SOME INEQUALITIES FOR THE MODULI OF CURVE FAMILIES

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1. Introduction. The modulus of a curve family is a basic tool in the theory of quasiconformal and quasiregular mappings in $\mathbf{R}^n$. The numerical value of the modulus is known only for a few curve families. Therefore good estimates are of importance. Several estimates are given in the literature ([1], [2], [4], [7], [9]). However, the known estimates are not adequate in all those cases which are relevant to the theory of quasiconformal and quasiregular mappings.

Let $E, F \subset \bar{\mathbf{R}}^n$ be non-empty sets and let $\Delta_{EF} = \Delta(E, F)$ be the family of all closed curves which join E to F in $\bar{\mathbf{R}}^n$. In this paper we shall study the problem of finding estimates for the modulus $M(\Delta_{EF})$ in terms of the 'sizes' of E and F, in particular, when E and F are 'small.' We list some well-known estimates.

(a) E and F are connected (cf. Gehring [1], [2], and Väisälä [9, pp. 27–40]). In this case $M(\Delta_{EF})$ has a lower bound, which depends on the dimension n and the spherical diameters $q(E)$ and $q(F)$.

(b) E is connected and $\operatorname{cap} F > 0$ (i.e., F is of positive conformal capacity). In this case $M(\Delta_{EF})$ has a lower bound, which depends on $q(E)$, F, and n (cf. Martio, Rickman, and Väisälä [4, 3.11]).

(c) $\operatorname{cap} E > 0$ and $\operatorname{cap} F > 0$. In this case $M(\Delta_{EF})$ has a lower bound depending only on E, F, and n (cf. Näkki [7]).

The lower bound in (a) is an increasing function of $\min\{q(E), q(F)\}$. It seems that in the cases (b) and (c) the dependence of the lower bound on the 'sizes' of E and F is more complicated and that a further study of this dependence is desirable. The condition of being of positive capacity in (b) and (c) measures the local structure of the set rather than its global size. In order to achieve a quantitative lower bound for $M(\Delta_{EF})$ also in cases (b) and (c) we introduce a set function $c(\cdot)$ with the following properties.

1.1 THEOREM. *There exists a set function* $c(\cdot)$ *in* $\bar{\mathbf{R}}^n$ *with the properties*

(1) $c(E) = c(hE)$ *whenever* $h: \bar{\mathbf{R}}^n \to \bar{\mathbf{R}}^n$ *is a spherically isometric Möbius transformation.*

(2) $c(\cdot)$ *is a quasiadditive (for definition cf.* 3.20*) outer measure with* $0 \leqslant c(E) \leqslant c(\bar{\mathbf{R}}^n) < \infty$.

(3) $c(E) > 0$ *if and only if* $\operatorname{cap} E > 0$.

(4) *If E is connected, then* $c(E) \geqslant a_n q(E)$, *where a_n is a positive number depending only on n.*

(5) $M(\Delta_{EF}) \geqslant \beta \min\{c(E), c(F)\}$ *where β is a positive number depending only on n.*

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(6) *If* $q(\bar{E},\bar{F})=t>0$, *then there exists a positive number* α *depending only on* n *and* t *such that* $M(\Delta_{EF})\leqslant\alpha\min\{c(E),c(F)\}$.

As a corollary we get a quantitative lower bound in each of the above cases (a)–(c) with a simple dependence on the sizes $c(E)$ and $c(F)$ of E and F. It should be emphasized that the main interest of Theorem 1.1 lies in the inequalities (5) and (6) when E and F are disconnected sets of positive capacity.

The proof of the theorem is based on a comparison principle for the modulus, which was first used in [4] and [7]. In order to give the reader some idea about the set function $c(E)$, we note that by Remark 3.12(2) below

$$c(E)\sim\max\{M(\Delta(S^{n-1}(2),E_1)),M(\Delta(S^{n-1}(\tfrac{1}{2}),E_2))\} \tag{1.2}$$

where $E_1=E\cap\bar{B}^n$, $E_2=E\setminus E_1$ and where $\sim$ indicates that the ratio of the left and right sides of (1.2) is bounded from above and below by numbers depending only on n. We shall apply Theorem 1.1 to prove a new distortion theorem for quasiregular mappings.

2. Preliminary results.

2.1 NOTATION. Throughout the paper we shall assume that n is a fixed integer and $n\geqslant 2$. We denote the n-dimensional Euclidean space by $\mathbf{R}^n$ and its one-point compactification by $\bar{\mathbf{R}}^n=\mathbf{R}^n\cup\{\infty\}$. All topological operations are performed with respect to $\bar{\mathbf{R}}^n$ unless otherwise mentioned. Balls and spheres centered at $x\in\mathbf{R}^n$ and with radius $r>0$ are denoted, respectively, by $B^n(x,r)=\{z\in\mathbf{R}^n: |z-x|<r\}$, $S^{n-1}(x,r)=\partial B^n(x,r)$. We employ the abbreviations $B^n(r)=B^n(0,r)$, $B^n=B^n(1)$, $S^{n-1}(r)=S^{n-1}(0,r)$, and $S^{n-1}=S^{n-1}(1)$. The standard unit coordinate vectors are $e_1,\ldots,e_n$.

