EXTRINSIC SPHERES IN A KÄHLER MANIFOLD

Seiichi Yamaguchi, Hiroaki Nemoto, and Nobuyuki Kawabata

1. Introduction. An $n(\geqslant 2)$-dimensional submanifold of an arbitrary Riemannian manifold is called an extrinsic sphere if it is totally umbilical and has nonzero parallel mean curvature vector [5]. An n-dimensional Riemannian manifold is called an intrinsic sphere if it is locally isometric to an ordinary sphere in a Euclidean space. Since extrinsic spheres are natural analogues of ordinary spheres in a Euclidean space from the extrinsic point of view, it is natural to ask when an extrinsic sphere is to be an intrinsic sphere. In this situation, it is well known that an extrinsic sphere in a Euclidean space is an intrinsic sphere. However, in general, an extrinsic sphere is not always an intrinsic sphere (see [4: p. 66], for example). On the other hand, when the ambient manifold is a Kähler manifold, B. Y. Chen has proved the following Theorem A:

THEOREM A [2]. *A complete, connected, simply connected and even-dimensional extrinsic sphere of a Kähler manifold is isometric to an ordinary sphere if its normal connection is flat.*

He has also given counterexamples which are not isometric to an ordinary sphere in odd-dimensional case [3].

In this paper, we shall try to classify a complete, connected and simply connected extrinsic sphere of a Kähler manifold. That is, we shall prove the following Theorem:

THEOREM. *A complete, connected and simply connected extrinsic sphere M^n in a Kähler manifold $\tilde{M}^{2m}$ is one of the following:*

(1) *M^n is isometric to an ordinary sphere,*
(2) *M^n is homothetic to a Sasakian manifold,*
(3) *M^n is a totally real submanifold and the f-structure is not parallel in the normal bundle.*

Here we note that case (2) *and* (3) *occur only when $n=$ odd and $m\geqslant n+1$, respectively.*

2. Preliminaries. Let $\tilde{M}$ be a Riemannian manifold of dimension m and M an n-dimensional submanifold of $\tilde{M}$. Let $\langle\ ,\ \rangle$ be the metric tensor field on $\tilde{M}$ as well as the induced metric on M. We denote by $\tilde{\nabla}$ the covariant differentiation in $\tilde{M}$ and by ∇ the covariant differentiation in M determined by the induced metric on M. Then the Gauss-Weingarten formulas are given by

$$\tilde{\nabla}_X Y=\nabla_X Y+h(X,Y),$$
$$\tilde{\nabla}_X N=-A_N X+\nabla^{\perp}_X N,$$

Received February 7, 1983. Revision received May 18, 1983.
Michigan Math. J. 31 (1984).

respectively, where X and Y are vector fields tangent to M and N normals to M. Moreover $\nabla^{\perp}$ is the linear connection induced in the normal bundle $T^{\perp}M$, called the normal connection, and h (resp. A_N) is called the second fundamental form (resp. the shape operator at N). Then h and A_N satisfy $\langle A_N X, Y\rangle = \langle h(X, Y), N\rangle$. If the second fundamental form h satisfies $h(X, Y) = \langle X, Y\rangle H$, then M is called a totally umbilical submanifold in $\tilde{M}$, where $H = (\text{trace}\, h)/n$ is the mean curvature vector of M in $\tilde{M}$. Then the Gauss-Weingarten formulas reduce to

$$\tilde{\nabla}_X Y = \nabla_X Y + \langle X, Y\rangle H,$$

$$\tilde{\nabla}_X N = -\langle N, H\rangle X + \nabla^{\perp}_X N.$$

If the mean curvature vector H of a totally umbilical submanifold is nonzero and is parallel with respect to the normal connection $\nabla^{\perp}$, then M is said to be an extrinsic sphere of $\tilde{M}$. The mean curvature $\mu = \sqrt{\langle H, H\rangle}$ of an extrinsic sphere is a nonzero constant.

Let $\tilde{M}$ be a real $2m$-dimensional (complex m-dimensional) almost Hermitian manifold with almost complex structure J and with Hermitian metric $\langle\ ,\ \rangle$. An n-dimensional Riemannian manifold M isometrically immersed in $\tilde{M}$ is called a totally real submanifold of $\tilde{M}$ if $JT_xM \subset T_x^{\perp}M$ for each point x of M, where T_xM denotes the tangent space of M at x and $T_x^{\perp}M$ the normal space of M at x. Let $N_x(M)$ be an orthogonal complement of JT_xM in $T_x^{\perp}M$. Then we have the decomposition $T_x^{\perp}M = JT_xM \oplus N_x(M)$. Thus we see that the space $N_x(M)$ is invariant under the action of J, that is, if $N \in N_x(M)$ then $JN \in N_x(M)$. If N is a vector field in the normal bundle $T^{\perp}M$, we put $JN = tN + fN$, where tN is the tangential part of JN and fN the normal part of JN. Then f is called an f-structure in the normal bundle $T^{\perp}M$. If $\nabla^{\perp}_X f = 0$ for any tangent vector field X, then the f-structure in the normal bundle is said to be parallel.

