ON COMPACT COMPLETELY BOUNDED MAPS OF C^*-ALGEBRAS

Tadasi Huruya

1. Introduction. A C^*-algebra is said to be subhomogeneous if all its irreducible representations are on Hilbert spaces of dimension at most some positive integer. In the proof of [5: Theorem 3] we proved that if a C^*-algebra A is infinite dimensional and a C^*-algebra B is not subhomogeneous, then there exists a compact linear map of A into B which is not completely bounded and which is not a linear combination of positive linear maps.

In this paper we show that if ϕ is a map of a C^*-algebra into a nuclear C^*-algebra such that there exists a sequence of linear maps of finite rank converging to ϕ in the completely bounded norm, then ϕ is a linear combination of compact, completely positive maps. We also study, in the completely bounded norm, the closure of the set of linear maps of finite rank between some C^*-algebras. As an application of a result of Smith [8: Theorem 2.8], we prove that if on the algebraic tensor product $A \odot B$ of two C^*-algebras A and B, the greatest cross norm γ is equivalent to the projective C^*-cross norm, then either A or B is finite dimensional.

2. Preliminaries. For C^*-algebras A and B, let $B(A,B)$, $K(A,B)$ and $F(A,B)$ denote the set of bounded linear maps of A into B, the set of compact linear maps of A into B and the set of linear maps of finite rank of A into B, respectively.

Let ϕ be a map in $B(A,B)$. It is possible to define associated maps $\phi \otimes \mathrm{id}_n : A \otimes M_n \to B \otimes M_n$, and ϕ is said to be completely positive if each $\phi \otimes \mathrm{id}_n$ is positive, and completely bounded if $\sup_n \|\phi \otimes \mathrm{id}_n\| < \infty$. This quantity is called the completely bounded norm $\|\phi\|_{\mathrm{cb}}$ when it exists. If ϕ is completely positive, $\|\phi\|_{\mathrm{cb}} = \|\phi\|$. Let $CB(A,B)$ denote the set of completely bounded maps of A into B. For ϕ in $B(A,B)$, if there exist completely positive maps ϕ_i of A into B such that $\phi = \phi_1 - \phi_2 + i(\phi_3 - \phi_4)$, then ϕ is said to have a completely positive decomposition. If a net $\{\phi_\beta\}$ in $CB(A,B)$ converges to ϕ in the norm $\|.\|_{\mathrm{cb}}$, we write $\text{cb-lim}_\beta\, \phi_\beta = \phi$.

Let $A \otimes_\alpha B$ denote the completion of the algebraic tensor product $A \odot B$ under a norm α. In particular $A \otimes_{\max} B$ and $A \otimes_{\min} B$ mean the projective and injective C^*-tensor products, respectively [9: Chapter IV, Section 4].

A C^*-algebra A is said to be nuclear if for every C^*-algebra B the C^*-norm on the algebraic tensor product $A \odot \mathrm{B}$ is uniquely determined [2].

For the theory of C^*-algebras, we refer to the book of Takesaki [9].

3. Nuclear range algebras. The following lemma is perhaps known. For completeness we include the proof.

Received February 25, 1983. Revision received May 31, 1983.
Michigan Math. J. 30 (1983).

LEMMA 1. *Let $\{f_i\}_{i=1}^n$ be a set of self-adjoint elements of A^* and let $\{a_i\}_{i=1}^n$ be a set of self-adjoint elements of B. Put $\phi(x)=\sum_{i=1}^n f_i(x)a_i$ for x in A. Then there exist completely positive maps ϕ^+,ϕ^- of finite rank such that $\phi=\phi^+-\phi^-$, $\|\phi^+\|_{\mathrm{cb}}\leqslant\sum_{i=1}^n\|f_i\|\|a_i\|$ and $\|\phi^-\|_{\mathrm{cb}}\leqslant\sum_{i=1}^n\|f_i\|\|a_i\|$.*

