WEIGHTED L^p ESTIMATES FOR THE CAUCHY INTEGRAL OPERATOR

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Introduction and statement of basic result. In 1977 A. P. Calderón proved that the Cauchy Integral Operator for a curve $(x, A(x))$ is bounded on L^2 provided that A' is in L^∞ with sufficiently small L^∞ norm. Four years later R. R. Coifman, Y. Meyer, A. McIntosh, and G. David developed new techniques and were able to remove the restriction on the size of the L^∞ norm of A'. Furthermore, as a result of the almost everywhere existence of the Cauchy Integral for rectifiable curves one can deduce the existence of a weighted L^2 estimate for such curves (see [2]). The main objective of this paper is the direct derivation of weighted L^p estimates for the Cauchy Integral Operator with weights that can be explicitly exhibited in a way that clarifies the role played by the geometry of the curve. We will prove the following:

THEOREM A. *There exist constants k_1 and k_2 such that for all $p>1$ there exists a constant C_p for which the following inequality holds:*

$$\int C_*^p(A,f)(x)\frac{dx}{(((1+S_q(A'))^{k_1})^*)^{k_2}} \leqslant C_p \int |f(t)|^p\,dt$$

where $C_*(A,f)=\sup_{\epsilon>0}|C_\epsilon(A,f)|$ and $C_\epsilon(A,f)(x)$ is the truncated operator corresponding to the Cauchy Singular Integral Operator

$$C(A,f)(x)=\text{p.v.}\int \frac{1+iA'(y)}{x-y+i(A(x)-A(y))}f(y)\,dy.$$

$S_q(A')$ denotes the q-sharp function of A':

$$S_q(A')(x)=\sup_Q\left(\frac{1}{|Q|}\int_Q |A'(y)-m_Q(A')|^q\,dy\right)^{1/q}$$

with $m_Q(.)$ denoting the mean over the specified interval, and the sup taken over all intervals containing x. $(.)^*$ denotes the Hardy-Littlewood Maximal Function and the variable t stands for arc-length.

The proof proceeds in three steps: (a) we use the Coifman-Meyer-McIntosh theorem (CMM) mentioned in the beginning as an a priori estimate to derive a provisional form of Theorem A via a good-λ inequality; (b) we use the measure theoretic and geometric techniques of step (a) again to show that the provisional result obtained there implies an improvement of itself, thus obtaining an L^p estimate; and (c) we prove a weak-type $(1,1)$ estimate which implies Theorem A by interpolation. The proof therefore contains a bootstrap argument from the CMM theorem following ideas used by G. David to derive the CMM theorem

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from the original result of Calderón. The proof provides a good illustration of the interplay between analysis and geometry.

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1. The first stage of the bootstrap argument.

PROPOSITION 1. *There exists a constant k such that for all positive ϵ we can find a constant C_ϵ for which the following is true:*

$$|\{x\colon C_*(A,f)(x)>(1+\epsilon)\lambda \text{ and } (1+S_q(A')(x))^k M_p(f)(x)\leqslant \lambda/C_\epsilon\}| \leqslant 0.9|\{x\colon C_*(A,f)(x)>\lambda\}|,$$

where $1/p+1/q=1/r$, r is positive and $M_p(f)$ is the p-maximal function of f, defined as follows:

$$M_p(f)(x)=\sup_Q\left(\frac{1}{|Q|}\int|f(y)|^p\,dy\right)^{1/p}$$

with the supremum taken over all intervals Q containing x.

Proof. We decompose the following open set as a union of pairwise disjoint open intervals: $\{x\colon C_*(A,f)(x)>\lambda\}=\bigcup Q_i=\bigcup (p_i,q_i)$. This implies that

$$|C_*(A,f)(p_i)|\leqslant\lambda \quad \text{for all } i. \tag{1.1}$$

It suffices to prove that

$$|\{x \text{ in } Q_i\colon C_*(A,f)>(1+\epsilon)\lambda \text{ and } (1+S_q(A'))^k M_p(f)\leqslant\lambda/C_\epsilon\}| \leqslant 0.9|Q_i|. \tag{1.2}$$

In fact we may also assume the following about Q_i:

$$|\{x \text{ in } Q_i\colon (1+S_q(A'))^k M_p(f)>\lambda/C_\epsilon\}|<0.1|Q_i|, \tag{1.3}$$

since otherwise there is nothing to prove. Let $\bar{Q}_i$ be an interval centered at p_i with length four times the length of Q_i. Write:

$$f=f_1+f_2 \quad \text{with} \quad f_1=f|_{\bar{Q}_i} \tag{1.4}$$

and let

$$\nu(Q_i)=\left(\frac{\lambda}{C_\epsilon}\,\frac{1}{\sup'_{Q\supset\bar{Q}_i}\left(\frac{1}{|Q|}\int_Q|f(y)|^p\,dy\right)^{1/p}}\right)^{1/k}-1. \tag{1.5}$$

We make one more decomposition of an open set into a union of pairwise disjoint open intervals:

$$\{x \text{ in } \bar{Q}_i\colon (1+S_q(A'))^k M_p(f)>\lambda/C_\epsilon \text{ or } M_1(A'-m_{Q_i}(A'))>\bar{C}\nu(Q_i)\} =\bigcup R_j. \tag{1.6}$$

