ON CURVATURE AND SIMILARITY

Douglas N. Clark and Gadadhar Misra

1. Introduction. The purpose of this note is to shed some light on the relationship between the Cowen–Douglas curvatures $\mathcal{K}_T$ and $\mathcal{K}_S$, for two similar operators T, S of class $B_1(\Omega)$, by making use of recent results on the similarity of Toeplitz operators [1].

To be specific, let Ω be a planar region. We say a bounded operator T on a Hilbert space H belongs to $B_1(\Omega)$ if $T-\lambda I$ is onto and has 1-dimensional kernel for $\lambda \in \Omega$, and if

$$\bigvee_{\lambda\in\Omega} \ker(T-\lambda I)$$

is dense in H. For $T \in B_1(\Omega)$, the curvature $\mathcal{K}_T$ is defined, for $\lambda \in \Omega$, by

$$\mathcal{K}_T(\lambda) = -\frac{\partial^2}{\partial\bar{\lambda}\partial\lambda} \log\|k_\lambda\|^2,$$

where $\{k_\lambda\}$ is an analytic determination of the set of null vectors of $T-\lambda I$, $\lambda \in \Omega$.

In [4], Cowen and Douglas introduce B_1 and $\mathcal{K}_T$ and prove, among other things, that $\mathcal{K}_T$ is a complete unitary invariant for $T \in B_1(\Omega)$. But for similar $S, T \in B_1(\Omega)$, the situation is not made so clear. In fact, the best analogue of the result for unitary equivalent S and T is left as a conjecture for the case of similarity. Let $S, T \in B_1(\mathbf{D})$, $\mathbf{D}$ the unit disk, and suppose the closure $\bar{\mathbf{D}}$ of $\mathbf{D}$ is a k-spectral set for S and T, for some k. The *Cowen–Douglas conjecture* ([4], p. 252) states that *if S and T are similar, then*

$$\lim_{\lambda\to\lambda_0\in\mathbf{T}} \mathcal{K}_T(\lambda)/\mathcal{K}_S(\lambda) = 1,$$

where $\mathbf{T}$ is the unit circle. (Actually, Cowen and Douglas also conjecture the converse statement; we shall have no further comment concerning the converse, however.)

In Section 2, using a "piece" of Toeplitz operator from [1], we show that the Cowen–Douglas conjecture is false. In Section 3, we investigate our example further, showing how the failure of the conjecture can be used to obtain a spectral set estimate. In Section 4, we describe a class of Toeplitz operators for which the Cowen–Douglas conjecture holds.

2. The example. Let T_F denote the Toeplitz operator with symbol

$$F(z) = z^2/(z-\beta) \quad \tfrac{1}{2} < \beta < 1,$$

so that, for $x \in H^2$,

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$$T_F x = PF(e^{it})x(e^{it}),$$

where P is the projection of L^2 on H^2. The function F maps the unit circle $\mathbf{T}$ to a "figure 8", sending the two arcs of $\mathbf{T}$ from $u_0 = (1+\sqrt{4\beta^2-1}\,i)/2\beta$ to $\bar{u}_0$ to simple closed curves. The image of the arc from u_0 to $\bar{u}_0$ containing -1 has winding number $+1$ with respect to its interior, which we denote ℓ, and the image of the complementary arc has winding number -1, with respect to its interior $\mathcal{L}$. By the standard index theory for Toeplitz operators, $T_F - \lambda I$ has one dimensional cokernel and is one-to-one (for $\lambda \in \ell$), and has one dimensional kernel and is onto (for $\lambda \in \mathcal{L}$).

Let $f(z)$ denote the rational function $\bar{F}(\bar{z}^{-1})$ so that $T_f = T_F^*$, and let $\mathfrak{M}$ denote the closed span of the eigenvectors of $T_f - \lambda I$, for $\lambda \in \ell$ (which is equivalent to $\bar{\lambda} \in \ell$). Let T_f' denote the restriction $T_f' = T_f|_{\mathfrak{M}}$. By [1, Theorem 1], T_f' is similar to the coanalytic Toeplitz operator T_τ^*, where τ is the Riemann mapping function from $|z|<1$ onto ℓ. Therefore the operator $T = \tau^{-1}(T_f')$ is well defined, and is similar to S, the adjoint of the unilateral shift.