2.2 THE SPHERICAL METRIC. The stereographic projection

$$f\colon \bar{\mathbf{R}}^n\to S^n(e_{n+1},\tfrac{1}{2})\subset\mathbf{R}^{n+1}$$

is defined by

$$f(x)=e_{n+1}+\frac{x-e_{n+1}}{|x-e_{n+1}|^2}.$$

The spherical (chordal) distance between two points $a,b\in\bar{\mathbf{R}}^n$ is the number $q(a,b)=|f(a)-f(b)|$. If $a\neq\infty\neq b$, then

$$q(a,b)=|a-b|(1+|a|^2)^{-1/2}(1+|b|^2)^{-1/2}$$

and $q(a,\infty)=(1+|a|^2)^{-1/2}$. It is clear that $q(a,b)\leqslant\min\{1,|a-b|\}$ holds for $a,b\in\bar{\mathbf{R}}^n$. For $x\in\bar{\mathbf{R}}^n$, $t\in(0,1)$, let $Q(x,t)=\{z\in\bar{\mathbf{R}}^n: q(z,x)<t\}$.

2.3 THE MODULUS OF A CURVE FAMILY. A *curve* is a non-constant mapping $\gamma\colon\Delta\to\bar{\mathbf{R}}^n$ where $\Delta\subset\mathbf{R}$ is an interval. The set $\gamma\Delta$ will be denoted by $|\gamma|$. Let Γ be a family of curves in $\bar{\mathbf{R}}^n$. The modulus of Γ is defined by

$$M(\Gamma)=\inf_{\rho}\int_{\mathbf{R}^n}\rho^n\,dm \tag{2.4}$$

where m is the n-dimensional Lebesgue measure and the infimum is taken over all non-negative Borel-functions $\rho: \mathbf{R}^n \to \mathbf{R}^1 \cup \{\infty\}$, with $\int_\gamma \rho\, ds \geqslant 1$ for all locally rectifiable $\gamma \in \Gamma$. For the properties of the modulus the reader is referred to [9]. If $E, F, G \subset \bar{\mathbf{R}}^n$, we denote by $\Delta(E,F;G)$ the family of all closed paths $\gamma: [a,b] \to \bar{\mathbf{R}}^n$ such that $\gamma(t) \in G$ for $t \in (a,b)$, and one of the points $\gamma(a), \gamma(b)$ belongs to E and the other to F. We also denote $\Delta(E,F) = \Delta(E,F;\bar{\mathbf{R}}^n)$. Let $u \in \mathbf{R}^n$ and $b > a > 0$ and let Γ be a curve family such that $|\gamma| \cap S^{n-1}(u,a) \neq \varnothing \neq |\gamma| \cap S^{n-1}(u,b)$ for every $\gamma \in \Gamma$. Then

$$M(\Gamma) \leqslant \omega_{n-1}\left(\log \frac{b}{a}\right)^{1-n} \tag{2.5}$$

holds (cf. [9, 7.5]) where ω_{n-1} is the $(n-1)$-dimensional surface area of S^{n-1}. Given $E \subset \mathbf{R}^n$, $t > r > 0$, and $x \in \mathbf{R}^n$ we denote

$$\begin{aligned} M_t(E,r,x) &= M(\Delta(S^{n-1}(x,t), \bar{B}^n(x,r) \cap E; \mathbf{R}^n)) \\ M(E,r,x) &= M_{2r}(E,r,x). \end{aligned} \tag{2.6}$$

If $t > s > r > 0$, then it follows from [6, 2.7] that

$$M_t(E,r,x) \leqslant M_s(E,r,x) \leqslant \left(\frac{\log(t/r)}{\log(s/r)}\right)^{n-1} M_t(E,r,x). \tag{2.7}$$

An important property of the modulus is the conformal invariance

$$M(\Gamma) = M(f\Gamma) \tag{2.8}$$

whenever $f: G \to G'$ is a conformal mapping and Γ is a curve family in G and $f\Gamma$ is its image under f (cf. [9, 8.2]).

2.9 THE CAPACITY OF A CONDENSER. A condenser in $\mathbf{R}^n$ is a pair (A,C) where A is open in $\mathbf{R}^n$ and C is a compact non-empty subset of A. The capacity of $E = (A,C)$ is defined by

$$\operatorname{cap} E = \inf_u \int_{\mathbf{R}^n} |\nabla u|^n \, dm \tag{2.10}$$

where u runs through all C^∞-functions with compact support in A and $u(x) \geqslant 1$ for $x \in C$. It follows from a result of Ziemer [14] that

$$\operatorname{cap} E = M(\Delta(C, \partial A; \bar{\mathbf{R}}^n)) = M(\Delta(C, \partial A; A)). \tag{2.11}$$

For bounded A, (2.11) was proved in [14], and the general case follows from it by a simple limiting argument. Let G be a domain in $\mathbf{R}^n$, let $f: G \to \mathbf{R}^n$ be a continuous open mapping, and let $(A,C) = E$ be a condenser with $A \subset G$. Then $fE = (fA, fC)$ is a condenser as well. If f is conformal, it follows from (2.8) and (2.11) that

$$\operatorname{cap} E = \operatorname{cap} fE. \tag{2.12}$$

A compact set $C \subset \mathbf{R}^n$ is said to be of capacity zero, denoted $\operatorname{cap} C = 0$, if $\operatorname{cap}(A,C) = 0$ for some bounded open $A \subset \mathbf{R}^n$. Otherwise C is of positive capacity, $\operatorname{cap} C>0$. The notion of a condenser can be readily extended to the case where A is an open subset of $\bar{\mathbf{R}}^n$ with $\partial A \neq \varnothing$, and where $C \subset A$ is compact. If (A, C) is a condenser in $\bar{\mathbf{R}}^n$, we define $\operatorname{cap}(A, C)=M(\Delta(C, \partial A;\ \bar{\mathbf{R}}^n))$. A compact set $C \subset \bar{\mathbf{R}}^n$ is said to be of capacity zero, if $\operatorname{cap}(A, C)=0$ for some open $A \subset \bar{\mathbf{R}}^n$ with $\bar{A} \neq \bar{\mathbf{R}}^n$. It is well known that the definition of capacity zero does not depend on the open set A; this fact will also follow from the results in this paper. Sets of capacity zero have zero Hausdorff dimension (cf. [8, p. 72]).