Set $\mu_i=m_{Q_i}(A')$. Since M_1 is of weak type $(1,1)$ we obtain

$$
\begin{aligned}
|\{x \text{ in } Q_i : M_1(A'-\mu_i) > \bar{C}\nu(Q_i)\}| &\leqslant \frac{C}{\bar{C}\nu(Q_i)} \int_{Q_i} |A'-\mu_i|\,dy \\
&\leqslant \frac{C}{\bar{C}\nu(Q_i)} |Q_i| \left(\frac{1}{|Q_i|} \int_{Q_i} |A'-\mu_i|^q\,dy \right)^{1/q} \qquad (1.7)\\
&\leqslant \frac{C}{\bar{C}\nu(Q_i)} |Q_i| S_q(A')(x_i) \text{ with } x_i \text{ in } Q_i.
\end{aligned}
$$

In view of (1.3) we can certainly choose x_i in Q_i so that

$$S_q(A')(x_i) \leqslant \nu(Q_i). \tag{1.8}$$

Returning to (1.7), and choosing $\bar{C} > 10C$, we obtain

$$|\{x \text{ in } Q_i : M_1(A'-\mu_i)(x) > \bar{C}\nu(Q_i)\}| < 0.1|Q_i|. \tag{1.9}$$

Note that C is a geometric constant. We combine (1.3) and (1.9) to obtain the following:

$$\Big| Q_i - \bigcup_{R_j \subset Q_i} R_j \Big| > 0.8|Q_i|. \tag{1.10}$$

We let $\bar{R}_j$ be intervals of length twice the length of R_j and centered around the same points. We define

$$F_i = Q_i - \bigcup_{R_j \subset Q_i} \bar{R}_j. \tag{1.11}$$

It is immediate that

$$|F_i| > 0.6|Q_i|. \tag{1.12}$$

We will now describe a modification A_1 of A, which agrees with A everywhere except for the intervals R_j.

$$A_1(x) = \begin{cases} A(x) - \mu_i x \quad \text{for } x \text{ in } \bar{Q}_i - \bigcup R_j \\ \left(\dfrac{A(b_j) - A(a_j)}{b_j - a_j} - \mu_i \right)(x - a_j) + A(a_j) - \mu_i \\ \qquad\qquad\qquad\qquad \text{for } x \text{ in } R_j = (a_j, b_j). \end{cases} \tag{1.13}$$

We claim that A_1 as defined above is Lipschitz. For x in $\bar{Q}_i - \bigcup R_j$, $|A_1'(x)| = |A'(x) - \mu_i| \leqslant M_1(A'-\mu_i) \leqslant \bar{C}\nu(Q_i)$. For x in R_j we have

$$|A_1'(x)| = \frac{1}{|R_j|} \left| \int_{R_j} (A'(u) - \mu_i)\,du \right| \leqslant M_1(A'-\mu_i)(a_j) \leqslant \bar{C}\nu(Q_i).$$

Consequently,

$$\|A_1'\|_\infty \leqslant \bar{C}\nu(Q_i). \tag{1.14}$$

Consider now

$$C_1(f_1)(x) = \text{p.v.} \int \frac{1 + i(A'(y) - \mu_i)}{x - y + i(A_1(x) - A_1(y))} f_1(y)\,dy \tag{1.15}$$

thinking of it as a Cauchy Integral for A_1 acting on the product $(1+i(A'(y)-\mu_i))\times f_1(y)$. Fixing $r>1$ we apply CMM.

$$\begin{aligned} \|C_{1,*}(f_1)\|_r^r &= \|C_*(A_1, (1+i(A'-\mu_i))f_1)\|_r^r \\ &\leq C_r(1+\nu(Q_i))^m \|(1+i(A'-\mu_i))f_1\|_r^r. \end{aligned} \tag{1.16}$$

We will now investigate what happens to this estimate when we add a constant w to both $g=A'-\mu_i$ and A_1'. We will let $B_w(y)=A_1(y)+wy$ and we split the argument into two cases according to the relative size of $\|A_1'\|_\infty$ and w.

Case I. $|w|<2\|A_1'\|_\infty+1$, which implies $\|B_w'\|_\infty \leq 3(\|A_1'\|_\infty+1)$. Therefore,

$$\begin{aligned} \|C_*(B_w, (1+i(g+w))f_1)\|_r^r &\leq C_r(1+\nu(Q_i))^m \|(1+|g|+|w|)|f_1|\|_r^r \\ &\leq C_r(1+\nu(Q_i))^{m+r}\|(1+|g|)|f_1|\|_r^r. \end{aligned}$$

Case II. $|w|>2\|A_1'\|_\infty+1$. By factoring out w from the denominator we obtain

$$C(B_w, (1+i(g+w))f_1) = \text{p.v.}\int \frac{1}{z(x)-z(y)} \frac{1+i(g(y)+w)}{w} f_1(y)\,dy$$

where

$$z(x) = \frac{x}{w} + i\left(\frac{A_1(x)}{w} + x\right).$$

Moreover, since

$$\left|\frac{A_1(x)-A_1(y)}{w(x-y)} + 1\right| \geq 1 - \frac{1}{|w|}\|A_1'\|_\infty > \frac{1}{2},$$

the curve defined by $z(x)$ is bi-Lipschitz. Hence

$$\begin{aligned} \|C_*(B_w, (1+i(g+w))f_1)\|_r^r &\leq C_r \left\| \frac{1+|g(y)|+|w|}{|w|} |f_1(y)| \right\|_r^r \\ &\leq C_r \|(1+|g|)|f_1|\|_r^r. \end{aligned}$$