Proof. Let f_i^+,f_i^- be positive linear functionals on A such that $f_i=f_i^+-f_i^-$, $\|f_i\|=\|f_i^+\|+\|f_i^-\|$ and let a_i^+,a_i^- be positive elements of B such that $a_i=a_i^+-a_i^-$, $a_i^+a_i^-=0$. Then $\|a_i^+\|\leqslant\|a_i\|$ and $\|a_i^-\|\leqslant\|a_i\|$. For x in A, we put $\phi_1(x)=f_1(x)a_1$, $\phi_1^+(x)=f_1^+(x)a_1^++f_1^-(x)a_1^-$, $\phi_1^-(x)=f_1^+(x)a_1^-+f_1^-(x)a_1^+$. Then $\phi_1=\phi_1^+-\phi_1^-$,

$$\|\phi_1^+\|\leqslant\max\{\|f_1^+\|\|a_1^+\|,\ \|f_1^-\|\|a_1^-\|\}\leqslant\|f_1\|\|a_1\|.$$

Similarly, $\|\phi_1^-\|\leqslant\|f_1\|\|a_1\|$.

Let $C^*(a_1)$ denote the C^*-subalgebra generated by a_1. We have $\phi_1^+(A)\subseteq C^*(a_1)$, $\phi_1^-(A)\subseteq C^*(a_1)$. Since $C^*(a_1)$ is commutative, both ϕ_1^+ and ϕ_1^- are completely positive [9: Chapter IV, Corollary 3.5].

We put, for x in A,

$$\phi^+(x)=\sum_{i=1}^n(f_i^+(x)a_i^++f_i^-(x)a_i^-),$$

$$\phi^-(x)=\sum_{i=1}^n(f_i^+(x)a_i^-+f_i^-(x)a_i^+).$$

The above argument implies that ϕ^+ and ϕ^- are desired completely positive maps. □

LEMMA 2. *Let ϕ be a self-adjoint linear map of finite rank of a C^*-algebra A into a nuclear C^*-algebra B. Given $\epsilon>0$ there exist completely positive maps ϕ^+,ϕ^- in $F(A,B)$ such that $\phi=\phi^+-\phi^-$, $\|\phi^+\|<\|\phi\|_{\mathrm{cb}}+\epsilon$ and $\|\phi^-\|<\|\phi\|_{\mathrm{cb}}+\epsilon$.*

Proof. We choose $\{f_i\}_{i=1}^n$ in A^* and $\{b_i\}_{i=1}^n$ in B such that $f_i^*=f_i$, $b_i^*=b_i$ $(i=1,\dots,n)$ and $\phi(x)=\sum_{i=1}^n f_i(x)b_i$ for each x in A. Let $\tilde{f}_i$ denote the σ-weakly continuous extension to A^{**} of f_i and for x in A^{**} we put $\tilde{\phi}(x)=\sum_{i=1}^n\tilde{f}_i(x)b_i$. Then $\tilde{\phi}$ is a σ-weakly continuous map of A^{**} into B. Also $(A\otimes M_m)^{**}$ can be identified with $A^{**}\otimes M_m$. Then Kaplansky's density theorem shows that $\|\tilde{\phi}\|_{\mathrm{cb}}=\|\phi\|_{\mathrm{cb}}$. We may then assume that A has a unit.

By a result of Choi-Effros [2: Theorem 3.1], there exist a matrix algebra M_k and completely positive contractions $\tau:B\to M_k$, $\psi:M_k\to B$ such that $\sum_{i=1}^n\|f_i\|\|\psi\circ\tau(b_i)-b_i\|<\epsilon$. By Wittstock's theorem [13: Satz 4.5] (see also [7]), there exist completely positive maps ϕ_1,ϕ_2 such that $\tau\circ\phi=\phi_1-\phi_2$, $\|\phi_j\|\leqslant\|\tau\circ\phi\|_{\mathrm{cb}}\leqslant\|\phi\|_{\mathrm{cb}}$ $(j=1,2)$. Since M_k is finite dimensional, both ϕ_1 and ϕ_2 are of finite rank.

