DOUBLY SLICED KNOTS AND DOUBLED DISK KNOTS

J. Levine

A *doubly sliced knot* is, roughly speaking, a knot which can be realized as a slice of a trivial knot. This notion was introduced by Fox [4] and Sumners [16]; Kearton [9] and Stoltzfus [18] have extracted algebraic obstructions from the middle dimensional homology of the infinite cyclic covering of the complement of the knot. In the case of higher dimensional simple knots the vanishing of this obstruction is necessary and sufficient for a knot to be doubly sliced ([16], [7], [19]). One can, in addition, ask whether there are further obstructions in the case of non-simple knots. In comparison, recall that there are no such further obstructions to a knot being sliced [12].

In his recent Ph.D. dissertation [15], D. Ruberman has used Casson–Gordon type invariants to define such obstructions and construct "algebraically" doubly sliced knots (i.e., satisfying the Sumners or Stoltzfus conditions) of every dimension which are not doubly sliced (also see [5]).

In the first part of this note we present a simpler approach by showing that the entire cohomology ring of the infinite cyclic covering of the complement of a knot represents a generalization of the Sumners and Stoltzfus obstructions. Examples in dimension 2 and 4 show this is a non-trivial generalization, but I have not yet found examples in higher dimensions.

The analogy between doubly sliced knots and codimension one submanifolds of Euclidean space, which is pointed out in [5] and [15], is also apparent in this result—compare [14].

We illustrate the usefulness of this approach by some examples: (1) the 2-twist spin of any 2-bridge knot and (2) the knots constructed by Cappell–Shaneson [3] are all shown to be not doubly sliced.

In the second part we show that a large number of doubly sliced knots can be generated by the process of "doubling" a disk knot. This generalizes the observation of Sumners [16] that the connected sum of any knot with its inverse is doubly sliced and, furthermore, includes all spun and super-spun knots [2]. It is not hard to find doubly sliced knots, in low dimensions, which are not doubled disk knots, but I have not been able to find higher-dimensional examples.

We also discuss, following suggestions to the author by D. Sumners, the related phenomenon of invertible disk knots (see [16]). The above result about doubled disk knots follow from the fact that "suspensions" of disk knots are invertible. On the other hand, examples of invertible knots which are not suspensions can be found in every dimension using the construction of [6].

However, I have found no examples which are "1-simple", i.e. the complements of the disk knot and its boundary knot have abelian fundamental group.

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1. A. An *n-knot* is defined to be a locally flat PL-submanifold $K^n \subseteq S^{n+2}$, where K^n is PL-homeomorphic to S^n. It is *doubly sliced* if there exists a *trivial* $(n+1)$-knot $\hat{K}^{n+1} \subseteq S^{n+3}$ such that $\hat{K}^{n+1} \cap S^{n+2} = K^n$. A knot is trivial if it bounds a PL-locally flat imbedded disk. We regard $S^{n+2} \subseteq S^{n+3}$ in the standard way (see [16]).

In [16] Sumners observes that the Blanchfield pairing of a doubly sliced n-knot, with n odd, is *hyperbolic*. In [18] Stoltzfus observes that the "torsion pairing" of a doubly sliced n-knot, with n even, is hyperbolic.

If X denotes the complement of the knot and $\tilde{X}$ the canonical infinite cyclic covering defined by $H_1(X) \approx Z$, then we have non-singular pairings:

$$H_i(\tilde{X}) \times H_{n+1-i}(\tilde{X}) \to Q(\Lambda)/\Lambda \quad \text{(Blanchfield pairing)}$$

$$T_i(\tilde{X}) + T_{n-i}(\tilde{X}) \to Q/Z \quad \text{(Torsion pairing)}$$

satisfying certain linearity and Hermitian properties (see [11]). $T_i(\tilde{X})$ is the Z-torsion submodule of $H_i(\tilde{X})$ and both are regarded as $\Lambda = Z[t, t^{-1}]$ modules induced by the group of covering translations.