By comparing the estimates obtained above we see that we have an L^r estimate in (1.16) which is independent of a constant added to both g and A_1'. We now let

$$w = m_{Q_i}(A') = \mu_i \quad \text{and} \quad B(y) = A_1(y) + \mu_i y. \tag{1.17}$$

Estimate (1.16) becomes:

$$\|C_*(B, (1+iA')f_1)\|_r^r \leq C_r(1+\nu(Q_i))^{m+r}\|(1+|A'-\mu_i|)|f_1|\|_r^r.$$

Consequently,

$$\begin{aligned} &|\{x \text{ in } \bar{Q}_i : C_*(B, (1+iA')f_1)(x) > \epsilon\lambda/5\}| \\ &\leq \frac{\|C_*(B, (1+iA')f_1)\|_r^r}{(\epsilon\lambda/5)^r} \leq C_r(1+\nu(Q_i))^{m+r}(\epsilon\lambda/5)^{-r}\|(1+|A'-\mu_i|)|f_1|\|_r^r \\ &\leq C_r(1+\nu(Q_i))^{m+r}(\epsilon\lambda/5)^{-r}|Q_i|\left(\frac{1}{|\bar{Q}_i|}\int_{\bar{Q}_i}(1+|A'-\mu_i|)^q\right)^{r/q}\left(\frac{1}{|\bar{Q}_i|}\int_{\bar{Q}_i}|f|^p\right)^{r/p}. \end{aligned}$$

Here we have applied Hölder's inequality. Moreover, the first of the two integrals of this inequality is dominated by $(1+S_q(A')(x_i))^r \leqslant (1+\nu(Q_i))^r$ by (1.8). By choosing k so that $m+2r<kr$, and $C_\epsilon > (10C)^{1/r}5/\epsilon$, we obtain

$$|\{x \text{ in } \bar{Q}_i : C_*(B, (1+iA')f_1(x) > \lambda\epsilon/5\}| \leqslant C(\lambda\epsilon/5)^{-r}(\lambda/C_\epsilon)^r|Q_i| \tag{1.18}$$
$$\leqslant C(5/\epsilon C_\epsilon)^r|Q_i| < 0.1|Q_i|$$

(by (1.5)). We now turn our attention to the set F_i. We will estimate the difference $h_\epsilon(x) = C_\epsilon(A, f_1)(x) - C_\epsilon(B, (1+iA')f_1)(x)$ for x in F_i. The estimate below is independent of ϵ, so we will drop ϵ from the formula. Since $A=B$ except on $\bigcup R_j$ (see (1.13) and (1.17)),

$$h(x) = \int_{\bigcup R_j}\left(\frac{1}{x-y+i(A(x)-A(y))} - \frac{1}{x-y+i(B(x)-B(y))}\right)(1+iA'(y))f(y)\,dy. \tag{1.19}$$

Since $A(x)=B(x)$ for x in F_i we obtain the estimate

$$\int_{F_i}|h(x)|\,dx \leqslant \int_{F_i}\sum_j\int_{R_j}\frac{|A(y)-B(y)|(1+|A'(y)|)|f_1(y)|}{|x-y|^2\left|1+i\left(\dfrac{A_1(x)-A_1(y)}{x-y}-\mu_i\right)\right|}\,dy\,dx. \tag{1.20}$$

Here we have used the following fact:

$$\frac{B(x)-B(y)}{x-y} = \frac{\int_y^x B'(u)\,du}{x-y} = \frac{\int_y^x (A_1'(u)+\mu_i)\,du}{x-y}$$
$$= \frac{A_1(x)-A_1(y)}{x-y} + \mu_i$$

(see (1.17)). Moreover,

$$|A(y)-B(y)| = \left|\int_{a_j}^y (A'(u)-m_{R_j}(A'))\,du\right|$$
$$\leqslant |R_j|\left(\frac{1}{|R_j|}\int_{R_j}|A'(u)-m_{R_j}(A')|\,du\right)$$
$$\leqslant |R_j|S_q(A')(a_j) \leqslant |R_j|(1+\nu(Q_i)).$$

We now claim that:

$$\frac{1+|A'-\mu_i|+|\mu_i|}{\left|1+i\left(\dfrac{A_1(x)-A_1(y)}{x-y}+\mu_i\right)\right|} \leqslant C(1+|A'(y)-\mu_i|)(1+\nu(Q_i)). \tag{1.21}$$