Let $\phi'=\phi-\psi\circ\tau\circ\phi$. Then ϕ' is finite dimensional. By Lemma 1, there exist completely positive maps ϕ_3 and ϕ_4 in $F(A,B)$ such that $\phi'=\phi_3-\phi_4$, $\|\phi_j\|\leqslant\sum_{i=1}^n\|f_i\|\|\psi\circ\tau(b_i)-b_i\|<\epsilon$ $(j=3,4)$.

We put $\phi^+ = \psi\circ\phi_1 + \phi_3$, $\phi^- = \psi\circ\phi_2 + \phi_4$. Then ϕ^+ and ϕ^- are completely positive maps in $F(A,B)$ such that $\phi = \phi^+ - \phi^-$,

$$\|\phi^+\| \leqslant \|\psi\circ\phi_1\| + \|\phi_3\| \leqslant \|\phi_1\| + \|\phi_3\| < \|\phi\|_{\mathrm{cb}} + \epsilon,$$

$$\|\phi^-\| \leqslant \|\psi\circ\phi_2\| + \|\phi_4\| \leqslant \|\phi_2\| + \|\phi_4\| < \|\phi\|_{\mathrm{cb}} + \epsilon.$$

THEOREM 3. *Let ϕ be a map of a C^*-algebra A into a nuclear C^*-algebra B. If there exists a sequence $\{\phi_n\}$ in $F(A,B)$ with* cb-$\lim_n \phi_n = \phi$, *then ϕ is a linear combination of* cb-*limit maps of sequences of completely positive maps in $F(A,B)$.*

Proof. We may assume that ϕ and all ϕ_n's are self-adjoint since each completely bounded map ψ satisfies $\|\mathrm{Re}(\psi)\|_{\mathrm{cb}} \leqslant \|\psi\|_{\mathrm{cb}}$, where $\mathrm{Re}(\psi)$ denotes the self-adjoint part of ψ. Using the sequence $\{\phi_n\}$, we have, by a standard argument, a sequence $\{\psi_n\}$ of self-adjoint linear maps in $F(A,B)$ such that $\|\psi_n\|_{\mathrm{cb}} \leqslant \epsilon/2^n$ $(n \geqslant 2)$ and $\sum_{n=1}^{\infty} \psi_n = \phi$ in $\|.\|_{\mathrm{cb}}$. Then

$$\|\psi_1\|_{\mathrm{cb}} \leqslant \|\phi\|_{\mathrm{cb}} + \sum_{n=2}^{\infty} \|\psi_n\|_{\mathrm{cb}} \leqslant \|\phi\|_{\mathrm{cb}} + (\epsilon/2).$$

By Lemma 2 we have completely positive maps in $F(A,B)$ such that $\psi_1 = \psi_1^+ - \psi_1^-$, $\|\psi_1^+\| \leqslant \|\psi_1\|_{\mathrm{cb}} + (\epsilon/2)$, $\|\psi_1^-\| \leqslant \|\psi_1\|_{\mathrm{cb}} + (\epsilon/2)$, $\psi_n = \psi_n^+ - \psi_n^-$, $\|\psi_n^+\| \leqslant \epsilon/2^n$, $\|\psi_n^-\| \leqslant \epsilon/2^n$ $(n \geqslant 2)$. For x in A, we put $\phi^+(x) = \sum_{n=1}^{\infty} \psi_n^+(x) \in B$, $\phi^-(x) = \sum_{n=1}^{\infty} \psi_n^-(x) \in B$. Then $\phi = \phi^+ - \phi^-$ and

$$\|\phi^i\| \leqslant \|\psi_1^i\| + \sum_{n=2}^{\infty} \|\psi_n^i\| \leqslant \|\phi\|_{\mathrm{cb}} + \epsilon \quad (i = +, -).$$

Then we have, in $\|.\|_{\mathrm{cb}}$,

$$\sum_{n=1}^{\infty} \psi_n^+ = \phi^+, \quad \sum_{n=1}^{\infty} \psi_n^- = \phi^-.$$

This completes the proof. □

With the same notation as in Theorem 3, we remark that the closure of $F(A,B)$ in the norm $\|.\|_{\mathrm{cb}}$ is the span of cb-limit maps of sequences of completely positive maps of finite rank.