When $n = 2i-1$, for the Blanchfield pairing, or $n = 2i$, for the torsion pairing, we are dealing with a pairing of a module with itself. Such a pairing, $\langle\ ,\ \rangle$ is *hyperbolic* if the underlying module $H = A \oplus B$ where A, B are self-annihilating submodules under $\langle\ ,\ \rangle$. By non-singularity, A and B are thus represented by $\langle\ ,\ \rangle$ as duals of each other.

Conversely, it has been proven by Sumners [16] and Kearton [7] that in the case of *simple*, higher odd-dimensional $(n>2)$ knots, these hyperbolic properties are sufficient to imply doubly sliced. A knot is *simple* if $\pi_i(X) \approx \pi_i(S^1)$ for $2i<n$. When $n=1,2$ the Casson–Gordon invariants imply the existence of counter-examples (see [15]).

We will regard the product structure in $\tilde{X}$ from a different point of view—that taken by Milnor [13]. We will proceed from Milnor's result [13] that the infinite cyclic covering $\tilde{X}$ of the complement X of a tubular neighborhood of an n-knot behaves, homologically, like a compact $(n+1)$-manifold. More specifically $H^{n+1}(\tilde{X}, \partial\tilde{X}) \approx F$, where coefficients are a field F, and the cup-products $\beta_i: H^i(\tilde{X}) \times H^{n+1-i}(\tilde{X}, \partial\tilde{X}) \to F$ are non-singular for all i. Note that $H^i(\tilde{X}) \approx H^i(\tilde{X}, \partial\tilde{X})$, for $0<i<n+1$, when X is a knot complement.

THEOREM A. *If X is the complement of a doubly sliced knot, then $H^*(\tilde{X})$ is hyperbolic in the following sense. There exist (graded) subalgebras $A, B \subseteq H^*(\tilde{X})$ (i.e., A, B are $F[t, t^{-1}]$—submodules and closed under cup-product), closed under cohomology operations, satisfying:*

(i) $H^i(\tilde{X}) = A^i \oplus B^i$ *for* $i>0$;

(ii) $\beta_i(A^i \times A^{n+1-i}) = 0 = \beta_i(B^i \times B^{n+1-i})$ *for all i.*

As a consequence, the pairing $A^i \times B^{n+1-i} \to F$, induced by β_i, is non-singular for $0<i<n+1$.

This theorem overlaps with the previous results ([16], [18]) since the Blanchfield pairing (over Q) can be derived from β_i, when $n=2i-1$, and the torsion pairing on elements of prime order p can be derived from β_i, when $n=2i$, and the Bockstein $H^i(\tilde{X}) \to H^{i+1}(\tilde{X})$, for $F=Z/p$.

The proof of theorem A is very easy. Let $\hat{K}^{n+1} \subseteq S^{n+3}$ be a trivial knot such that $\hat{K}^{n+1} \cap S^{n+2} = K^n$ a doubly sliced knot. Let T be a tubular neighborhood of $\hat{K}^{n+1}$ in S^{n+3} so that $T \cap S^{n+2}$ is a tubular neighborhood of K^n in S^{n+2}. Set $X = S^{n+2} - (T \cap S^{n+2})$, $X_\pm = D_\pm^{n+2} - (T \cap D_\pm^{n+2})$. A Mayer–Vietoris argument gives isomorphisms:

$$H^i(\tilde{X}_+) \oplus H^i(\tilde{X}_-) \xrightarrow{\approx} H^i(\tilde{X}), \quad \text{for } i>0,$$

induced by the inclusion maps $i_\pm : X \to X_\pm$, since $X_+ \cup X_-$ is the complement of a trivial knot and so $H^i(\tilde{X}_+ \cup \tilde{X}_-) = 0$ for $i>0$.