This follows from an argument similar to that for (1.16) by distinguishing two cases, $|\mu_i| \leqslant 2\|A_1'\|_\infty + 1$ and $|\mu_i| > 2\|A_1'\|_\infty + 1$. In view of this, (1.20) yields

$$\int_{F_i} |h(x)|\,dx \leqslant C(1+\nu(Q_i))^2 \int_{\bar{Q}_i} (1+|A'(y)-\mu_i|)^s |f(y)|\,dy$$

where $s=q/p'>1$ if $1/p+1/p'=1$ and $1/p+1/q=1/r$ with $r>1$. Therefore, by Hölder's inequality,

$$\int_{F_i} |h(x)|\,dx \leqslant C(1+\nu(Q_i))^2 |Q_i|(1+S_q(A')(x_i))^s \left(\frac{1}{|\bar{Q}_i|}\int_{\bar{Q}_i}|f|^p\right)^{1/p}$$
$$\leqslant C(1+\nu(Q_i))^{2+s}|Q_i|\left(\frac{1}{|\bar{Q}_i|}\int_{\bar{Q}_i}|f|^p\right)^{1/p}.$$

Hence,

$$|\{x \text{ in } F_i : |h(x)|>\lambda\epsilon/5\}| \leqslant (5/\lambda\epsilon)\int_{F_i}|h(x)|\,dx$$
$$\leqslant C(5/\lambda\epsilon)|Q_i|(1+\nu(Q_i))^{2+s}\left(\frac{1}{|\bar{Q}_i|}\int_{\bar{Q}_i}|f|^p\right)^{1/p} \tag{1.22}$$
$$\leqslant (5C/\epsilon C_\epsilon)|Q_i| \leqslant 0.1|Q_i|.$$

We have taken $k>2+s$ and $C_\epsilon>50C/\epsilon$.

By combining (1.18) and (1.22) we obtain:

$$|\{x \text{ in } F_i : C_*(A,f_1)(x)>2\lambda\epsilon/5\}|$$
$$\leqslant |\{x \text{ in } F_i : \sup_{\epsilon>0}|h_\epsilon(x)|>\lambda\epsilon/5\}| + |\{x \text{ in } F_i : \sup_{\epsilon>0}|C_\epsilon(B,(1+A')f_1)|(x)>\lambda\epsilon/5\}|$$
$$\leqslant 0.2|Q_i|.$$

Since $|F_i|>0.6|Q_i|$ we obtain the following estimate for f_1:

$$|\{x \text{ in } Q_i : C_*(A,f_1)(x)\leqslant 2\lambda\epsilon/5\}| \geqslant 0.4|Q_i|. \tag{1.23}$$

For f_2 we have the following estimate:

LEMMA 1. *For all x in F_i,*

$$C_*(A,f_2)(x)\leqslant\lambda(1+(\epsilon/5)). \tag{1.24}$$

(1.23) and (1.24) immediately imply Proposition 1:

$$|\{x \text{ in } Q_i : C_*(A,f)(x)\leqslant\lambda(1+(3\epsilon/5))\}| \geqslant 0.4|Q_i|.$$

Proof of Lemma 1. Recall that $Q_i=(p_i,q_i)$. From (1.1) it follows that $C_*(A,f_2)(p_i)\leqslant\lambda$. It suffices to prove the following estimate:

$$\sup_{\epsilon>0}|C_\epsilon(A,f_2)(x)-C_\epsilon(A,f_2)(p_i)|\leqslant\lambda\epsilon/5 \quad \text{for all } x \text{ in } F_i. \tag{1.25}$$

Fix an interval J_x centered at x and an interval $J_{p_i}=J_i$ centered around p_i and of the same length as J_x. We will estimate

$$\left|\int_{R-J_x}K(x,y)f_2(y)\,dy-\int_{R-J_i}K(p_i,y)f_2(y)\,dy\right| \leqslant \tag{1.26}$$

$$\leqslant \left|\int_{R-J_x\cup J_i\cup \bar{Q}_i}(K(x,y)-K(p_i,y))f(y)\,dy\right|$$
$$+\int_{J_x\Delta J_i}|K(x,y)||f_2(y)|\,dy+\int_{J_x\Delta J_i}|K(p_i,y)||f_2(y)|\,dy$$

where

$$K(x,y)=\frac{1+iA'(y)}{x-y+i(A(x)-A(y))}.$$

The estimates for the last two integrals are the same. Here is how we estimate one of them.

Because f_2 is supported on $\bar{Q}_i^c$ we have $|x-y|>(1/5)|J|$ with $J=J_x\cup J_i$, for all x in F_i and all y in $\bar{Q}_i^c\cap(J_x\Delta J_i)$.

$$\begin{aligned}(1.27)\quad &\int_{J_x\Delta J_i}\frac{1+|A'(y)|}{|x-y|\left|1+i\left(\dfrac{A(x)-A(y)}{x-y}\right)\right|}|f_2(y)|\,dy\\ &\leqslant\frac{5}{|J|}\int_{J_x\Delta J_i}\frac{(1+|A'(y)-m_J(A')|+|m_J(A')|)|f_2(y)|}{\left|1+i\dfrac{1}{x-y}\displaystyle\int_y^x(A'(u)-m_J(A'))\,du+im_J(A')\right|}\,dy.\end{aligned}$$