If we remove the condition "nuclear" from Theorem 3, we have the following situation. Its proof is a slight improvement of [5: Example 12]. The author is grateful to the referee for suggesting a simplification of the original argument.

EXAMPLE 4. *Let A be the reduced group C^*-algebra of the free group on two generators and let $M = \sum_{n=1}^{\infty} \oplus A_n$, $A_n = A$, the $C(\infty)$-direct sum. Then there exists a map ϕ of A into M having the following properties:*

(1) *There exists a sequence of linear maps of finite rank which converges to ϕ in the norm $\|.\|_{\mathrm{cb}}$;*

(2) *There exists no bounded linear map $\phi \otimes_{\max} \mathrm{id} : A \otimes_{\max} A \to M \otimes_{\max} A$ satisfying $\phi \otimes_{\max} \mathrm{id}(a \otimes b) = \phi(a) \otimes b$;*

(3) *ϕ has no completely positive decomposition.*

Proof. By De Cannière and Haagerup's theorem [1: Corollary 3.11] we have a sequence $\{\phi_n\}$ in $F(A,A)$ with $\|\phi_n\|_{cb}\leq 1$ and $\lim_n \phi_n(x)=x$ for all x in A. Since each ϕ_n is of finite rank, we can define the bounded linear map $\phi_n\otimes_{\max}\mathrm{id}: A\otimes_{\max}A\to A\otimes_{\max}A$. If $\{\|\phi_n\otimes_{\max}\mathrm{id}\|\}$ is bounded, then

$$\lim_n \phi_n\otimes_{\max}\mathrm{id}(x)=x$$

for all $x\in A\otimes_{\max}A$. Let $\rho: A\otimes_{\max}A\to A\otimes_{\min}A$ be the *-homomorphism and let $x\in\ker\rho$.

If $f\in(A\otimes_{\max}A)^*$, then $f\circ(\phi_n\otimes_{\max}\mathrm{id})\in A^*\odot A^*$, the algebraic tensor product, and $f(x)=\lim_n f\circ(\phi_n\otimes_{\max}\mathrm{id})(x)=0$ since $g\in A^*\odot A^*$ implies that $g(\ker\rho)=\{0\}$ by the commutativity of

$$\begin{array}{ccc} A^*\odot A^* & \to & (A\otimes_{\max}A)^* \\ & \searrow & \uparrow \rho^* \\ & & (A\otimes_{\min}A)^*. \end{array}$$

Hence $x=0$, a contradiction. Therefore we may assume that $\{\phi_n\}$ satisfies $n^3\leq\|\phi_n\otimes_{\max}\mathrm{id}\|$ for all n. We put $\phi=\sum_{n=1}^{\infty}\oplus(1/n)^2\phi_n$. Since $\|\phi\|_{cb}\leq\sum_{n=1}^{\infty}(1/n)^2$, (1) holds.

Suppose that there exists the bounded linear map

$$\phi\otimes_{\max}\mathrm{id}: A\otimes_{\max}A\to M\otimes_{\max}A.$$

Let ψ_n be the *-homomorphism of M onto A_n defined by $\psi_n(\sum_{i=1}^{\infty}\oplus x_i)=x_n$. Using the map $\psi_n\otimes_{\max}\mathrm{id}: M\otimes_{\max}A\to A_n\otimes_{\max}A$, we have $\|(1/n)^2\phi_n\otimes_{\max}\mathrm{id}\|=\|(\psi_n\otimes_{\max}\mathrm{id})\circ(\phi\otimes_{\max}\mathrm{id})\|\leq\|\phi\otimes_{\max}\mathrm{id}\|$, so that $n^3\leq n^2\|\phi\otimes_{\max}\mathrm{id}\|$. Hence $n\leq\|\phi\otimes_{\max}\mathrm{id}\|$, a contradiction. Consequently (2) holds.