If we now set $A = i_+^* H^*(\tilde{X}_+)$, $B = i_-^* H^*(\tilde{X}_-)$, all the conclusions of the theorem, except (ii), are immediate. To see (ii), let $V_\pm = D_\pm^{n+3} \cap \partial T$ and consider the exact homology sequence:

$$H^{n+1}(\tilde{X}_+, \tilde{V}_+) \xrightarrow{i_+^*} H^{n+1}(\partial \tilde{X}_+, \tilde{V}_+) \xrightarrow{\delta^*} H^{n+2}(\tilde{X}_+, \partial \tilde{X}_+) \to H^{n+2}(\tilde{X}_+, \tilde{V}_+).$$

Note that $\tilde{V}_+$ is contractible and, by [13], $H^{n+2}(\tilde{X}_+) = 0$ and $H^{n+2}(\tilde{X}_+, \partial \tilde{X}_+) = F$. Also $H^{n+2}(\partial \tilde{X}_+, \tilde{V}_+) \approx H^{n+1}(\tilde{X}, \partial \tilde{X}) = F$ by excision. Thus, δ^* is an isomorphism and so $i_+^* = 0$. Since $\beta_i \mid A^i \times A^{n+1-i}$ pulls back, under i_+^*, to the cup-product $H^i(\tilde{X}_+) \times H^{n+1-i}(\tilde{X}_+, \tilde{V}_+) \to H^{n+1}(\tilde{X}_+, \tilde{V}_+)$, it follows that $\beta_i(A^i \times A^{n+1-i}) = 0$. A similar argument shows $\beta_i(B^i \times B^{n+1-i}) = 0$. □

B. We illustrate Theorem A by several examples.

1) We show that the 2-twist spin of any 2-bridge knot K is not doubly sliced. This example could also be deduced from the hyperbolic property of the torsion pairing.

According to Zeeman [20] these knots are fibred with fiber the double branched cover of K. According to Schubert [17], this cover is a lens space $L(r,s)$ for some odd integer r. Thus, if we choose p a prime dividing r and $F=Z/p$, then $H^1(\tilde{X}) \approx F$ generated by α, and $H^2(\tilde{X}) \approx F$ generated by the Bockstein of α. The impossibility of a decomposition of $H^*(\tilde{X})$ required by Theorem A is clear.

2) We consider the knots constructed by Cappell–Shaneson [3].

Let M be an $(n+1) \times (n+1)$ integer matrix satisfying the following properties:

(i) $\det M = +1$,

(ii) $\det(\lambda^i M - I) = \pm 1$ for $0 < i < n+1$,

where $\lambda^i M$ is the ith exterior power of M, and I the identity matrix. M defines a linear automorphism of $\mathbf{R}^{n+1}$ preserving the integer lattice, and therefore, an automorphism μ of the $(n+1)$-torus $T = S^1 \times \cdots \times S^1$. The induced automorphism on $H_i(T)$ is given by $\lambda^i M$. By taking the mapping torus of μ, [3] constructs a fibred n-knot with fiber $T^* = T$-point and monodromy μ.

Conditions (i) and (ii) are actually conditions on the characteristic polynomial $\Delta(t) = \det(tI - M)$. For the cases $n = 2, 3, 4$ only $i = 1, 2$ must be checked in (ii), since $\lambda^i M$ is the transpose of $\lambda^{n+1-i} M$. Particular examples can be found in [3].

PROPOSITION. *None of the* CS-*knots are doubly sliced.*

This will follow from the observation that $\Delta(t)$ must be irreducible for any CS-knot, for then $H^1(\tilde{X}) \approx H^1(T)$ would admit no proper submodules, and we could therefore assume that, e.g., $A^1 = H^1(\tilde{X})$, $B^1 = 0$ (coefficients in Q). But, since $H^*(\tilde{X}) \approx H^*(T^*)$ is generated, as an algebra, by $H^1(\tilde{X})$, we would conclude $A = H^*(\tilde{X})$, $B^i = 0$ for $i > 0$.

To see that $\Delta(t)$ must be irreducible, suppose $f(t)$ is a factor of degree $d < n+1$. Consider the characteristic polynomial $\phi(t)$ of $\lambda^d M$; $\phi(t)$ is the product of linear terms $(t - \xi_1 \dots \xi_d)$, where $\xi_1, \dots, \xi_d$ are roots of $\Delta(t)$. If we choose $\xi_1, \dots, \xi_d$ to be the roots of $f(t)$, we see that $t \pm f(0)$ is a factor of $\phi(t)$. But since $f(t) \mid \Delta(t)$, $f(0) = \pm 1$ and so $\phi(1)$ has a factor which is either zero or 2. In either case $\phi(1) = \pm 1$ will be impossible.