The fact that T is similar to S implies $T \in B_1(\mathbf{D})$, since the intertwining similarity must preserve essential spectrum, index and dimension of kernel. We also use $T = L^{-1}SL$ to show that $\bar{\mathbf{D}}$ is a k-spectral set for T. In fact, for any polynomial p, $p(T) = L^{-1}p(S)L$, so that

$$\|p(T)\| \leq \|L^{-1}\|\|L\|\|p(S)\| = k\|p\|_\infty,$$

where $k = \|L^{-1}\|\|L\|$.

In order to compute the curvature of T, we note that, by Wiener–Hopf factorization, the eigenvectors $h_\lambda(z)$ of T_f satisfying $h_\lambda(0) = 1$ are given by

$$h_\lambda(z) = (1-\beta z)(1-\lambda z+\lambda\beta z^2)^{-1} \quad \lambda \in \ell. \tag{2.1}$$

The eigenvectors of T are of course $h_{\tau(\lambda)}$, for $\lambda \in \mathbf{D}$. Factoring the denominator in (2.1) and expanding in partial fractions, we have

$$h_\lambda(z) = \frac{1-\beta z}{(1-d_+(\lambda)z)(1-d_-(\lambda)z)} = \frac{1}{d_+-d_-}\left(\frac{d_+-\beta}{1-d_+z} - \frac{d_--\beta}{1-d_-z}\right), \tag{2.2}$$

where $d_\pm(\lambda) = 2\beta\lambda/[\lambda \pm (\lambda^2-4\beta\lambda)^{1/2}]$. We compute the norm of h_λ by taking the inner product of the two expressions (2.1) and (2.2) and using the reproducing property of $(1-d_\pm(\lambda)z)^{-1}$ in H^2. After rearrangement of terms, we have

$$\begin{aligned}\log\|h_\lambda\|^2 &= \log[\beta^2(1-|\beta\lambda|^2)+|1-\beta\lambda|^2] - \log(1-|d_+(\lambda)|^2)\\ &\quad - \log(1-|d_-(\lambda)|^2) - \log|1-d_+(\lambda)\bar{d}_-(\lambda)|^2.\end{aligned} \tag{2.3}$$

We want to compute the asymptotic behavior of $\mathcal{K}_T(\lambda)$, as $\lambda \to 1/\beta$, λ real and $\lambda \in \ell$ (i.e., $\lambda < 1/\beta$). For the first term in (2.3), it is easily seen that

$$-\frac{\partial^2}{\partial\bar{\lambda}\partial\lambda}\log[\beta^2(1-|\beta\lambda|^2)+|1-\beta\lambda|^2] = \beta^2(1-\beta^2\lambda^2)^{-2} + o[(1-\beta\lambda)^{-2}].$$

For the second two terms on the right of (2.3), we can verify directly that $|d_\pm'(1/\beta)|^2 = \beta^4/(4\beta^2-1)$ and that, for λ real, $1-|d_\pm(\lambda)|^2 = 1-\beta\lambda$. This shows

$$\frac{\partial^2}{\partial\bar{\lambda}\partial\lambda}\log(1-|d_\pm(\lambda)|^2)=\beta^4(4\beta^2-1)^{-1}(1-\beta\lambda)^{-2}+o[(1-\beta\lambda)^{-2}].$$

Since $|1-d_+(\lambda)\bar{d}_-(\lambda)|$ tends to a nonzero limit as $\lambda\to 1/\beta$ (in fact, $d_+\to\bar{u}_0$ and $d_-\to u_0$), we see that the last term in (2.3) contributes $o(1)$ to the curvature, and we have

$$-\frac{\partial^2}{\partial\bar{\lambda}\partial\lambda}\log\|h_\lambda\|^2=\beta^2(1-\beta^2\lambda^2)^{-2}-2\beta^4(4\beta^2-1)^{-1}(1-\beta\lambda)^{-2}+o[(1-\beta\lambda)^{-2}]$$
$$=-\beta^2(4\beta^2+1)/[4(4\beta^2-1)(1-\beta\lambda)^2]+o[(1-\beta\lambda)^{-2}].$$