If (3) holds, (2) holds [9: Chapter IV, Proposition 4.23]. This is a contradiction. □

THEOREM 5. *For C^*-algebras A and B, the following are equivalent:*

(1) *Either A is finite dimensional or B is subhomogeneous;*

(2) $B(A,B)=CB(A,B)$;

(3) $K(A,B)\subseteq CB(A,B)$;

(4) *$K(A,B)$ is the span of compact, completely positive maps;*

(5) *$K(A,B)$ is the closure of $F(A,B)$ in the norm $\|.\|_{cb}$.*

Proof. It is easy to check that (2)⇒(3), (4)⇒(3) and (5)⇒(3) as $CB(A,B)$ is a Banach space in the norm $\|.\|_{cb}$ (cf. the proof of [11: Proposition 1]). The remark in the introduction implies that (3)⇒(1).

By [8: Theorem 2.8] we have (2)⇒(1). Let $\psi: A\to B$ be a bounded linear map. If A is finite dimensional, then ψ is of finite rank. Hence ψ is completely bounded. If B is subhomogeneous, it follows from [8: Theorem 2.10] that $\|\psi\|_{cb}=\|\psi\otimes\mathrm{id}_n\|$ where n is the greatest number in the set of dimensions of irreducible representations of B. Hence ϕ is completely bounded. Consequently, (1)⇒(2).

Suppose that (1) holds. We then show that both (4) and (5) hold. If A is finite dimensional, then (4) and (5) are clearly true. We assume next that B is subhomogeneous. Then $B(A,B)=CB(A,B)$ by the above argument. The open mapping theorem implies that there exists a positive real number R such that $\|\phi\|_{cb} \leqslant R\|\phi\|$ for all ϕ in $B(A,B)$. Since B has the approximation property (see, for example, [2: Theorem 3.1]), for ϕ in $K(A,B)$, there exists a sequence in $F(A,B)$ converging to ϕ in the norm $\|\phi\|_{cb}$. Then (4) holds. Also it follows from Theorem 3 that ϕ is a linear combination of compact, completely positive maps. Therefore (5) holds. □

4. Dual range algebras. A C^*-algebra A is said to be dual if there exists a *-isomorphism from A to a C^*-subalgebra of the C^*-algebra of compact linear operators on some Hilbert space. We show that if ϕ is a compact, completely positive map of a dual C^*-algebra A into another C^*-algebra B, there exists a sequence in $F(A,B)$ converging to ϕ in the norm $\|.\|_{cb}$. Moreover if B is nuclear, the closure of $F(A,B)$ in $\|.\|_{cb}$ is the span of compact, completely positive maps.

LEMMA 6. *Let ϕ be a compact, completely positive map between C^*-algebras A and B and let $\{u_\beta\}$ be an approximate unit for A. Let ϕ_β be the map defined by $\phi_\beta(x)=\phi(u_\beta x u_\beta)$ for each x in A. Then* cb-$\lim_\beta \phi_\beta=\phi$.

Proof. There exist a bounded linear map V and a representation π of A such that $\phi(x)=V^*\pi(x)V$ for each x in A [9: Chapter IV, Theorem 3.6]. Since ϕ is compact and $\pi(u_\nu)\leqslant\pi(u_\mu)\leqslant I$ for $\nu\leqslant\mu$, $\{\phi(u_\beta)\}$ converges to V^*V in the operator norm. Hence $\{\|(I-\pi(u_\beta))^{1/2}V\|\}$ converges to 0 as $\|(I-\pi(u_\beta))^{1/2}V\|^2=\|V^*(I-\pi(u_\beta))V\|$. Since $\{(I-\pi(u_\beta))^{1/2}\}$ is bounded, $\{\pi(u_\beta)V\}$ converges to V in the operator norm. We then have

$$\begin{aligned}\|\phi_\beta-\phi\|_{cb}&=\|(V^*\pi(u_\beta)-V^*)\pi(.)\pi(u_\beta)V+V^*\pi(.)(\pi(u_\beta)V-V)\|_{cb}\\&\leqslant\|V^*\pi(u_\beta)-V^*\|\|\pi(u_\beta)V\|+\|V^*\|\|\pi(u_\beta)V-V\|\\&\leqslant 2\|\pi(u_\beta)V-V\|\|V\|.\end{aligned}$$