We shall first list some well-known lemmas. The next result is a simple corollary to the spherical cap-inequality (cf. [9, 10.12]) and it was proved in [12, 1.10].

2.13 LEMMA. *Let $E \subset \mathbf{R}^n$, $t>r>0$, and let $E_r \subset E \cap \bar{B}^n(r)$ be connected. Then*

$$M_t(E, r, 0) \geqslant c_n \log \frac{2t+d(E_r)}{2t-d(E_r)}$$

where c_n is the positive constant in [9, 10.12].

The next result will be called the comparison principle for the modulus. It originates from a result of Martio, Rickman and Väisälä [4, 3.11] and Näkki [7]. The formulation differs from [7, 3.1] because we allow four sets instead of three, but with obvious modifications the proof given in [7] remains valid in the present case as well.

2.14 LEMMA. *Let G be a domain in $\bar{\mathbf{R}}^n$, let $F_j \subset G$, $j=1,2,3,4$, and let $\Gamma_{ij}=\Delta(F_i, F_j; G)$, $1 \leqslant i, j \leqslant 4$. Then*

$$M(\Gamma_{12}) \geqslant 3^{-n} \min\{M(\Gamma_{13}), M(\Gamma_{24}),\ \inf M(\Delta(|\gamma_{13}|, |\gamma_{24}|;\ G))\}$$

where the infimum is taken over all rectifiable curves $\gamma_{13} \in \Gamma_{13}$ and $\gamma_{24} \in \Gamma_{24}$.

2.15 COROLLARY. *Let $F_1, F_2, F_3, F_4 \subset \bar{\mathbf{R}}^n$ and let $\Gamma_{ij}=\Delta(F_i, F_j)$, $1 \leqslant i, j \leqslant 4$. Then*

$$M(\Gamma_{12}) \geqslant 3^{-n} \min\{M(\Gamma_{13}), M(\Gamma_{24}), \delta_n(r)\}$$

where $r=\min\{q(F_1, F_3), q(F_2, F_4)\}$ and

$$\delta_n(r)=\inf M(\Delta(F, F^*)) \tag{2.16}$$

where the infimum is taken over all continua F and F^* in $\bar{\mathbf{R}}^n$ with $q(F) \geqslant r$, $q(F^*) \geqslant r$.

It was proved by Väisälä [9, §12] (cf. also Gehring [2]) that the number $\delta_n(r)>0$ for $r>0$ and $\delta_n(0)=0$. We shall give here a different proof of this fact, which is based on the following corollary to 2.14.

2.17 COROLLARY. *Let $0<a<b$, $u \in \mathbf{R}^n$ and let $F_1, F_2 \subset \bar{B}^n(u, a)$, $F_3 \subset \bar{\mathbf{R}}^n \backslash B^n(u, b)$. Then*

$$M(\Gamma_{12}) \geqslant 3^{-n} \min\left\{M(\Gamma_{13}), M(\Gamma_{23}), c_n \log \frac{b}{a}\right\}.$$

Proof. The proof follows from 2.14 and [9, 10.12]. □

2.18 LEMMA. (1) *Let* $E, F \subset \bar{B}^n(u)$ *be connected sets with* $d(E), d(F) \geqslant tu$. *Then* $M(\Delta(E,F)) \geqslant d_1 t$, *where* d_1 *is a positive constant depending only on n.*

(2) *Let* $E, F \subset \bar{\mathbf{R}}^n$ *be connected sets with* $q(E), q(F) \geqslant a$. *Then* $M(\Delta(E,F)) \geqslant \delta_n(a) \geqslant Da$ *where D is a positive number depending only on n.*

Proof. (1) Apply 2.13 and 2.17 with $F_1 = F$, $F_2 = E$, *and* $F_3 = S^{n-1}(2u)$ to get the desired estimate with $d_1 = 3^{-n} c_n (\log 2)/2$.

(2) By performing a preliminary spherically isometric Möbius transformation (cf. [9, 12.2]) if necessary we may assume by (2.8) that $-re_1 \in E$ and $re_1 \in F$. Since $q(B^n) = 1$ we may assume that $r \in [0,1]$. It follows that each of the sets $\bar{B}^n(2) \cap E$ and $\bar{B}^n(2) \cap F$ must contain a component E_1 and F_1, respectively, with

$$d(E_1) \geqslant q(E_1) \geqslant \min\{a, q(S^{n-1}, S^{n-1}(2))\} \geqslant a/\sqrt{10}$$

and similarly for F_1. The proof follows now from part (1) with $D = d_1/(2\sqrt{10})$. □

The next result is the Main lemma.