Selecting $\tau(\lambda)$ to be real if λ is real and using $h_{\tau(\lambda)}$, we obtain

$$\mathcal{K}_T(\lambda)=-\beta^2(4\beta^2+1)\tau'(\lambda)^2/[4(4\beta^2-1)(1-\beta\tau(\lambda))^2]+o[\tau'^2(1-\beta\tau)^{-2}].$$

As is well known, the curvature of the backward shift S is $-(1-|\lambda|^2)^{-2}$, and so if λ is real, we have, for the similar operators S and T,

$$\mathcal{K}_T/\mathcal{K}_S=\beta^2(4\beta^2+1)\tau'(\lambda)^2(1-\lambda^2)^2/[4(4\beta^2-1)(1-\beta\tau(\lambda))^2]$$
$$+o[\tau'^2(1-\lambda^2)^2(1-\beta\tau)^{-2}].$$

A theorem of Warschawski [6] tells us the behavior of τ and τ' near a singularity of $\tau(\mathbf{T})$. Indeed, if the inner angle of $\partial\ell$ at $1/\beta$ is $\alpha\pi$ $(0<\alpha\leqslant 2)$, and if $\tau(1)=1/\beta$, then, by [6],

$$\lim_{z\to 1}(z-1)[\tau(z)-\beta^{-1}]^{-1/\alpha}=\alpha\lim_{z\to 1}[\tau(z)-\beta^{-1}]^{1-1/\alpha}/\tau'(z),$$

or

$$\lim_{z\to 1}(z-1)^{-1}[\tau(z)-\beta^{-1}]/\tau'(z)=\alpha^{-1}.$$

It is a matter of elementary analytic geometry to check that $\alpha\pi=2\cos^{-1}(1/2\beta)$, and so we have proved

(2.4) $$\lim_{\lambda\to 1}\mathcal{K}_T/\mathcal{K}_S=4\pi^{-2}[\cos^{-1}(1/2\beta)]^2(4\beta^2+1)/(4\beta^2-1).$$

The right side of (2.4) cannot equal 1 for $\frac{1}{2}<\beta<1$. Indeed, the limit as $\beta\to\frac{1}{2}$ is $8/\pi^2<1$ and the function is decreasing on $(\frac{1}{2},1)$.

3. *k*-spectral sets. In this section we prove a proposition which sheds some additional light on the example of the previous section. Recall that a compact planar set Σ is called a k-spectral set for a bounded operator $\mathfrak{Z}$ if $\|f(\mathfrak{Z})\|\leqslant k\|f\|_\Sigma$ for all polynomials, f, where $\|\ \ \|_\Sigma$ is the sup norm on Σ.

PROPOSITION 1. *If Ω is simply connected, if $\mathfrak{Z}\in B_1(\Omega)$, and if $\bar{\Omega}$ is a k-spectral set for $\mathfrak{Z}$, then*

(3.1) $$|\mathcal{K}_{\mathfrak{Z}}(\omega)/\mathcal{K}_{\mathcal{S}}(\omega)|\geqslant k^{-2}$$

for $\omega\in\Omega$, where $\mathcal{S}$ is the adjoint of multiplication by z on $H^2(\Omega)$.

Proof. For the proof, we need to recall two facts which characterize the curvatures of $\mathfrak{I}$ and $\mathfrak{S}$ respectively. First, if $\omega \in \Omega$, the operator $\mathfrak{I}$, restricted to its invariant subspace $\ker(\mathfrak{I}-\omega I)^2$, has the 2×2 matrix representation

$$\mathfrak{I}|_{\ker(\mathfrak{I}-\omega I)^2} = \begin{bmatrix} \omega & H_{\mathfrak{I}}(\omega) \\ 0 & \omega \end{bmatrix} \tag{3.2}$$

where

$$H_{\mathfrak{I}}(\omega)^2 = -1/\mathcal{K}_{\mathfrak{I}}(\omega) \tag{3.3}$$

(see [4], p. 195). Second, the curvature of $\mathfrak{S}$ is given by

$$\mathcal{K}_{\mathfrak{S}}(\omega) = -\sup|f'(\omega)|^2 \quad \omega \in \Omega \tag{3.4}$$

where the supremum is over the class $\mathrm{Hol}_\omega(\Omega, \mathbf{D})$ of all holomorphic f having sup norm 1 in Ω and vanishing at ω (see [5], "Schwartz Lemma," preceding Corollary 1.1′).

