+1}{2^k},\ 1-\frac{1}{2^k}\leq r<1\right\}$$

and its "top half"

$$T_I = \left\{ re^{i\theta} \in S_I : r < 1 - \frac{1}{2^{k+1}} \right\}.$$

Note that by the condition $\rho(z_m, z_n) > 99/100$ if $m \neq n$, there is at most one z_n in any T_I.

Our first task will be to construct a BMO_d function F such that $|F_I - \alpha_n| < C$ if $z_n \in T_I$. To this end we define the collection $\mathcal{I} \subseteq \mathfrak{D}$ as follows. $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2$ where $\mathcal{I}_1 = \{I \in \mathfrak{D} : \exists z_n \in T_I\}$ and $\mathcal{I}_2 = \{I \in \mathfrak{D} : \exists J \in \mathcal{I}_1$ such that $|J| = |I|$ and J is adjacent to $I\}$. The reason for considering the extra arcs $\mathcal{I}_2$ will become apparent at the end of the proof.

Let $I \in \mathfrak{D}$ and pick a point $z_I \in T_I$. To I we will then associate the number $\alpha_I = \beta(z_I)$. Note that if $z_n \in T_I$ or if $|I| = |J|$, J is adjacent to I, and $z_n \in T_J$, then $\rho(z_I, z_n) < 999/1000$. Hence from the hypothesis

$$\exp[\lambda|\alpha_n - \beta(z_I)|] \frac{1}{1000} < \exp[\lambda|\alpha_n - \beta(z_I)|][1 - \rho(z_I, z_n)^2] \leqslant M,$$

so that $|\alpha_n - \alpha_I| < C$. Since $z_0 = 0$ and $\alpha_0 = 0$ we may also assume that $\alpha_{\partial \mathbf{D}} = 0$.

CLAIM. If $J \in \mathfrak{D}$ then $\sum_{\substack{I \subseteq J \\ I \in \mathcal{I}}} \exp(\lambda|\alpha_I - \alpha_J|)(|I|/|J|) \leqslant C$.

To prove this Claim we first note that an easy and well-known estimate shows that if $z \in T_J$ and $z_n \in S_J$, then $(1 - |z|^2)/|1 - \bar{z}_n z| > 1/10$. Hence

$$\begin{aligned}
\sum_{\substack{I \subseteq J \\ I \in \mathcal{I}_1}} \exp[\lambda|\alpha_I - \alpha_J|] \frac{|J|}{|I|} &\leqslant C \sum_{z_n \in S_J} \exp[\lambda|\alpha_n - \beta(z_J)|] \frac{1 - |z_n|^2}{1 - |z_J|^2} \\
&\leqslant 100C \sum_{z_n \in S_J} \exp[\lambda|\alpha_n - \beta(z_J)|] \frac{(1 - |z_n|^2)(1 - |z_J|^2)}{|1 - \bar{z}_n z_J|^2} \\
&= 100C \sum_n \exp[\lambda|\alpha_n - \beta(z_J)|][1 - \rho(z_J, z_n)^2] \leqslant CM.
\end{aligned}$$

To handle the arcs in $\mathcal{I}_2$ write J_1, J_2 for the two arcs adjacent to J for which $|J_1| = |J| = |J_2|$. If $I \in \mathcal{I}_2$, let I' be adjacent to I, $|I'| = |I|$, and $z_n \in T_{I'}$. Then $z_n \in S_{J_1} \cup S_J \cup S_{J_2}$. Hence, as is easily shown, $(1 - |z_J|^2)/|1 - \bar{z}_n z_J| > 1/100$, and we have already observed that $|\alpha_n - \alpha_I| < C$. Combining these observations, we see that

$$\begin{aligned}
\sum_{\substack{I \subseteq J \\ I \in \mathcal{I}_2}} e^{\lambda|\alpha_I - \alpha_J|} \frac{|I|}{|J|} &\leqslant C \sum_{z_n \in S_{J_1} \cup S_J \cup S_{J_2}} e^{\lambda|\alpha_n - \beta(z_J)|} \frac{1 - |z_n|^2}{1 - |z_J|^2} \\
&\leqslant 10000C \sum_{z_n \in S_{J_1} \cup S_J \cup S_{J_2}} e^{\lambda|\alpha_n - \beta(z_J)|} \frac{(1 - |z_n|^2)(1 - |z_J|^2)}{|1 - \bar{z}_n z_J|^2} \\
&\leqslant 10000CM.
\end{aligned}$$