2.19 LEMMA. *Let* $E \subset \bar{B}^n$ *and let* $G_t = \bigcup_{x \in E} Q(x,t)$ *for* $t \in (0,1)$, *and let* $r \in (1, \infty)$. *The following estimate holds, if t is so small that* $q(G_t, S^{n-1}(r)) \geqslant t$,

$$M(\Delta(E, \partial G_t)) \leqslant a(t) M(\Delta(E, S^{n-1}(r))),$$

where $a(t)$ *is a positive number depending only on n, r, and t.*

Proof. Fix $r > 1$ and $t \in (0,1)$ with $q(G_t, S^{n-1}(r)) \geqslant t$. Set $F_1 = E$, $F_2 = S^{n-1}(r)$, and $F_3 = \partial G_t = F_4$. It follows from 2.15 and 2.18(2) that

$$M(\Gamma_{12}) \geqslant 3^{-n} \min\{M(\Gamma_{13}), M(\Gamma_{23}), Dt\} \tag{2.20}$$

where Γ_{ij} are as in 2.15. We shall first find a lower bound for $M(\Gamma_{23})$. It is easy to show that F_3 contains a continuum of Euclidean diameter at least $t\sqrt{2}$. Hence we obtain, by 2.13 and [9, 6.2],

$$M(\Gamma_{23}) \geqslant c_n \log \frac{2r + t\sqrt{2}}{2r - t\sqrt{2}} \geqslant (c_n \log 2)\frac{t}{r} \geqslant D\frac{t}{r}. \tag{2.21}$$

Here D is the number in 2.18(2) and (2.20). Next observe that $d(|\gamma|) \geqslant q(|\gamma|) \geqslant t$ and $|\gamma| \subset B^n(r)$ for every $\gamma \in \Gamma_{13}$, and thus we get the following upper bound by [9, 7.1], $M(\Gamma_{13}) \leqslant \Omega_n r^n / t^n$ where $\Omega_n = m(B^n)$. Now (2.20) and (2.21) yield

$$M(\Gamma_{12}) \geqslant 3^{-n} \min\{M(\Gamma_{13}), Dt/r\} \geqslant 3^{-n} \min\left\{M(\Gamma_{13}), \frac{Dt^{n+1}}{\Omega_n r^{n+1}} M(\Gamma_{13})\right\}$$

$$\geqslant 3^{-n} M(\Gamma_{13}) \min\{1, Dt^{n+1}/(\Omega_n r^{n+1})\} = M(\Gamma_{13})/a(t). \qquad \square$$

2.22 REMARKS. (1) Let $E \subset \bar{B}^n$ be a compact set of positive capacity and $G_t = \bigcup_{x \in E} Q(x,t)$, $t \in (0,1)$. It follows from a result of J. Väisälä [11] that

$$M(\Delta(E, \partial G_t))/M(\Delta(E, S^{n-1}(2))) \to \infty$$

as $t \to 0$. Therefore the function $a(t)$ in 2.19 must be unbounded whenever $\operatorname{cap} E > 0$.

(2) Choosing $E=\bigcup_{k=1}^{\infty} S^{n-1}(1-2^{-k})$ in 2.19 one can show by [9, 6.7 and 7.5] that $a(t) \geqslant \text{const.}\, t^{1-n}\log(1/t)$ for this choice of E. The rate of growth of the upper bound for $a(t)$, which can be obtained from the proof of 2.19, is probably not the best possible.

The following subadditivity property of the modulus will be applied in what follows. The proof follows directly from [9, 6.2].

2.23 REMARK. Let $E=E_1\cup E_2$ and $F=F_1\cup F_2$ be sets in $\bar{\mathbf{R}}^n$, and let $\Gamma_1=\Delta(E_1,F_1)$, $\Gamma_2=\Delta(E_2,F_2)$, $\Gamma_3=\Delta(E_1,F_2)$, $\Gamma_4=\Delta(E_2,F_1)$. Then

$$M(\Delta(E,F)) \leqslant 4\max\{M(\Gamma_j): j=1,2,3,4\}.$$

3. The construction of $c(E)$. Throughout this section we shall identify $\bar{\mathbf{R}}^n$ with the Riemann sphere $S^n(e_{n+1},\frac{1}{2})$ via the stereographic projection 2.2. Let $\tilde{x}=-x/|x|^2$ for $x\in\mathbf{R}^n\setminus\{0\}$ and $\tilde{0}=\infty$, $\tilde{\infty}=0$. Then $f(x)$ and $f(\tilde{x})$ are antipodal points on the Riemann sphere, i.e. $q(x,\tilde{x})=1$, where f is the stereographic projection 2.2. The spherical balls $Q(x,r)$ are convex for $r\in(0,1/\sqrt{2})$; $\bar{\mathbf{R}}^n\setminus\bar{Q}(x,r)=Q(\tilde{x},\sqrt{1-r^2})$ is convex for $r\in[1/\sqrt{2},1)$. Moreover, $fQ(x,1/\sqrt{2})$ is a hemisphere of the Riemann sphere centered at $f(x)$. For each $x\in\bar{\mathbf{R}}^n$ there exists a Möbius transformation $h_x\colon \bar{\mathbf{R}}^n\to\bar{\mathbf{R}}^n$ such that

$$\left\{\begin{array}{l} h_x(x)=0 \\ q(h_x(z),h_x(y))=q(z,y) \end{array}\right. \tag{3.1}$$

for all $z,y\in\bar{\mathbf{R}}^n$ (cf. [9, 12.2]). Further, $h_xQ(x,r)=B^n(r/\sqrt{1-r^2})$ for any $x\in\bar{\mathbf{R}}^n$ and $r\in(0,1)$ and $h_xQ(x,1/\sqrt{2})=B^n$ by the Pythagorean theorem, $q(0,y)^2+q(y,\infty)^2=1$.