Now let $f \in \mathrm{Hol}_\omega(\Omega, \mathbf{D})$. Since $f(\mathfrak{I})|_{\ker(\mathfrak{I}-\omega I)^2} = f(\mathfrak{I}|_{\ker(\mathfrak{I}-\omega I)^2})$, it follows from (3.2) that

$$\left\| \begin{bmatrix} 0 & f'(\omega)H_{\mathfrak{I}}(\omega) \\ 0 & 0 \end{bmatrix} \right\| = \left\| \begin{bmatrix} f(\omega) & f'(\omega)H_{\mathfrak{I}}(\omega) \\ 0 & f(\omega) \end{bmatrix} \right\| = \|f(\mathfrak{I}|_{\ker(\mathfrak{I}-\omega I)^2})\| \leqslant \|f(\mathfrak{I})\| \leqslant k.$$

Therefore $H_{\mathfrak{I}}(\omega) \leqslant k/\sup|f'(\omega)|$, for $f \in \mathrm{Hol}_\omega(\Omega, \mathbf{D})$ and, by (3.3) and (3.4),

$$\mathcal{K}_{\mathfrak{I}}(\omega) \leqslant -k^{-2}\sup|f'(\omega)|^2 = k^{-2}\mathcal{K}_{\mathfrak{S}}(\omega),$$

proving (3.1).

The result of the proposition is to be compared with an inequality of Cowen and Douglas [4, Corollary 4.30] which implies that if $\mathfrak{S}$ and $\mathfrak{I}$ lie in $B_1(\Omega)$, are similar, and have $\bar{\Omega}$ as a k-spectral set for some k, then

$$(\|L\|\|L^{-1}\|)^{-2} \leqslant \mathcal{K}_{\mathfrak{I}}(\omega)/\mathcal{K}_{\mathfrak{S}}(\omega) \leqslant (\|L\|\|L^{-1}\|)^2 \tag{3.5}$$

for any L satisfying

$$\mathfrak{I} = L^{-1}\mathfrak{S}L. \tag{3.6}$$

If, in addition, $\mathfrak{S}$ has $\bar{\Omega}$ as a 1-spectral set (as in the case for the $\mathfrak{S}$ of the proposition), then (3.6) implies that $\mathfrak{I}$ has $\bar{\Omega}$ as a $\|L\|\|L^{-1}\|$-spectral set. Therefore inequality (3.1) is sharper than the left inequality in (3.5).

As applied to the examples T and S of Section 2, the proposition implies that $\bar{\mathbf{D}}$ is a k-spectral set for T only if

$$k \geqslant \pi[2\cos^{-1}(1/2\beta)]^{-1}[(4\beta^2-1)/(4\beta^2+1)]^{1/2}.$$

In particular, $\bar{\mathbf{D}}$ is not a 1-spectral set for T.

4. A positive result. In this section, we show that the Cowen–Douglas conjecture holds for the class of Toeplitz operators considered in [3]. First, we need a version of the classical theorem on the angular derivative. In this case, we make a strong *local* hypothesis and obtain *unrestricted* approach to the derivative.

LEMMA. *Let* $z_0 \in \mathbf{T}$, $\delta > 0$, $g(z)$ *analytic in* $\Delta = \{|z - z_0| < \delta\}$, $|g(z)| = 1$ *on* $\mathbf{T} \cap \Delta$ *and* $|g(z)| < 1$ *on* $\mathbf{D} \cap \Delta$. *Then*

$$\lim_{z \to z_0} (1 - |g(z)|^2)/(1 - |z|^2) = |g'(z_0)| \tag{4.1}$$

for unrestricted approach from $z \in \mathbf{D}$.