Note that a trivial consequence of the claim is that

$$\sum_{\substack{I \subseteq J \\ I \in \mathcal{I}}} \frac{|I|}{|J|} \leqslant C.$$

Using the language of Garnett (Chapter VII, Section 3 of [5]) we now define the generations $G_0, G_1, G_2, \ldots$ as follows. G_0 is the singleton set $\{\partial\mathbf{D}\}$, and for $j>0$

$$G_j=\{I\in\mathcal{G}: I \text{ is a maximal arc in } \mathcal{G} \text{ satisfying } I\subsetneq J \text{ where } J\in G_{j-1}\}.$$

Set $\Omega_1=\bigcup\{I: I\in G_1\}$ and define f_1 on Ω_1 by setting $f_1\equiv\alpha_I$ on $I\in G_1$. Let $K\in\mathfrak{D}$. If $K\subseteq I\in G_1$, then $f_1\equiv\alpha_I$ on K, so if we set $a_K=\alpha_I$ we see that

$$\frac{1}{|K|}\int_{\Omega_1\cap K}\exp(\lambda|f_1-a_K|)=1.$$

If K is not contained in a member of G_1 set $a_K=\alpha_K$; then

$$\frac{1}{|K|}\int_{\Omega_1\cap K}\exp(\lambda|f_1-a_K|)=\frac{1}{|K|}\sum_{\substack{I\in G_1\\ I\subseteq K}}\exp(\lambda|\alpha_I-\alpha_K|)|I|\leqslant C$$

by our above Claim. Hence by Wolff's Theorem, Theorem 3.2, there is a function $F_1\in\mathrm{BMO}_d$ such that $F_1\equiv\alpha_I$ on $I\in G_1$, $\|F_1\|_{\mathrm{BMO}_d}\leqslant C$, and $\int_{\partial\mathbf{D}}|F_1|\leqslant C\sum_{I\in G_1}|\alpha_I||I|$ since $a_{\partial\mathbf{D}}=\alpha_{\partial\mathbf{D}}=0$.

Now for $J\in G_1$ define $\Omega_2^J=\bigcup\{I: I\in G_2,\ I\subseteq J\}$ and define f_2^J on Ω_2^J by setting $f_2^J\equiv\alpha_I-\alpha_J$ on I for $I\in G_2$, $I\subseteq J$. We verify the hypothesis of Wolff's Theorem in the same way as above and hence we can find $F_2^J\in\mathrm{BMO}_d(J)$ such that $F_2^J\equiv\alpha_I-\alpha_J$ on I for $I\in G_2$, $I\subseteq J$, $\|F_2^J\|_{\mathrm{BMO}_d(J)}\leqslant C$, and $\int_J|F_2^J|\leqslant C\sum_{\substack{I\in G_2\\ I\subseteq J}}|\alpha_I-\alpha_J||I|$. Define F_2 on $\partial\mathbf{D}$ by setting $F_2=F_2^J$ on $J\in G_1$, $F_2=0$ off $\bigcup\{J: J\in G_2\}$.

Continuing in this fashion we obtain functions $\{F_j\}_{j=1}^{\infty}$ satisfying the following properties:

(a) If $I\in G_j$, $I\subseteq J\in G_{j-1}$, then $F_j\equiv\alpha_I-\alpha_J$ on K.
(b) If $J\in G_{j-1}$ then $F_j\in\mathrm{BMO}_d(J)$ with $\|F_j\|_{\mathrm{BMO}_d(J)}\leqslant C$.
(c) $F_j=0$ off $\bigcup\{J: J\in G_{j-1}\}$.
(d) If $J\in G_{j-1}$ then

$$\int_J|F_j|\leqslant C\sum_{\substack{I\in G_j\\ I\subseteq J}}|\alpha_I-\alpha_J||I|.$$