Since

$$\begin{aligned}\frac{1}{|x-y|}\left|\int_y^x(A'(u)-m_J(A'))\,du\right|&\leqslant\frac{5}{|J|}\int_J|A'(u)-m_J(A')|\,du\\ &\leqslant 5S_q(A')(x)\leqslant 5\nu(Q_i)\end{aligned}$$

we can estimate the integrand in (1.27) by distinguishing two cases as for (1.16) depending on whether $|m_J(A')|>2(5\nu(Q_i))+1$ or not. In either case we obtain the following estimate which is independent of the size of $m_J(A')$:

$$\begin{aligned}(1.28)\quad &\int_{J_x\Delta J_i}|K(x,y)||f_2(y)|\,dy\\ &\leqslant C(1+\nu(Q_i))\frac{1}{|J|}\int_J(1+|A'-m_J(A')|)^s|f_2|\,dy\\ &\leqslant C(1+\nu(Q_i))^{1+s}\left(\frac{1}{|J|}\int_J|f_2(y)|^p\,dy\right)^{1/p}\quad\text{(Hölder's inequality)}\\ &\leqslant C(\lambda/C_\epsilon)\leqslant\lambda\epsilon/15\end{aligned}$$

where k and s are taken as in (1.22), and $C_\epsilon>15C/\epsilon$.

We now turn our attention to the first integral in (1.26). It suffices to obtain an estimate for

$$\begin{aligned}C_K&=\int_{R-\bar{Q}_i}|K(x,y)-K(p_i,y)||f(y)|\,dy\\ &\leqslant\int_{R-\bar{Q}_i}\frac{|p_i-x+i(A(p_i)-A(x))|(1+|A'(y)|)|f(y)|}{|x-y+i(A(x)-A(y))||p_i-y+i(A(p_i)-A(y))|}\,dy.\end{aligned}$$

Since $\frac{1}{2}|p_i-y|<|x-y|<(3/2)|p_i-y|$ we obtain: $C_K \leqslant$

$$C\int_{R-\bar{Q}_i} \frac{\left(1+\left|\frac{1}{p_i-x}\int_{p_i}^{x}|A'-\mu_i|\right|+\mu_i\right)(1+|A'-\mu_i|+|\mu_i|)|f|}{|x-y|^2\left|1+i\frac{1}{x-y}\int_y^x (A'-\mu_i)+i\mu_i\right|\left|1+i\frac{1}{p_i-y}\int_{p_i}^{y}(A'-\mu_i)+i\mu_i\right|}\,dy.$$

The integrals inside this integrand are all dominated by $M_1(A'-\mu_i)$ which in turn is dominated by $\bar{C}\nu(Q_i)$. We are again in position to apply an argument similar to that for (1.16) by distinguishing two cases depending on whether $|\mu_i|>2\bar{C}\nu(Q_i)+1$ or not. In both cases we obtain

$$C_K \leqslant C(1+\nu(Q_i))^2 \int_{R-Q_i} \frac{|Q_i|(1+|A'(y)-\mu_i|)|f(y)|}{|x-y|^2}\,dy.$$

By a standard argument involving the decomposition

$$R-\bar{Q}_i \subset \bigcup_{j=1}^{\infty} \{2^j|Q_i|>|x-y|>2^{j-1}|Q_i|\}$$

we find that

$$\begin{aligned} C_K &\leqslant C(1+\nu(Q_i))^3(1+S_q(A')(x_i))^s \sup_{Q\supset \bar{Q}_i}\left(\frac{1}{|Q|}\int_Q |f|^p\right)^{1/p} \\ &\leqslant C(1+\nu(Q_i))^k \sup_{Q\supset \bar{Q}_i}\left(\frac{1}{|Q|}\int_Q |f|^p\right)^{1/p} \quad (\text{for } k>s+3) \\ &\leqslant \lambda C/C_\epsilon \leqslant \lambda\epsilon/15 \quad \text{if } C_\epsilon > 15C/\epsilon. \end{aligned}$$

The proof of Lemma 1 is now complete. □

COROLLARY 1. $\|C_*(A,f)\|_s^s \leqslant C_s\|(1+S_q(A'))^k M_p(f)\|_s^s$.

Proof. Standard argument using the good-λ inequality of Proposition 1. □

THEOREM 1. *Let* $q>1$, $1/p+1/q=1/r$, $r>1$, $p_1>p$. *If*

$$\omega(x)=(((1+S_q(A'))^{kp_1+1})^*(x))^{kp_1/(kp_1+1)}$$

then $\|C_*(A,f)\|_{p_1}^{p_1} \leqslant C_{p_1}\int |f|^{p_1}\omega(y)\,dy$.

Proof. ω is a maximal function to a power less than 1 so it is a weight of class A_p for $p\geqslant 1$. Moreover, ω clearly dominates $(1+S_q(A'))^{kp_1}$. Theorem 1 follows from Corollary 1 by an application of the Weighted Norm Inequality for the Maximal Function.

2. The second stage of the bootstrap argument. The inequality in Proposition 1 contains the constant 0.9. The proof of the proposition could not be directly improved to allow for the replacement of that constant by a parameter that could be made arbitrarily small. Proposition 2 below moves in that direction and follows from Proposition 1.