In the cases $n = 2, 4$ these results could not be obtained from the torsion pairing, since there are no Z-torsion elements in $H_*(\tilde{X})$. In the case $n = 3$, however, it can be checked that the characteristic polynomial of $\lambda^2 M$ is irreducible whenever $\Delta(t)$ is and, therefore, the Blanchfield pairing cannot even be metabolic—thus these CS-knots are not even sliced.

2. A. An *n-disk knot* $\Delta^n \subseteq D^{n+2}$ is a locally flat, properly imbedded n-disk. Its *boundary* is the $(n-1)$-knot $\dot{\Delta}^n \subseteq \dot{D}^{n+2}$. The *double* of Δ^n is the n-knot obtained by identifying two copies of (D^{n+2}, Δ^n) along their boundary.

THEOREM B. *The double of any disk knot is doubly sliced.*

We put this in a somewhat more general context. A disk knot Δ^n is *invertible* if there exists another disk knot Δ_0^n such that $\dot{\Delta}^n = \dot{\Delta}_0^n$ and the n-knot created by the union $\Delta^n \cup \Delta_0^n \subseteq D^{n+2} \cup D^{n+2} = S^{n+2}$ is unknotted (see [16]). Clearly the boundary of an invertible disk knot is doubly sliced, whereas every doubly sliced knot is the boundary of an invertible disk knot. The *suspension* of an n-disk knot Δ^n is the $(n+1)$-disk knot $I \times \Delta^n \subseteq I \times D^{n+2} \approx D^{n+3}$; clearly the boundary of the suspension of a disk knot is its double.

THEOREM C. *The suspension of any disk knot is invertible.*

This is proved in [16] when the boundary is unknotted. Theorem B is an obvious consequence of Theorem C.

LEMMA. *A disk knot* Δ^n *is invertible if and only if the complement* $X = D^{n+2} - \Delta^n$ *imbeds in* $S^1 \times S^{n+1}$ *inducing an isomorphism* $H_1(X) \xrightarrow{\approx} H_1(S^1 \times S^{n+1})$.

Proof. If Δ^n is invertible then the pair (D^{n+2}, Δ^n) imbeds in (S^{n+2}, S^n), the trivial knot. A surgery along S^n converts S^{n+2} to $S^1 \times S^{n+1}$ and will display $D^{n+2} - \Delta^n$ imbedded, as desired.

Conversely, assume $X \subseteq S^1 \times S^{n+1}$, where X will now denote the complement of an open tubular neighborhood of Δ^n in D^{n+2}. Choose a meridian of Δ^n in ∂X. A surgery along this curve (with an appropriate framing) will convert $S^1 \times S^{n+1}$ into S^{n+2}. If we insist that the framing contain the normal field to ∂X in X, then the surgery will add a handle of index 2 to X along the meridian. This converts X into D^{n+2}. Furthermore the transverse sphere K^n of the surgery, which is unknotted in S^{n+2}, will intersect D^{n+2} in Δ^n. Thus (D^{n+2}, Δ^n) is realized as a submanifold of the trivial n-knot (S^{n+2}, S^n), completing the proof of the lemma.

Now suppose the n-disk knot Δ^n is the suspension of a disk-knot Δ^{n-1}. Then the complement $X = D^{n+2} - \Delta^n = I \times X_0$, where $X_0 = D^{n+1} - \Delta^{n-1}$. There is an obvious imbedding $X_0 \subset D^{n+1} \subset S^{n+1}$, denoted by i. Choose any smooth map $\phi: X_0 \to S^1$ representing a generator of $H^1(X_0; Z)$ and consider the imbedding $X_0 \subset S^{n+1} \times S^1$ defined by $x \mapsto (i(x), \phi(x))$. The normal bundle is clearly trivial and so there is an extension to an imbedding of $X = I \times X_0 \subseteq S^1 \times S^{n+1}$. The proof of Theorem C is now completed by invoking the Lemma. □

B. We now pose the problem of finding invertible disk-knots which are not suspensions, and doubly sliced knots which are not doubles of disk knots. We give some examples drawn from the constructions of [6]. We begin by recalling the main result of [6] in a somewhat more general and explicit form.