For $E\subset\bar{\mathbf{R}}^n$, $x\in\bar{\mathbf{R}}^n$, $0<r<t<1$, denote

$$\left\{\begin{array}{l} m_t(E,r,x)=M(\Delta(\partial Q(x,t),E\cap\bar{Q}(x,r)) \\ m(E,x)=m_t(E,1/\sqrt{2},x) \ \text{ where } t=(\sqrt{3})/2. \end{array}\right. \tag{3.2}$$

3.3 DEFINITION. We define

$$\left\{\begin{array}{l} c(E,x)=\max\{m(E,x),m(E,\tilde{x})\}, \\ c(E)=\inf\{c(E,x): x\in\bar{\mathbf{R}}^n\}. \end{array}\right. \tag{3.4}$$

3.5 REMARK. We shall now find an upper bound for the numbers in (3.2). Since the modulus is invariant under conformal mappings (2.8) it follows that

$$\begin{aligned} m_t(E,r,x)&=M(\Delta(S^{n-1}(t/\sqrt{1-t^2}),(h_xE)\cap\bar{B}^n(r/\sqrt{1-r^2}),0)) \\ &\leqslant \omega_{n-1}\left(\log\left(\frac{t}{r}\sqrt{\frac{1-r^2}{1-t^2}}\right)\right)^{1-n} \end{aligned} \tag{3.6}$$

by (2.5), where h_x is as in (3.1). In particular,

$$m(E,x)\leqslant m(\bar{\mathbf{R}}^n,x)\leqslant\omega_{n-1}(\log\sqrt{3})^{1-n}. \tag{3.7}$$

If $F\subset Q(x,r)$, where $r\in(0,1/\sqrt{2}]$, it follows from (3.6) that

(3.8) $$m(F,x)\leqslant\omega_{n-1}(\log((\sqrt{3}/r)\sqrt{1-r^2}))^{1-n}.$$

In particular, $c(F,x)\to 0$ as $q(F)\to 0$.

3.9 LEMMA. *There exists a positive number b_n depending only on n such that for $E\subset\bar{\mathbf{R}}^n$, $n\geqslant 2$, $c(E,x)\leqslant b_n c(E,y)$ whenever $x,y\in\bar{\mathbf{R}}^n$ and $c(E)\leqslant c(E,z)\leqslant b_n c(E)$ for all $z\in\bar{\mathbf{R}}^n$.*

Proof. Observe first that we obtain from (3.6) the following equality:

(3.10) $$\begin{aligned} M(\Delta(\partial Q(x,\sqrt{3}/2),\partial Q(x,1/\sqrt{2})))&=M(\Delta(\partial Q(x,1/\sqrt{2}),\partial Q(x,1/2)))\\ &=\omega_{n-1}(\log\sqrt{3})^{1-n}=a.\end{aligned}$$

Let $x,y\in\bar{\mathbf{R}}^n$. In what follows we shall assume that

(3.11) $$c(E,x)=m(E,x).$$

The case $c(E,x)=m(E,\tilde{x})$ can be dealt with exactly in the same way; even the constants will be the same in this case. Let $E_1^*=E\cap\bar{Q}(x,1/\sqrt{2})\cap\bar{Q}(y,1/\sqrt{2})$, $E_2^*=(E\backslash E_1^*)\cap\bar{Q}(x,1/\sqrt{2})$. It follows from [9, 6.2] and (3.11) that either

$$M(\Delta(\partial Q(x,\sqrt{3}/2),E_1^*))\geqslant c(E,x)/2 \quad\text{or}\quad M(\Delta(\partial Q(x,\sqrt{3}/2),E_2^*))\geqslant c(E,x)/2.$$

In the first case denote $F_1=E_1^*$, $F_2=\partial Q(y,\sqrt{3}/2)$, $F_3=\partial Q(x,\sqrt{3}/2)$, and $F_4=\partial Q(y,1/\sqrt{2})$. In the second case denote $F_1=E_2^*$, $F_2=\partial Q(y,1/\sqrt{2})$, $F_3=\partial Q(x,\sqrt{3}/2)$, and $F_4=\partial Q(y,\sqrt{3}/2)$. In both cases

$$\begin{aligned}\min\{q(F_1,F_3),q(F_2,F_4)\}&\geqslant q(\partial Q(x,\sqrt{3}/2),\partial Q(x,1/\sqrt{2}))\\ &=q(S^{n-1}(\sqrt{3}),S^{n-1})=(\sqrt{3}-1)/\sqrt{8}=\delta.\end{aligned}$$

Hence we obtain, by 2.15, 2.18(2) and (3.10), (3.7):

$$\begin{aligned} c(E,y)&\geqslant M(\Gamma_{12})\geqslant 3^{-n}\min\{M(\Gamma_{13}),M(\Gamma_{24}),D\delta\}\\ &\geqslant 3^{-n}\min\{c(E,x)/2,a,D\delta\}=3^{-n}\min\{c(E,x)/2,D\delta\}\\ &\geqslant b_n^{-1}c(E,x);\ b_n^{-1}=3^{-n}\min\left\{\frac{1}{2},D\delta\frac{(\log\sqrt{3})^{n-1}}{\omega_{n-1}}\right\}.\end{aligned}$$

3.12 REMARKS. (1) The set functions $c(\cdot,\ x)$ are usually not invariant under spherically isometric Möbius transformations while $c(\cdot)$ has this invariance property. This fact motivates the infimum in (3.4) (supremum would lead to an essentially similar notion by 3.9). However, $c(\cdot)$ is not invariant under stretchings (cf. (3.6)).