PROPOSITION 2. *Let $p_1>p$ and ω be as in Theorem 1. Then there exists a constant C such that*

$$|\{x: C_*(A,f)>2\lambda \text{ and } ((M_{p_1}^{p_1}(f)\omega)^*)^{1/p_1}\leqslant \lambda/\beta\}|\leqslant (C/\beta)|\{x: C_*(A,f)>\lambda\}|.$$

Proof. Let $\{x: C_*(A,f)(x)>\lambda\}=\bigcup_i Q_i$, where the Q_i are pairwise disjoint open intervals. It suffices to prove

$$|\{x \text{ in } Q_i: C_*(A,f)>2\lambda \text{ and } ((M_{p_1}^{p_1}(f)\omega)^*)^{1/p_1}\leqslant \lambda/\beta\}|\leqslant (C/\beta)|Q_i|. \tag{2.1}$$

We may also assume that there exists x_i in Q_i such that

$$((M_{p_1}^{p_1}(f)\omega)^*(x_i))^{1/p_1}\leqslant \lambda/\beta \tag{2.2}$$

since otherwise there is nothing to prove. Moreover, if $Q_i=(a_i,b_i)$,

$$C_*(A,f)(a_i)\leqslant \lambda. \tag{2.3}$$

Let $\bar{Q}_i$ be an open interval centered at a_i with length four times the length of Q_i. Write $f=f_1+f_2$ with $f_1=f|_{\bar{Q}_i}$.

First we obtain an estimate for f_1:

$$\begin{aligned}|\{x: C_*(A,f_1)(x)>\lambda/2\}| &\leqslant (C/\lambda^{p_1})\int C_*^{p_1}(A,f_1)(y)\,dy\\ &\leqslant (C/\lambda^{p_1})\int |f_1(y)|^{p_1}\omega(y)\,dy \quad \text{(Theorem 1)}\\ &= (C/\lambda^{p_1})\int_{\bar{Q}_i} |f(y)|^{p_1}\omega(y)\,dy\\ &\leqslant (C/\lambda^{p_1})|Q_i|(|f|^{p_1}\omega)^*(x_i)\\ &\leqslant (C/\lambda^{p_1})|Q_i|(M_{p_1}^{p_1}(f)\omega)^*(x_i)\\ &\leqslant (C/\beta^{p_1})|Q_i| \quad \text{(by 2.2).}\end{aligned} \tag{2.4}$$

Before proceeding with the estimate for f_2 we need to make some comments. (2.2) implies that $M_{p_1}(f)(x_i)(1+S_q(A')(x_i))^k\leqslant \lambda/\beta$ and consequently

$$S_q(A')(x_i)\leqslant (\lambda/\beta)^{1/k}\left(\sup_{J\supset Q_i}\left(\frac{1}{|J|}\int_J |f(y)|^{p_1}\,dy\right)^{-1/p_1}\right)^{1/k}-1 = \nu(Q_i) \tag{2.5}$$

Therefore, if $\mu_i=m_{Q_i}(A')$,

$$\begin{aligned}|\{x \text{ in } Q_i: (A'-\mu_i)^*(x)>\beta_1\nu(Q_i)\}| &\leqslant \frac{C}{\beta_1\nu(Q_i)}\int_{Q_i}|A'-\mu_i|\\ &\leqslant \frac{C}{\beta_1\nu(Q_i)}|Q_i|S_q(A')(x_i)\\ &\leqslant \frac{C}{\beta_1}|Q_i|.\end{aligned} \tag{2.6}$$

Letting $F_i=\{x \text{ in } Q_i: (A'-\mu_i)^*(x)\leqslant \beta_1\nu(Q_i)\}$ we obtain

$$(2.7)\qquad |F_i| > (1-(C/\beta_1))|Q_i|.$$

We now obtain the estimate for f_2. From (2.3) we see that $C_*(A,f_2)(a_i) \leqslant \lambda$. As in the proof of Lemma 1 we need to estimate

$$\Delta(x) = \sup_{\epsilon>0}|C_\epsilon(A,f_2)(x) - C_\epsilon(A,f_2)(a_i)| \quad \text{for all } x \text{ in } F_i.$$

In view of (2.5) and (2.7) we obtain as in Lemma 1 the following:

$$\Delta(x) \leqslant C\beta_1^2(1+\nu(Q_i))^k \sup_{J\supset \bar{Q}_i}\left(\frac{1}{|J|}\int_J |f|^p\right)^{1/p}$$

and, by (2.5),

$$(2.8)\qquad \Delta(x) \leqslant C\lambda\beta_1^2/\beta \quad \text{for all } x \text{ in } F_i$$

Therefore,

$$(2.9)\qquad C_*(A,f_2)(x) \leqslant C_*(A,f_2)(a_i) + \Delta(x) \leqslant \lambda(1+(C\beta_1^2/\beta))$$

By combining (2.4), (2.6), and (2.9) we obtain

$$|\{x \text{ in } Q_i : C_*(A,f)(x) \leqslant \lambda(1+\tfrac{1}{2}+(C\beta_1^2/\beta))\}|$$
$$\geqslant |\{x \text{ in } Q_i : C_*(A,f_1)(x) \leqslant \lambda/2 \text{ and } C_*(A,f_2)(x) \leqslant \lambda(1+(C\beta_1^2/\beta))\}|$$
$$\geqslant (1-(C/\beta^{p_1}) - (C/\beta_1))|Q_i|.$$

By choosing β_1 and β so that $\frac{1}{4} \leqslant C\beta_1^2/\beta \leqslant \frac{1}{2}$ we obtain

$$(2.10)\qquad |\{x \text{ in } Q_i : C_*(A,f)(x) > 2\lambda\}| \leqslant (C/\beta)|Q_i|.$$