THEOREM [6]. *Let $r < n$, positive integers, and $\alpha \in \pi_r(S^1 \vee S^r)$ such that, upon projection $S^1 \vee S^r \to S^r$, $\alpha \mapsto$ identity in $\pi_r(S^r)$. Then there exists an invertible disk $\Delta^n \subseteq D^{n+2}$ whose complement is homotopy equivalent to the adjunction space $S^1 \vee S^r \cup_\alpha e^{r+1}$.*

The construction in [6] will suffice to prove this result except that the authors impose more severe restrictions on r than ours. The necessary modifications are as follows, using the notation of [6]. The required imbedding $f: S^r \to \delta K - \delta A$ may be constructed by taking connected sums of copies of $f_1: S^r \to \delta K$, and its inverse, using tubes which go around δA in δK. The imbedding $g: B^{r+1} \to L - E^{q-2}$ ($q = n+2$ in our notation) is constructed by taking boundary connected sums of g_1 and its inverse which extend the tubes used in the construction of f. To see that f and f_1 are diffeotopic in δK and g and g_1 are diffeotopic in L, it suffices to notice that the tubes may be reeled in so that f and g are diffeotopic (in δK and L) to trivial connected sums of a number of copies of f_1 and g_1 and their inverses which add up, algebraically, to $+1$; but a trivial connected sum of f_1 or g_1 with its inverse is diffeotopic to the trivial imbedding and so we have the required conclusion.

C. We now apply this theorem to prove:

THEOREM D. *For any $n > 1$, there exists an invertible n-disk knot which is not a suspension.*

THEOREM E. *For $n = 1, 2$ there exists a doubly sliced n-knot which is not the double of a disk knot.*

If X denotes the complement of an n-disk knot, and $\tilde{X}$ its infinite cyclic cover, then the theorem of [6] asserts the existence of invertible disk-knots such that $H_r(\tilde{X})$ is a cyclic $Z[t, t^{-1}]$-module of order $\lambda(t)$, where $\lambda(t)$ is any element of $\Lambda = Z[t, t^{-1}]$ satisfying $\lambda(1) = 1$, and $H_i(\tilde{X}) = 0$ for $i \neq 0, r$. In particular we may choose $r=n-1$. But it is not hard to see that $H_{n-1}(\tilde{X})$ must be zero if the knot is a suspension. If $X=I\times X_0$ where X_0 is the complement of an $(n-1)$-disk knot, then $H_{n-1}(\tilde{X})\approx H_{n-1}(\tilde{X}_0)\approx \overline{H_e^2(\tilde{X}_0,\partial\tilde{X}_0)}$, by duality, where the latter is equivariant cohomology with a dual Λ-module structure (see [11]). It follows from the universal coefficient arguments of [11] that $H_e^2(\tilde{X}_0,\partial\tilde{X}_0)$ is an extension of $e^2(H_0(\tilde{X}_0,\partial\tilde{X}_0))$ by $e^1(H_1(\tilde{X}_0,\partial\tilde{X}_0))$. But $H_0(\tilde{X}_0,\partial\tilde{X}_0)=0=H_1(\tilde{X}_0,\partial\tilde{X}_0)$ since $H_1(\tilde{X}_0)\approx H_1(\tilde{X})=0$.

This proves Theorem D. □

To prove Theorem E for $n=1$ is easy. The double of a 1-disk knot must be of the form $K\#(-K)$, for some 1-knot K, but the knot 9_{46} is not of this form (its Alexander polynomial is $(2-t)(2t-1)$) and is constructed in [6]—corrected in [16]—as a doubly sliced knot. For $n=2$ we consider the 3-twist spin K of 9_{46}, which is doubly sliced as a consequence of the easy fact that any twist-spin of a doubly sliced knot is again doubly sliced. To see that K is not the double of a disk knot we consider $H_1(\tilde{X})$, where $\tilde{X}$ is the infinite cyclic cover of the complement X of K in S^4. According to [20], $\tilde{X}$ is equivariantly homotopy equivalent to the cyclic 3-fold branched cover of 9_{46} and so

$$H_1(\tilde{X})\approx\Lambda/(2-t,t^3-1)\oplus\Lambda/(2t-1,t^3-1).$$

On the other hand we will show that, when $H_1(\tilde{X})$ is finite and K is the double of a 2-disk knot, then $H_1(\tilde{X})\approx B\oplus B$, for some finite Λ-module B. To see that this is not the case for our example notice that $H_1(\tilde{X})/(t-2)H_1(\tilde{X})\approx Z/7$, which is indecomposable.