Next, let us recall the definition of a Sasakian manifold (for details, see [9]). Let M be a Sasakian manifold with structure tensors $(\phi, \xi, \eta, \langle\ ,\ \rangle)$. Then they satisfy

$$\phi^2 X = -X + \eta(X)\xi,$$

$$\phi\xi = 0, \qquad \eta(\phi X) = 0, \qquad \eta(\xi) = 1,$$

$$\langle \phi X, \phi Y\rangle = \langle X, Y\rangle - \eta(X)\eta(Y),$$

$$\eta(X) = \langle X, \xi\rangle, \qquad \nabla_X \xi = -\phi X,$$

$$(\nabla_X \phi) Y = \langle X, Y\rangle \xi - \eta(Y) X,$$

for any vector fields X and Y on M. A Sasakian manifold is odd-dimensional and orientable. The following Theorem B is known:

THEOREM B [7]. *Let M be a Riemannian manifold. If M admits a Killing vector field ξ of constant length satisfying*

$$k^2(\nabla_X \nabla_Y \xi - \nabla_{\nabla_X Y} \xi) = \langle Y, \xi\rangle X - \langle X, Y\rangle \xi$$

for a nonzero constant k and any vector fields X and Y on M, then M is homothetic to a Sasakian manifold.

The next Theorem C plays a fundamental role in this paper.

THEOREM C [6], [10]. *Let $M^n, n \geqslant 2$, be a complete, connected and simply connected Riemannian manifold. Then M^n admits a non-trivial solution ρ of*

$$(\nabla\nabla\omega)(X;Y;Z)+k^2(2\omega(Z)\langle X,Y\rangle+\omega(Y)\langle Z,X\rangle+\omega(X)\langle Y,Z\rangle)=0$$

for any vector fields X, Y and Z on M^n, where $\omega=d\rho$, if and only if M^n is isometric to an ordinary sphere.

3. Proof of Theorem. Let M^n be an extrinsic sphere in a Kähler manifold $\tilde{M}^{2m}$ with complex structure J. Let ν_x be the vector subspace of the normal space of M^n at $x \in M^n$ spanned by the mean curvature vector field H and $\nu_x^\perp$ be the orthogonal complement of ν_x in the normal space. Then ν and $\nu^\perp$ are differentiable vector bundles over M^n such that the normal bundle is the direct sum of theirs. For any tangent vector field X of M^n and a unit normal vector field $e=H/\mu$, we put

$$JX=\phi X-\eta(X)e+PX, \tag{3.1}$$

$$Je=\xi+\zeta, \tag{3.2}$$

where ϕX (resp. ξ) denotes the tangential component of JX (resp. Je), PX (resp. ζ) denotes the $\nu^\perp$-component of JX (resp. Je). Notice that e is a parallel unit normal vector field and

$$\tilde{\nabla}_X e=-\mu X. \tag{3.3}$$

It is clear from (3.1) and (3.2) that

$$\begin{aligned}\eta(X)&=-\langle JX,e\rangle=\langle X,Je\rangle=\langle X,\xi\rangle,\\ \langle\phi X,Y\rangle&=\langle JX,Y\rangle=-\langle X,JY\rangle=-\langle X,\phi Y\rangle.\end{aligned} \tag{3.4}$$

Differentiating (3.2) covariantly and making use of (3.1), (3.2) and (3.3), we obtain $\nabla_X\xi+\nabla_X^\perp\zeta=-\mu(\phi X+PX)$, from which

$$\nabla_X\xi=-\mu\phi X, \tag{3.5}$$

$$\nabla_X^\perp\zeta=-\mu PX. \tag{3.6}$$

It follows from (3.4) and (3.5) that ξ is a Killing vector field on M^n. Differentiating (3.1) covariantly and making use of (3.1), (3.2), (3.3), (3.4) and (3.5), we obtain

$$\nabla_X(\phi Y)+\mu\eta(Y)X+\nabla_X^\perp(PY)=\phi\nabla_X Y+P\nabla_X Y+\mu\langle X,Y\rangle\xi+\mu\langle X,Y\rangle\zeta,$$

which implies that

$$(\nabla_X\phi)Y=\nabla_X(\phi Y)-\phi\nabla_X Y=\mu(\langle X,Y\rangle\xi-\eta(Y)X), \tag{3.7}$$

$$P\nabla_X Y=\nabla_X^\perp(PY)-\mu\langle X,Y\rangle\zeta. \tag{3.8}$$

The equations (3.5) and (3.7) tell us that

$$\nabla_X \nabla_Y \xi - \nabla_{\nabla_X Y} \xi = \mu^2(\langle Y, \xi \rangle X - \langle X, Y \rangle \xi). \tag{3.9}$$