This completes the proof of Proposition 2. □

COROLLARY 2. *Let W be a weight of class A_∞. Then*

$$W(\{x : C_*(A,f) > 2\lambda \text{ and } ((M_{p_1}^{p_1}(f)\omega)^*)^{1/p_1} \leqslant \lambda/\beta\}) \leqslant (C/\beta^\delta)\, W(\{C_*(A,f) > \lambda\}).$$

Proof. Recall that there exist constants C and δ such that for all intervals Q and all measurable subsets E of Q, $W(E)/W(Q) \leqslant C(|E|/|Q|)^\delta$ where $W(E) = \int_E W(x)\,dx$. The corollary now follows directly from (2.10).

COROLLARY 3. *If W is in A_∞ then*

$$\int C_*^s(A,f)(y)\,W(y)\,dy \leqslant C_s \int ((M_{p_1}^{p_1}(f)\omega)^*(y))^{s/p_1}\omega(y)\,dy.$$

Proof. Follows from Corollary 2 by standard argument. □

PROPOSITION 3. *There exist constants k_1 and k_2 such that the following estimate holds for s sufficiently large* ($s > p_1$):

$$\int \frac{C_*^s(A,f)(x)}{(((1+S_q(A'))^{k_1})^*)^{k_2}}\,dx \leqslant C_s \int |f(x)|^s\,dx.$$

Proof. ω is as in the statement of Theorem 1. Write $kp_1/(kp_1+1) = k_1k_2$ with $0<k_1$, $k_2<1$ and set

(2.11) $$\omega_1=(((1+S_q(A'))^{kp_1+1})^*)^{k_2}, \quad \text{so that} \quad \omega=\omega_1^{k_1}.$$

Both ω and ω_1 are in A_1. In particular ω_1 is in A_z where $z=(s/p_1)'=s/(s-p_1)$. By the theory of weights, $W=\omega_1^{-(s-p_1)/p_1}$ is in $A_{z'}=A_{s/p_1}$. By Corollary 3 we obtain (by taking $s>p_1$ and using the Weighted Norm Inequality on the right-hand side):

$$\begin{aligned}\int C_*^s(A,f)(y)\,W(y)\,dy &\leq C_s \int M_{p_1}^s(f)\,\omega^{s/p_1}W(y)\,dy \\ &\leq C_s \int M_{p_1}^s(f)\,\omega_1^{k_1 s/p_1}\omega_1^{-(s-p_1)/p_1}\,dy \\ &\leq C_s \int M_{p_1}^s(f)(y)\,dy \leq C_s \int |f|^s\end{aligned}$$

for $s>p_1/(1-k_1)$. This completes the proof. □

3. The weak type (1, 1) estimate.

PROPOSITION 4. *Let $W(x)$ be as in Proposition 3. The following weak type $(1,1)$ estimate holds:*

$$W(\{x\colon C_*(A,f)(x)>\lambda\}) \leq (C/\lambda)\int |f(t)|\,dt$$

where t stands for arc-length.

Proof. In the arc-length parameterization the Cauchy Integral can be written as follows:

$$C(A,f)(s)=\lim_{\epsilon\to 0}\int \frac{f(x(t))}{z(s)-z(t)}\,z'(t)\alpha_\epsilon(s,t)\,dt$$

where $\alpha_\epsilon(s,t)$ is the characteristic function of the set $\{t\colon |x(s)-x(t)|>\epsilon\}$, $s(x)=\int_0^x(1+A'(u)^2)^{1/2}\,du$, and $z(x)=x+iA(x)$.

We perform a Calderón–Zygmund decomposition for $f_1(t)=f\circ x(t)z'(t)$ as follows:

(3.1) $$f_1(t)=g(t)+b(t)$$

(3.2) $$g(t)=\begin{cases} f_1(t) & \text{for } t \text{ in } F \\ m_{Q_j}(f_1) & \text{for } t \text{ in } Q_j^0 \text{ with } F^c=\bigcup Q_j\end{cases}$$

(3.3) $$|F^c|\leq (C/\lambda)\int |f(t)|\,dt, \quad |f_1(t)|\leq\lambda \quad \text{for } t \text{ in } F,\ m_{Q_j}(|f_1|)\leq C\lambda.$$

We will produce estimates for g and b. We start with the estimate for g.

$$\int \frac{g(t)\alpha_\epsilon(s,t)}{z(s)-z(t)}\,dt=C_\epsilon(A,g_1) \quad \text{with} \quad g_1(x)=\frac{g(t(x))(1-iA'(x))}{(1+(A'(x))^2)^{1/2}}.$$

This is the result of a change of variables for the integral. Therefore,

$$
\begin{aligned}
W(\{C_*(A,g_1)>\lambda/2\}) &\leqslant (C/\lambda^s)\int C_*^s(A,g_1)\,W(x)\,dx \\
&\leqslant (C/\lambda^s)\int |g_1(x)|^s\,dx \quad \text{(Proposition 3)} \\
&\leqslant (C/\lambda)\int |g(t(x))|\,dx \quad (\text{since } g(t)\leqslant\lambda) \\
&\leqslant (C/\lambda)\int |g(t)|\,dt \leqslant (C/\lambda)\int |f(t)|\,dt.
\end{aligned} \tag{3.4}
$$