(2) Lemma 3.9 provides the following simple estimate for $c(E)$. It follows from (2.7) and (3.2) that

$$M(E,1,0)\leqslant m(E,1/\sqrt{2},0)=M_u(E,1,0)\leqslant\left(\frac{\log 2}{\log\sqrt{3}}\right)^{n-1}M(E,1,0),$$

where $u=\sqrt{3}$. Next let $h=h_\infty$ be the mapping (3.1), i.e. $h(x)=x/|x|^2$ for $x\in\mathbf{R}^n\backslash\{0\}$, $h(0)=\infty$, $h(\infty)=0$. Define $a(E)=\max\{M(E,1,0),M(hE,1,0)\}$.

Then the preceding inequality, together with (2.8), yields the following estimate:

$$a(E) \leqslant c(E,0) \leqslant \left(\frac{\log 2}{\log \sqrt{3}}\right)^{n-1} a(E)$$

and further, by 3.9,

$$a(E)/b_n \leqslant c(E) \leqslant \left(\frac{\log 2}{\log \sqrt{3}}\right)^{n-1} b_n a(E).$$

This inequality reduces the calculation of $c(E)$ to the calculation of $a(E)$, which clearly is a much simpler task.

It follows from (3.8) and 3.9 that $c(E) \to 0$ as $q(E) \to 0$. For connected sets the converse also holds true, as the following result shows.

3.13 COROLLARY. *There exists a positive number a_n depending only on n such that $c(F) \geqslant a_n q(F)$ whenever F is a connected set in $\bar{\mathbf{R}}^n$.*

Proof. Since both sides of the inequality are invariant under spherically isometric Möbius transformations, we may assume $0 \in F$. Then $F \cap \bar{B}^n$ contains a connected component F_1 with

$$d(F_1) \geqslant q(F_1) \geqslant \min\{1/\sqrt{2}, q(F)\} \geqslant q(F)/\sqrt{2}.$$

By 2.13 and (3.6) we get

$$c(F,0) \geqslant M_{\sqrt{3}}(F_1,1,0) \geqslant c_n \log \frac{2\sqrt{3}+q(F)/\sqrt{2}}{2\sqrt{3}-q(F)/\sqrt{2}}$$
$$\geqslant \frac{\sqrt{2}c_n}{\sqrt{3}} q(F).$$

The proof follows now from 3.9. □

3.14 THEOREM. *There exists a positive number β depending only on n such that for $E, F \subset \bar{\mathbf{R}}^n$ we have $M(\Delta(E,F)) \geqslant \beta \min\{c(E), c(F)\}$.*

Proof. Fix $x \in \bar{\mathbf{R}}^n$. If $m(E,x) = c(E,x)$ let

$$F_1 = E \cap \bar{Q}(x, 1/\sqrt{2}), \qquad F_3 = \partial Q(x, \sqrt{3}/2).$$

Otherwise $m(E,\tilde{x}) = c(E,x)$ and then we set

$$F_1 = E \cap \bar{Q}(\tilde{x}, 1/\sqrt{2}), \qquad F_3 = \partial Q(\tilde{x}, \sqrt{3}/2).$$

The sets F_2 and F_4 are defined in the same way as follows. If $m(F,x) = c(F,x)$ let $F_2 = F \cap \bar{Q}(x, 1/\sqrt{2})$, $F_4 = \partial Q(x, \sqrt{3}/2)$. Otherwise $m(F,\tilde{x}) = c(F,x)$ and then set $F_2 = F \cap \bar{Q}(\tilde{x}, 1/\sqrt{2})$ and $F_4 = \partial Q(\tilde{x}, \sqrt{3}/2)$. In any case

$$\min\{q(F_1,F_3), q(F_2,F_4)\} \geqslant q(S^{n-1}(\sqrt{3}), S^{n-1}) = \frac{\sqrt{3}-1}{\sqrt{8}} = \delta.$$

Set $\Gamma_{ij} = \Delta(F_i, F_j)$. It follows from 2.15 and 2.18(2) that

$$M(\Delta(E,F)) \geqslant M(\Gamma_{12}) \geqslant 3^{-n} \min\{c(E,x), c(F,x), D\delta\}$$
$$\geqslant \beta \min\{c(E,x), c(F,x)\}$$

where $\beta = 3^{-n} \min\{1, D\delta(\log\sqrt{3})^{n-1}/\omega_{n-1}\} > 0$ and the last inequality follows from (3.7). Since $x \in \bar{\mathbf{R}}^n$ was arbitrary, the proof follows. □

3.15 THEOREM. *Let $E, F \subset \bar{\mathbf{R}}^n$ be sets with $q(\bar{E}, \bar{F}) \geqslant t > 0$. Then*

$$M(\Delta(E,F)) \leqslant \alpha \min\{c(E), c(F)\}$$

where α is a positive number depending only on n and t.