Proof. First we claim that

$$z_0 \bar{g}(z_0) g'(z_0) \text{ is a real number.} \tag{4.2}$$

Let $z_0 = e^{i\theta_0}$, $g(z_0) = e^{i\psi_0}$ and $g(e^{i\theta}) = e^{i\psi}$, for $e^{i\theta} \in \Delta \cap \mathbf{T}$. We have

$$\begin{aligned} g'(z_0) &= \lim_{\theta \to \theta_0} (e^{i\psi_0} - e^{i\psi})/(e^{i\theta_0} - e^{i\theta}) \\ &= \lim_{\theta \to \theta_0} \exp[\tfrac{1}{2} i(\psi_0 + \psi - \theta - \theta_0)] \sin \tfrac{1}{2}(\psi_0 - \psi)/\sin \tfrac{1}{2}(\theta_0 - \theta) \\ &= \bar{z}_0 g(z_0) \lim_{\theta \to \theta_0} \sin \tfrac{1}{2}(\psi_0 - \psi)/\sin \tfrac{1}{2}(\theta_0 - \theta), \end{aligned}$$

which proves (4.2).

To prove the lemma, we note that the quotient on the left of (4.1) can be written as

$$\begin{aligned} [|g(e^{i\theta})|^2 - |g(re^{i\theta})|^2](1 - r^2)^{-1} &= g(e^{i\theta}) e^{-i\theta} (1 + r)^{-1} \bar{h}(e^{i\theta}, re^{i\theta}) \\ &\quad + \bar{g}(re^{i\theta}) e^{i\theta} (1 + r)^{-1} h(e^{i\theta}, re^{i\theta}), \end{aligned} \tag{4.3}$$

where $h(z, \omega)$ is defined by $h(z, \omega) = [g(z) - g(\omega)]/(z - \omega)$, if $z, \omega \in \Delta$, $z \neq \omega$ (and $h(z, z) = g'(z)$). Since $h(z, \omega)$ is uniformly continuous on compact subsets of $\Delta \times \Delta$, the right side of (4.3), as $r \to 1$, approaches the real part of $z_0 \bar{g}(z_0) g'(z_0)$ which, by (4.2) and its proof, is equal to $|g'(z_0)|$, proving the lemma.

Now let $F(z)$ be a rational function mapping $\mathbf{T}$ in an orientation preserving manner to a simple closed curve $F(\mathbf{T})$, and assume F is 1-to-1 in some annulus $\{r \leqslant |z| \leqslant 1\}$. By [3, Theorem 1], T_F is similar to T_τ, the Toeplitz operator associated with the mapping function τ from $\mathbf{D}$ to the interior of $F(\mathbf{T})$. In order to work in the disk $\mathbf{D}$, we set $T = \tau^{-1}(T_F)^*$. Then $T \in B_1(\mathbf{D})$ and T is similar to the backward shift S. Our result is:

PROPOSITION 2.

$$\lim_{\lambda \to \lambda_0 \in \mathbf{T}} \mathcal{K}_T(\lambda)/\mathcal{K}_S(\lambda) = 1.$$

Proof. Let $f(z) = \bar{F}(\bar{z}^{-1})$ be the rational function satisfying $T_f = T_F^*$, and write, for λ interior to $f(\mathbf{T})$,

$$f(z) - \lambda = a(\lambda) \prod (1 - d_i(\lambda) z) \prod (1 - e_i(\lambda) z)/[\prod (z - \delta_i) \prod (z - \gamma_i)],$$

where $|d_i(\lambda)|<1<|e_i(\lambda)|$ and $|\gamma_i|<1<|\delta_i|$. The eigenvectors of T are given by

$$k_\lambda(z)=\prod(1-\delta_i^{-1}z)/\prod(1-d_i(\bar{\tau}(\bar{\lambda}))z)$$