Next, we estimate b. Let $d_j = \operatorname{diam}(Q_j)$, and let $\bar{Q}_j$ be an interval centered at t_j, which is a fixed point of Q_j, with length four times the length of Q_j. If $\bar{F} = R - \bigcup \bar{Q}_j$, then

$$
|\bar{F}^c| \leqslant 4|F^c| \leqslant (C/\lambda)\int |f(t)|\,dt. \tag{3.5}
$$

As for g, we have

$$
\int \frac{b(t)\alpha_\epsilon(s,t)}{z(s)-z(t)}\,dt = C_\epsilon(A,b_1),
$$

and

$$
\begin{aligned}
&W\{C_*(A,b_1)(x)>\lambda/2\} \\
&\leqslant W\{x\{s \text{ in } \bar{F}: \sup_\epsilon |C_\epsilon(A,b_1)(x(s))|>\lambda/2\}\} + W\{x\{\bar{F}^c\}\} \\
&= W\{F_1\} + W\{F_2\}.
\end{aligned} \tag{3.6}
$$

But,

$$
W\{x\{\bar{F}^c\}\} \leqslant |x\{\bar{F}^c\}| \leqslant |\bar{F}^c| \leqslant (C/\lambda)\int |f(t)|\,dt, \tag{3.7}
$$

$$
W\{F_1\} = \int_{F_1} W(x)\,dx = \int_{s(F_1)} dW(s) \quad \text{where} \quad dW(s) = \frac{W(x(s))\,ds}{(1+(A'(x(s))^2)^{1/2}}
$$

$$
W\{F_1\} \leqslant (C/\lambda)\int_{\bar{F}} \sup_\epsilon \left| \int \frac{b(t)\alpha_\epsilon(s,t)}{z(s)-z(t)}\,dt \right| dW(s).
$$

Recall now that b is supported on $\bigcup Q_j$ and that $m_{Q_j}(b)=0$.

$$
\begin{aligned}
W\{F_1\} &\leqslant (C/\lambda)\int_{\bar{F}} \sup_\epsilon \left| \sum_j \int_{Q_j} \left(\frac{\alpha_\epsilon(s,t)}{z(s)-z(t)} - \frac{\alpha_\epsilon(s,t_j)}{z(s)-z(t_j)} \right) b(t)\,dt \right| dW(s) \\
&\leqslant (C/\lambda)\int_{\bar{F}} \sum_j \int_{Q_j} \frac{|z(t)-z(t_j)|\,|b(t)|}{|z(s)-z(t)|\,|z(s)-z(t_j)|}\,dt\,dW(s) \\
&\quad + (C/\lambda)\int_{\bar{F}} \sum_j \sup_\epsilon \int_{Q_j} \frac{|\alpha_\epsilon(s,t)-\alpha_\epsilon(s,t_j)|}{|z(s)-z(t_j)|}\,|b(t)|\,dt\,dW(s).
\end{aligned} \tag{3.8}
$$

But,

$$\frac{|s-t|}{|z(s)-z(t)|} \leqslant C(1+S_q(A'))(x(s)), \tag{3.9}$$

$$(1+S_q(A'))^2\, dW(s) \leqslant C\frac{(1+S_q(A'))^2}{(1+S_q(A'))^{k_1k_2}}\, ds \leqslant ds, \tag{3.10}$$

$$\int_{Q_j} |\alpha_\epsilon(s,t)-\alpha_\epsilon(s,t_j)|\, dW(s) \leqslant |x\{Q_j\}| \leqslant d_j. \tag{3.11}$$

Returning to (3.8), and using (3.9), (3.10) and (3.11) we obtain

$$\begin{aligned} W\{F_1\} &\leqslant (C/\lambda)\sum_j \int_{Q_j}\int_{|s-t_j|>2d_j} \frac{|b(t)|d_j}{(s-t)^2}\, ds\, dt \\ &\quad + (C/\lambda)\sum_j \int_{Q_j} \sup_\epsilon \int_{|s-t_j|>2d_j} \frac{|\alpha_\epsilon(s,t)-\alpha_\epsilon(s,t_j)|}{|s-t|}\, |b(t)|\, dW(s)\, dt \qquad (3.12)\\ &\leqslant (C/\lambda)\int |b(t)|\, dt \leqslant (C/\lambda)\int |f(t)|\, dt \end{aligned}$$

(3.12) and (3.7) now yield the desired estimate for b. And since we already have the estimate for g in (3.4) we see that the proof of Proposition 4 is complete. Theorem A now follows from Propositions 3 and 4 and standard interpolation arguments. □

In higher dimensions one can study the Double Layer Potential Operators. These have the form

$$C_\epsilon^j(A,f)(x) = \int_{|x-y|>\epsilon} \frac{x_j-y_j}{(|x-y|^2+(A(x)-A(y))^2)^{(n+1)/2}} f(y)\, dy$$

where x, y are in $\mathbf{R}^n$.

Instead of using a sharp function, we use a maximal function, namely $M_p(|\nabla A|)$ with $p>n$, and the techniques of this paper extend to yield similar weighted L^p estimates (see [7]).

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