Proof. Fix $x \in \bar{\mathbf{R}}^n$. Let $E_1 = E \cap \bar{Q}(x, 1/\sqrt{2})$, $E_2 = E \setminus E_1$, $F_1 = F \cap \bar{Q}(x, 1/\sqrt{2})$, $F_2 = F \setminus F_1$. Let $\Gamma_1 = \Delta(E_1, F_1)$, $\Gamma_2 = \Delta(E_1, F_2)$, $\Gamma_3 = \Delta(E_2, F_1)$, and $\Gamma_4 = \Delta(E_2, F_2)$. It follows from 2.23 that

$$M(\Delta(E,F)) \leqslant 4 \max\{M(\Gamma_j) : j = 1,2,3,4\}.$$

Without loss of generality we may assume that the maximum on the right side of this inequality is equal to $M(\Gamma_2)$, because the proof is similar in the other cases. Let

$$E_1^t = \bigcup \{Q(x, t/2) : x \in E_1\}, \qquad F_2^t = \bigcup \{Q(x, t/2) : x \in F_2\}.$$

If $\gamma \in \Gamma_2$, then clearly $|\gamma| \cap \partial E_1^t \neq \varnothing \neq |\gamma| \cap \partial F_2^t$, and hence by [9, 6.4]

$$\tfrac{1}{4} M(\Delta(E,F)) \leqslant M(\Gamma_2) \leqslant \min\{M(\Delta(E_1, \partial E_1^t)), M(\Delta(F_2, \partial F_2^t))\}. \tag{3.16}$$

We shall next find an upper bound for $M(\Delta(E_1, \partial E_1^t))$. By performing an auxiliary spherically isometric Möbius transformation if necessary we may assume $x = 0$. We shall apply 2.19 with $r = \sqrt{3}$. First observe that $q(E_1^s, S^{n-1}(\sqrt{3})) \geqslant s$ whenever $s \in (0, \delta/2)$, where $\delta = q(S^{n-1}, S^{n-1}(\sqrt{3}))$. Next fix $s = t\delta/2$. Then we obtain, by 2.19 and [9, 6.4]:

$$M(\Delta(E_1, \partial E_1^t)) \leqslant M(\Delta(E_1, \partial E_1^s)) \leqslant a(s) M(\Delta(S^{n-1}(\sqrt{3}), E_1)),$$
$$a(s) = 3^n/(\min\{1, D(t\delta/2)^{n+1}/(\Omega_n(\sqrt{3})^{n+1})\}).$$

A similar estimate holds for $M(\Delta(F_2, \partial F_2^t))$ as well. In conclusion, we get by (3.16) and 3.9

$$M(\Delta(E,F)) \leqslant 4a(s) \min\{c(E,x), c(F,x)\}$$
$$\leqslant 4b_n a(s) \min\{c(E), c(F)\}.$$

3.17 COROLLARY. *Let $E, F \subset \bar{\mathbf{R}}^n$ be sets with $q(\bar{E}, \bar{F}) \geqslant t > 0$. Then*

$$M(\Delta(E,F)) \leqslant d_1,$$

where d_1 depends only on n and t.

Proof. The proof follows from 3.15 and (3.7). □

3.18 COROLLARY. *Let $G \subset B^n$ be a domain, $E \subset G$ a compact set with $d(E, \partial G) \geqslant t$. Then there is a number β depending only on n and a number α depending only on n and t such that $\beta c(E) \leqslant \operatorname{cap}(G, E) \leqslant \alpha c(E)$.*

Proof. Because $G \subset B^n$ it follows that $q(E, \partial G) \geqslant t/3$. Hence it follows from (2.11) and (3.15) that

$$\operatorname{cap}(G, E) = M(\Delta(E, \mathbf{R}^n \setminus G)) \leqslant \alpha \min\{c(E), c(\mathbf{R}^n \setminus G)\}.$$

Because $\bar{\mathbf{R}}^n \setminus \bar{B}^n = Q(\infty, 1/\sqrt{2}) \subset \mathbf{R}^n \setminus G$ it follows that

$$c(\mathbf{R}^n \setminus G) = c(\bar{\mathbf{R}}^n) = \omega_{n-1}(\log \sqrt{3})^{1-n}$$

(cf. (3.7)). In particular $c(E) \leqslant c(\mathbf{R}^n \setminus G)$. Hence we get the desired upper bound,

$$\operatorname{cap}(G, E) \leqslant \alpha c(E).$$

The proof of the lower bound follows from 3.14. □

3.19 COROLLARY. *A compact set $E \subset \mathbf{R}^n$ is of capacity zero if and only if $c(E) = 0$.*

Proof. The proof follows from 3.18 and (2.12). □

3.20 DEFINITION. Let μ be a non-negative set function defined in $\operatorname{pot}(\bar{\mathbf{R}}^n) = \{E : E \subset \bar{\mathbf{R}}^n\}$. Then μ is said to be a quasiadditive outer measure if it has the following properties: (a) $\mu(\varnothing) = 0$; (b) $A \subset B \subset \bar{\mathbf{R}}^n$ implies $\mu(A) \leqslant \mu(B)$; (c) there exists a positive number $\lambda > 0$ such that $\mu(\bigcup_{j=1}^{\infty} E_j) \leqslant \lambda \sum_{j=1}^{\infty} \mu(E_j)$ whenever $E_j \subset \bar{\mathbf{R}}^n$, $j = 1, 2, \ldots$.

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. (1) follows directly from (3.4).

(2) It is clear by (3.7) that $c(\cdot)$ is bounded. It remains to prove that $c(\cdot)$ is a quasiadditive outer measure. The properties (a) and (b) in 3.20 are obvious (cf. [9, 6.2]). In order to prove (c), choose sets $E_j \subset \bar{\mathbf{R}}^n$, $j = 1, 2, \ldots$. We obtain, by 3.9 and by the fact that the modulus is subadditive [9, 6.2(3)],

$$c\left(\bigcup_{j=1}^{\infty} E_j\right) \leqslant b_n c\left(\bigcup_{j=1}^{\infty} E_j, 0\right) \leqslant b_n \sum_{j=1}^{\infty} c(E_j, 0) \leqslant b_n^2 \sum_{j=1}^{\infty} c(E_j).$$

The proofs of (3), (4), (5) and (6) follow, respectively, from 3.19, 3.13, 3.14 and 3.15. □

4. Applications to quasiregular mappings. In this section we shall give an application of Theorem 1.1 to quasiregular and quasimeromorphic mappings. For the definitions and basic properties of these mappings the reader is referred to [3]–[5], [8], [10], and [13].