If K is the double of a 2-disk knot with complement X_0, then $X=X_0\cup X_0$ where the two copies of X_0 are identified along the complement X_1 of the boundary 1-knot. If $H_1(\tilde{X})$ is finite, then $H_2(\tilde{X})=0$ by duality (see [11]) and the Mayer–Vietoris sequence yields the short exact sequence:

$$0\to H_1(\tilde{X}_1)\to H_1(\tilde{X}_0)\oplus H_1(\tilde{X}_0)\to H_1(\tilde{X})\to 0.$$

Since X_1 is the complement of a 1-knot, $H_1(\tilde{X}_1)$ is a Z-torsion free and so $H_1(\tilde{X})\approx tH_1(\tilde{X}_0)\oplus tH_1(\tilde{X}_0)$, where t denotes Z-torsion submodule.

D. On the positive side we prove:

THEOREM F. *Any simple odd-dimensional doubly sliced knot, of dimension* >2, *is the double of a disk knot.*

Recall [10] that a simple $(2q-1)$-knot K with complement X is defined by the condition $\pi_i(X)\approx\pi_i(S^1)$ for $i<q$, and K is determined by $\pi_q(X)\approx H_q(\tilde{X})$ with the Blanchfield pairing [8]. If the knot is doubly sliced, then $H_q(\tilde{X})$ is hyperbolic, i.e., $H_q(\tilde{X})\approx \overline{B\oplus e^1(B)}$, where $e^1(B)=\mathrm{Ext}^1_\Lambda(B,\Lambda)\approx\mathrm{Hom}_\Lambda(B,Q(\Lambda)/\Lambda)$ and the Blanchfield pairing is the obvious one (see [7], [18])—B is a Z-torsion free Λ-module of type K. To realize such a module by a double of a disk knot it will suffice to construct a $(2q-1)$-disk knot with complement X_0 such that

$$H_i(\tilde{X}_0)=\begin{cases} B & i=q\\ 0 & i\neq 0,q.\end{cases}$$

The construction of X_0 follows from [11: Proposition 12.5]. To most easily achieve the desired Blanchfield pairing on $\tilde{X}$, we should specify it to be zero on $\tilde{X}_0$. To see that X_0 is the complement of a disk knot, observe that X_0 is a homology circle and ∂X_0 is a homology $S^1 \times S^{2q-1}$; if we attach a handle of index 2 to X_0 along a generator of $\pi_1(\partial X_0)$, the result is a $(2q+1)$-disk and the desired disk knot is the transverse disk of the handle.

3. We note several questions raised by our results.

1) Find examples of knots of dimensions $n>2$ satisfying the middle-dimensional hyperbolic properties of Sumners or Stoltzfus, such that $H^*(\tilde{X})$ is not hyperbolic. For example, one can hope for such examples among CS knots (although, as noted, this is impossible for $n=3$). This would require the construction of $(n+1)\times(n+1)$ matrices M satisfying (i), (ii) of B-2 such that $\lambda^q M$ is hyperbolic when $n=2q-1$.

2) Is the hyperbolic property of Theorem A a condition for *stably* doubly sliced? A knot K is stably doubly sliced if there exists a doubly sliced knot L such that the connected sum $K\#L$ is doubly sliced. Stably doubly sliced knots make up the *zero* class in the Stoltzfus double null-cobordism group ([18], [19]).

This question is related to the purely algebraic question of whether the hyperbolic property of Theorem A is stable, i.e., if H and H^1 are algebras of the sort we are considering and H^1 and $H\oplus H^1$ are hyperbolic, must H be hyperbolic?

3) Are there doubly sliced knots of dimension >2 which are not doubles of disk knots?

Are there "1-simple" disk-knots (see introduction) which are not suspensions?

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Department of Mathematics
Brandeis University
Waltham, Massachusetts 02254