Now, let us consider a function ρ and a 1-form ω defined by $\rho = \langle \xi, \xi \rangle$ and $\omega(X) = d\rho(X) = X\rho$, respectively. Then we have from (3.5)

$$\omega(X) = 2\langle \tilde{\nabla}_X \xi, \xi \rangle = 2\langle \nabla_X \xi, \xi \rangle = -2\mu \langle \phi X, \xi \rangle. \tag{3.10}$$

Operating Y to $\omega(X)$ and using (3.5), (3.7) and (3.10), we get

$$(\nabla_Y \omega)X = -2\mu^2(\rho \langle X, Y \rangle - \eta(X)\eta(Y) - \langle \phi X, \phi Y \rangle). \tag{3.11}$$

Moreover, taking account of (3.5), (3.7), (3.10) and (3.11), we can see that the 1-form ω satisfies

$$(\nabla\nabla\omega)(X; Y; Z) + \mu^2(2\omega(Z)\langle X, Y \rangle + \omega(Y)\langle X, Z \rangle + \omega(X)\langle Y, Z \rangle) = 0.$$

Its calculations are simple but lengthy, so we omit the calculations. Thus, if the function ρ is non-trivial, that is, ρ is not a constant, M is isometric to an ordinary sphere by means of Theorem C. This proves Case (1) of our Theorem.

Next, if the function $\rho = \langle \xi, \xi \rangle$ is a nonzero constant, then Theorem B with (3.9) implies that M^n is homothetic to a Sasakian manifold since ξ is a Killing vector field on M^n. This proves Case (2) of our Theorem.

Lastly, if the function ρ vanishes identically, i.e., $\xi \equiv 0$, then from (3.1) and (3.5) we see that $JX = PX$, which means that M^n is a totally real submanifold. In this case, the vector bundle $\nu^\perp$ is decomposed more finely. Since $\langle JTM, J\nu \rangle = 0$ and $J\nu \subset \nu^\perp$, we find, for each point x of M^n, $T_x^\perp M = JT_x M \oplus \nu_x \oplus J\nu_x \oplus \tau_x$, where τ_x is the orthogonal complement of $JT_x M \oplus \nu_x \oplus J\nu_x$ in $T_x^\perp M$ which is invariant under the complex structure J. Thus we have $\dim(\tau_x) = 2m - 2n - 2 \geqslant 0$, that is, $m \geqslant n+1$. Next we show that the f-structure is not parallel in the normal bundle. If N is a vector field in the normal bundle $T^\perp M$, we put $JN = tN + fN$, where tN is the tangential part of JN and fN the normal part of JN. Operating X to the above equation, we have

$$\nabla_X(tN) + \mu\langle X, tN \rangle e - \mu\langle fN, e \rangle X + \nabla_X^\perp(fN) = -\mu\langle N, e \rangle PX + t\nabla_X N + f\nabla_X^\perp N,$$

from which $(\nabla_X^\perp f)N = -\mu(\langle N, e \rangle PX + \langle X, tN \rangle e)$ or

$$\|(\nabla_X^\perp f)N\|^2 = \mu^2(\langle N, e \rangle^2 \|PX\|^2 + \langle X, tN \rangle^2.$$

If we assume that the f-structure is parallel in the normal bundle, then we have $PX = 0$ and $tN = 0$. This is a contradiction. This proves Case (3) of our Theorem and completes the proof of our Theorem. □

REFERENCES

1. B. Y. Chen, *Geometry of submanifolds*, Dekker, New York, 1973.
2. ———, *Extrinsic spheres in Kähler manifolds*, Michigan Math. J. 23 (1976), 327–330.

3. ———, *Odd-dimensional extrinsic spheres in Kähler manifolds,* Rend. Mat. 12 (1979), 201-207.
4. ———, *Geometry of submanifolds and its applications,* Sci. Univ. Tokyo, Tokyo, 1981.
5. K. Nomizu and K. Yano, *On circles and spheres in Riemannian geometry,* Math. Ann. 210 (1974), 163-170.
6. M. Obata, *Riemannian manifolds admitting a solution of a certain system of differential equations.* Proc. U.S.-Japan Seminar in Differential Geometry (Kyoto, 1965), 101-114, Nippon Hyoronsha, Tokyo, 1966.
7. M. Okumura, *Certain almost contact hypersurfaces in Kaehlerian manifolds of constant holomorphic sectional curvatures,* Tôhoku Math. J. (2) 16 (1964), 270-284.
8. ———, *Totally umbilical submanifolds of a Kaehlerian manifold,* J. Math. Soc. Japan 19 (1967), 317-327.
9. S. Sasaki, *Almost contact manifolds I.* Proc. U.S.-Japan Seminar in Differential Geometry (Kyoto, 1965), 128-136, Nippon Hyoronsha, Tokyo, 1966.
10. S. Tanno, *Some differential equations on Riemannian manifolds,* J. Math. Soc. Japan 30 (1978), 509-531.
11. K. Yano and M. Kon, *Anti-invariant submanifolds,* Dekker, New York, 1976.

Department of Mathematics
Faculty of Science
Science University of Tokyo
Tokyo, Japan, 162