For what follows we shall need some facts about the hyperbolic geometry of B^n. The hyperbolic metric ρ is defined by the element of length

$$d\rho = 2|dx|/(1 - |x|^2).$$

For $x \in B^n$ and $M > 0$ we denote $D(x, M) = \{z \in B^n : \rho(z, x) < M\}$. It is well known that

$$(4.1)\qquad D(x,M)=B^n(y,r)\begin{cases} y=\dfrac{x(1-t^2)}{1-|x|^2t^2}, \\ r=\dfrac{(1-|x|^2)t}{1-|x|^2t^2} \end{cases}\qquad t=\tanh(M/2).$$

4.2 THEOREM. *Let* $E\subset\bar{\mathbf{R}}^n$ *be a compact set of positive capacity, and let* $f\colon B^n\to\bar{\mathbf{R}}^n\setminus E$ *be a quasimeromorphic mapping. Then*

$$q(f(x),f(y))\,c(E)\leqslant d(n)K_I(f)\left(-\log\tanh\frac{\rho(x,y)}{4}\right)^{1-n}$$

holds for $x,y\in B^n$, *where* $d(n)$ *is a positive number depending only on* n.

Proof. We may clearly assume that f is non-constant. Fix $x,y\in B^n$ and let J be the geodesic segment in the hyperbolic geometry joining x to y. Denote $\Gamma'=\Delta(fJ,E)$ and let Γ be the family of all maximal liftings of the elements of Γ' starting at J (for more details see [5, 3.11 and 3.12]). Since f is open it follows that $\overline{|\gamma|}\cap\partial B^n\neq\varnothing$ for each $\gamma\in\Gamma$ (cf. [5, 3.12]). For $w\in B^n$ denote $\Delta_w=\Delta(D(w,\rho(x,y)/2),\partial B^n;\,B^n)$. It follows from the conformal invariance of the modulus (2.8), (2.5), and (4.1) that

$$(4.3)\qquad M(\Delta_w)=M(\Delta_0)=\omega_{n-1}\left(-\log\tanh\frac{\rho(x,y)}{4}\right)^{1-n}$$

for all $w\in B^n$. Next we choose $z\in J$ such that $J\subset\bar{D}(z,\rho(x,y)/2)$. It follows from [9, 6.4 and 7.10] that

$$(4.4)\qquad M(\Gamma)\leqslant M(\Delta_z).$$

On the other hand, we get by 3.14, 3.13, and (3.7) that

$$(4.5)\qquad \begin{aligned} M(\Gamma')&\geqslant\beta\min\{c(E),c(fJ)\}\\ &\geqslant\beta c(E)a_nq(fJ)/a_1\geqslant c(E)\,q(f(x),f(y))/d\end{aligned}$$

where $a_1=\omega_{n-1}(\log\sqrt{3})^{1-n}$ and $d=a_1/(\beta a_n)$. Because $f\Gamma<\Gamma'$ it follows from the K_I-inequality [10, 3.1] and from [9, 6.4] that $M(\Gamma')\leqslant M(f\Gamma)\leqslant K_I(f)M(\Gamma)$. This inequality, together with (4.3)-(4.5), yields the desired estimate with $d(n)=\omega_{n-1}d$. □

4.6 COROLLARY. *Let* $E\subset\bar{\mathbf{R}}^n$ *be a compact connected set with* $q(E)>0$ *and let* $f\colon B^n\to\bar{\mathbf{R}}^n\setminus E$ *be a quasimeromorphic mapping. Then*

$$q(f(x),f(y))\,q(E)\leqslant b(n)K_I(f)\left(-\log\tanh\frac{\rho(x,y)}{4}\right)^{1-n}$$

holds for $x,y\in B^n$, *where* $b(n)$ *is a positive number depending only on* n.

Proof. The proof follows from 4.2 and 3.13 with $b(n)=d(n)/a_n$. □

4.7 REMARKS. (1) Corollary 4.6 is a generalization of a result of Gehring [2, Theorem 1, p. 233] to the case of quasimeromorphic mappings. One can prove a result somewhat similar to 4.2 by combining the two results [4, 3.1] and [4, 3.11] of Martio, Rickman, and Väisälä—however, with the following important difference. In 4.2 it is made explicit (as in [2, p. 233]) in which way $q(f(x), f(y))$ depends on the size of the omitted set E, while the result obtainable from [4, 3.1 and 3.11] seems to give only the existence of some bound for $q(f(x), f(y))$ without quantitative dependence on the size of E.

(2) For small values of $\rho(x, y)$ one can improve 4.2 and 4.6 as indicated in [13, §5].

(3) The following twofold invariance property of the estimate in 4.2 should be observed. Firstly, in view of the invariance properties of ρ, Theorem 4.2 yields the same upper bound for $q(f(x), f(y))$ and for $q(f(g(x)), f(g(y)))$ whenever $g: B^n \to B^n$ is a Möbius self-mapping of B^n. Secondly, in view of Theorem 1.1(1), Theorem 4.2 yields the same upper bound for

$$q(f(x), f(y)) \quad \text{and} \quad q(h(f(x)), h(f(y)))$$

whenever $h: \bar{\mathbf{R}}^n \to \bar{\mathbf{R}}^n$ is a spherically isometric Möbius transformation.

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