We now set $F=\sum_{j=1}^{\infty}F_j$. Let $I\in\mathfrak{D}$ and let J be the highest generation arc in $\mathcal{G}$ containing I, say $J\in G_{j_0}$. Notice that then $F_1,\ldots,F_{j_0}$ are constant on I and that by (b), $\|F_{j_0+1}\|_{\mathrm{BMO}_d(J)}\leqslant C$. Hence

$$\begin{aligned}\frac{1}{|I|}\int_I|F-F_I|&\leqslant\sum_{j=1}^{\infty}\frac{1}{|I|}\int_I|F_j-(F_j)_I|\\ &=\frac{1}{|I|}\int_I|F_{j_0+1}-(F_{j_0+1})_I|+2\sum_{j=j_0+2}^{\infty}\frac{1}{|I|}\int_I|F_j|\\ &\leqslant C+2\sum_{j=j_0+2}^{\infty}\sum_{\substack{J\in G_{j-1}\\ J\subseteq I}}\frac{1}{|I|}\int_J|F_j|\\ &\leqslant C+2C\sum_{j=j_0+2}^{\infty}\sum_{\substack{J\in G_{j-1}\\ J\subseteq I}}\frac{1}{|I|}\sum_{\substack{K\in G_j\\ K\subseteq J}}|\alpha_K-\alpha_J||K|,\quad\text{by (d).}\end{aligned}$$

Now by our above Claim,

$$\sum_{\substack{K\in G_j\\ K\subseteq J}} \exp(\lambda|\alpha_K-\alpha_J|)\frac{|K|}{|J|}\leqslant C, \quad \text{so} \quad \sum_{\substack{K\in G_j\\ K\subseteq J}} |\alpha_K-\alpha_J||K|\leqslant C|J|.$$

Hence the above sum is bounded by

$$C+C\sum_{j=j_0+2}^{\infty}\sum_{\substack{J\in G_{j-1}\\ J\subseteq I}}\frac{|J|}{|I|}\leqslant C+C\sum_{\substack{J\in \mathcal{G}\\ J\subseteq I}}\frac{|J|}{|I|}\leqslant C,$$

by the remark following our Claim. This shows that $\|F\|_{\mathrm{BMO}_d}\leqslant C$. Furthermore if $I\in\mathcal{G}$, say $I\in G_{j_0}$, then $F_1+\cdots+F_{j_0}\equiv\alpha_I$ on I, hence

$$\begin{aligned}|F_I-\alpha_I|&\leqslant\sum_{j=j_0+1}^{\infty}\frac{1}{|I|}\int_I|F_j|=\sum_{j=j_0+1}^{\infty}\sum_{\substack{J\in G_{j-1}\\ J\subseteq I}}\frac{1}{|I|}\int_J|F_j|\\ &\leqslant C\sum_{j=j_0+1}^{\infty}\sum_{\substack{J\in G_{j-1}\\ J\subseteq I}}\frac{1}{|I|}\sum_{\substack{K\in G_j\\ K\subseteq J}}|\alpha_K-\alpha_J||K|\\ &\leqslant C\end{aligned}$$

in exactly the same manner as above.

Bearing in mind our definition of the numbers α_I and the remarks immediately following that definition, we can summarize what we have done so far in the following manner. For $\{z_n\}$, $\{\alpha_n\}$, λ, M as in the statement of the Proposition we have constructed a function $F\in\mathrm{BMO}_d$ such that $\|F\|_{\mathrm{BMO}_d}\leqslant C$ and such that if $I\in\mathfrak{D}$ and $z_n\in T_I$ then $|F_I-\alpha_n|<C$. Furthermore because of our use of the extra arcs $\mathcal{G}_2$, if $I\in\mathfrak{D}$, $z_n\in T_I$, and $J\in\mathfrak{D}$ is adjacent to I with $|J|=|I|$, then $|F_J-\alpha_n|<C$.