[3, Corollary 2.1] and the d_i can be renumbered so that $|d_i(\bar{\tau}(\bar{\lambda}))|\to 1$ as $\lambda\to\lambda_0\in\mathbf{T}$ if and only if $i=1$ [3, Lemmas 3.1 and 4.1]. Expanding k_λ in partial fractions and computing the norm as we did for h_λ in Section 2, we obtain

$$\|k_\lambda\|^2=\sum_j \frac{\prod_i[(d_j(\bar{\tau}(\bar{\lambda}))-\bar{\delta}_i^{-1})(1-\delta_i^{-1}d_j(\bar{\tau}(\bar{\lambda})))]}{\prod_{i\neq j}[\bar{d}_j(\bar{\tau}(\bar{\lambda}))-\bar{d}_i(\bar{\tau}(\bar{\lambda}))]\prod_i[1-d_i(\bar{\tau}(\bar{\lambda}))\bar{d}_j(\bar{\tau}(\bar{\lambda}))]}. \tag{4.4}$$

If we rewrite the right side of (4.4) with a common denominator, we have that
(1) the products over $i\neq j$ (in (4.4)) divide the numerator, and
(2) the numerator tends to a nonzero limit as $\lambda\to\lambda_0\in\mathbf{T}$.

To prove (1), fix $p\neq q$ and note that in the resolution of (4.4) into a single fraction exactly two terms in the numerator fail to contain a factor of d_p-d_q: those coming from the terms on the right of (4.4) with $j=p$ and $j=q$. It is easy to see that d_p-d_q divides the numerator of the sum of these two terms.

To prove (2), note that all terms in the numerator of the resolution of (4.4) into a single fraction tend to 0 except the one arising from the term $j=1$ (and the term arising from that one does *not* tend to 0).

By (1) and (2), we can write

$$\|k_\lambda\|^2=A(\lambda)\Big/\prod_{i,j}[1-\bar{d}_i(\bar{\tau}(\bar{\lambda}))d_j(\bar{\tau}(\bar{\lambda}))],$$

where $A(\lambda)\not\to 0$ as $\lambda\to\lambda_0\in\mathbf{T}$. Therefore

$$\begin{aligned}-\frac{\partial^2}{\partial\bar{\lambda}\partial\lambda}\log\|k_\lambda\|^2&=\frac{\partial^2}{\partial\bar{\lambda}\partial\lambda}\log[1-|d_1(\bar{\tau}(\bar{\lambda}))|^2]+O(1)\\&=\left|\frac{d}{d\lambda}d_1(\bar{\tau}(\bar{\lambda}))\right|^2[1-|d_1(\bar{\tau}(\bar{\lambda}))|^2]^{-2}+O(1).\end{aligned}$$

By [3, Lemma 3.2], $g(\lambda)=d_1(\bar{\tau}(\bar{\lambda}))$ satisfies the lemma in a sufficiently small neighborhood Δ of λ_0 and we have

$$\mathcal{K}_T/\mathcal{K}_S=\left|\frac{d}{d\lambda}d_1(\bar{\tau}(\bar{\lambda}))\right|^2[1-|\lambda|^2]^2[1-|d_1(\bar{\tau}(\bar{\lambda}))|^2]^{-2}\to 1$$

as $\lambda\to\lambda_0$. This proves Proposition 2.

REMARK 1. If $F(z)=z^2(z-a)/(1-az)$, $a>1$, then the hypotheses of Proposition 2 are satisfied if $a>3$. If $a=3$, the annulus hypothesis is violated but it can be shown that the conclusion of Proposition 2 is still valid.

REMARK 2. By Proposition 1, the Cowen–Douglas conjecture remains open if $\bar{\mathbf{D}}$ is a 1-spectral set for both S and T. On the other hand, Proposition 2 gives examples where the conjecture is true but $\bar{\mathbf{D}}$ is not a 1-spectral set for T [2, §3].

REMARK 3. A local version of Proposition 2 is easily obtained and implies, for the examples S and T of Section 2, that the ratio of the curvatures tends to 1 if λ tends to $\lambda_0 \in \mathbf{T}$, $\lambda_0 \neq 1$.

REMARK 4. It is evident that our results on the Cowen–Douglas conjecture have relied heavily upon the behavior of the mapping function τ from $\mathbf{D}$ to certain planar regions. Conversely, a proof of some reasonable modification of the conjecture, say, for a Toeplitz operator similar to an analytic function τ of the backward shift, would supply information about the derivative of the function τ.

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Department of Mathematics
University of Georgia
Athens, Georgia 30602