Now for $0\leqslant\tau<2\pi$ define $z_n^{(\tau)}=z_ne^{i\tau}$ and construct a function $F^{(\tau)}\in\mathrm{BMO}_d$ as above corresponding to the points $\{z_n^{(\tau)}\}$ and the sequence $\{\alpha_n\}$. Define $u(e^{i\theta})=(1/2\pi)\int_0^{2\pi}F^{(\tau)}(e^{i(\theta+\tau)})\,d\tau$. Then by the Davies-Garnett-Jones result, Theorem 3.1, we have that $u\in\mathrm{BMO}$, $\|u\|_*\leqslant C$. Let $z_n\in T_I$ where $I\in\mathfrak{D}$. We have

$$\begin{aligned}u_I&=\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{|I|}\int_{e^{i\theta}\in I}F^{(\tau)}(e^{i(\theta+\tau)})\,d\theta\,d\tau\\ &=\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{|I|}\int_{e^{i\theta}\in Ie^{i\tau}}F^{(\tau)}(e^{i\theta})\,d\theta\,d\tau.\end{aligned}$$

We examine the inner integral. Say $|I|=2\pi/2^j$. Then $I\exp[i(2\pi k/2^j)]\in\mathfrak{D}$ and $z_n^{(2\pi k/2^j)}\in T_{I\exp[i(2\pi k/2^j)]}$. Hence by the construction

$$\left|\frac{1}{|I|}\int_{e^{i\theta}\in Ie^{i\tau}}F^{(\tau)}(e^{i\theta})\,d\theta-\alpha_n\right|<C$$

if $\tau=2\pi k/2^j$. If $\tau=(2\pi k/2^j)+\tau_1$, $0\leqslant\tau_1<2\pi/2^j$, set $I_1=I\exp(i2\pi k/2^j)$, $I_2=I\exp(i2\pi(k+1)/2^j)$, $J_1=I_1\cap Ie^{i\tau}$, $J_2=I_2\cap Ie^{i\tau}$. Then $I_1,I_2\in\mathfrak{D}$. Notice that for

τ in the range we are considering, either $z_n^{(\tau)} \in T_{I_1}$, or $z_n^{(\tau)} \in T_{I_2}$. By the comments in the preceding paragraph, in either case we have $|F_{I_1}^{(\tau)} - \alpha_n| < C$ and $|F_{I_2}^{(\tau)} - \alpha_n| < C$. Now a well-known consequence of the dyadic version of the John–Nirenberg Theorem tells us that

$$\left(\frac{1}{|I|}\int_{I_1} |F^{(\tau)} - \alpha_n|^2\right)^{1/2} < C \quad \text{and} \quad \left(\frac{1}{|I|}\int_{I_2} |F^{(\tau)} - \alpha_n|^2\right)^{1/2} < C.$$

Hence

$$\begin{aligned}\frac{1}{|I|}\int_{J_1} |F^{(\tau)} - \alpha_n| &\leqslant \left(\frac{1}{|I|}\int_{I_1} |F^{(\tau)} - \alpha_n|^2\right)^{1/2} \left(\frac{|J_1|}{|I_1|}\right)^{1/2} \\ &\leqslant C\left(\frac{|J_1|}{|I_1|}\right)^{1/2} \leqslant C\end{aligned}$$

and similarly $(1/|I|)\int_{J_2} |F^{(\tau)} - \alpha_n| \leqslant C$. Therefore

$$\begin{aligned}&\left|\frac{1}{|I|}\int_{Ie^{i\tau}} F^{(\tau)}(e^{i\theta})\, d\theta - \alpha_n\right| \\ &\quad = \left|\frac{1}{|I|}\int_{J_1} (F^{(\tau)}(e^{i\theta}) - \alpha_n)\, d\theta + \frac{1}{|I|}\int_{J_2} (F^{(\tau)}(e^{i\theta}) - \alpha_n)\, d\theta\right| \\ &\quad \leqslant C.\end{aligned}$$

This bound holds for all $\tau \in [0, 2\pi)$, and by integrating in τ we thus see that $|u_I - \alpha_n| < C$.

Now for $z_n \in T_I$, $I \in \mathfrak{D}$, write z_n' for the midpoint of the inner boundary of S_I. Since $u \in \mathrm{BMO}$, $\|u\|_* \leqslant C$, we have that $|u(z_n') - u(z_n)| < C$, and a well-known estimate (see Chapter 4 of [8]) tells us that $|u(z_n') - u_I| < C$. Hence $|u(z_n) - \alpha_n| < C$, completing the proof of the Proposition and of Theorem 2.1. □

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Department of Mathematics
University of Tennessee
Knoxville, Tennessee 37916