Hence cb-$\lim_\beta \phi_\beta=\phi$. □

PROPOSITION 7. *If ϕ is a compact, completely positive map of a dual C^*-algebra A into another C^*-algebra B, then there exists a sequence in $F(A,B)$ converging to ϕ in $\|.\|_{cb}$.*

Proof. Since A is dual, there exists an approximate unit $\{u_\beta\}$ such that each $u_\beta A u_\beta$ is finite dimensional. Let $\phi_\beta(x)=\phi(u_\beta x u_\beta)$ for x in A. Then ϕ_β is of finite rank. Lemma 6 implies that the desired sequence exists in $F(A,B)$. □

This proposition has applications to C^*-algebras considered in [4, 5].

PROPOSITION 8. *Let A be the C^*-algebra of all compact linear operators on an infinite dimensional Hilbert space and let $\tilde{A}$ be the C^*-algebra generated by A and the identity operator I. If ϕ is a compact, completely positive map of $\tilde{A}\otimes_{\min}\tilde{A}$ to $\tilde{A}$, there exists a sequence in $F(\tilde{A}\otimes_{\min}\tilde{A},\tilde{A})$ converging to ϕ in $\|.\|_{cb}$.*

Proof. Let f be a state on $\tilde{A}$ such that $f(A)=\{0\}$. Let R_f and L_f be bounded linear maps of $\tilde{A}\otimes_{\min}\tilde{A}$ into $\tilde{A}$ satisfying $R_f(a\otimes b)=f(a)b$ and $L_f(a\otimes b)=$

$f(b)a$ [10]. Then R_f and L_f are completely positive. For each x in $\tilde{A}\otimes_{\min}\tilde{A}$, put $\psi(x)=x-I\otimes(R_f(x))-(L_f(x))\otimes I\in A\otimes_{\min}A+\mathbf{C}(I\otimes I)$. Then $\|\psi\|_{\text{cb}}\leqslant 3$.

Let $\phi_R(x)=\phi(I\otimes x)$ and $\phi_L(x)=\phi(x\otimes I)$ for each x in $\tilde{A}$. By Proposition 7, there exists a sequence $\{\phi_n\}$ in $F(A,\tilde{A})$ converging to $\phi_R\,|\,A$ in $\|.\|_{\text{cb}}$. For each $x+cI$ in $\tilde{A}$, put $\psi_n(x+cI)=\phi_n(x)+c\phi(I\otimes I)$, where c is a complex number. Every ψ_n is of finite rank and cb-$\lim_n \psi_n=\phi_R$. Hence there exists a sequence in $F(\tilde{A}\otimes_{\min}\tilde{A},\tilde{A})$ converging to $\phi_R\circ R_f$ in $\|.\|_{\text{cb}}$.

Similarly, there exists a sequence in $F(\tilde{A}\otimes_{\min}\tilde{A},\tilde{A})$ converging to $\phi_L\circ L_f$ in $\|.\|_{\text{cb}}$.

Since $A\otimes_{\min}A$ is dual and the quotient algebra

$$[A\otimes_{\min}A+\mathbf{C}(I\otimes I)]/(A\otimes_{\min}A)=\mathbf{C},$$

there exists, by a similar argument, a sequence in $F(\tilde{A}\otimes_{\min}\tilde{A},\tilde{A})$ converging to $(\phi\,|\,A\otimes_{\min}A+\mathbf{C}(I\otimes I))\circ\psi$ in $\|.\|_{\text{cb}}$.

Consequently, there exists a sequence in $F(\tilde{A}\otimes_{\min}\tilde{A},\tilde{A})$ converging to $\phi=\phi_R\circ R_f+\phi_L\circ L_f+\phi\circ\psi$ in $\|.\|_{\text{cb}}$.

PROPOSITION 9. *Let A be a C^*-algebra with a closed two-sided ideal J such that A/J is finite dimensional and J is dual. If ϕ is compact, completely positive map of A into another C^*-algebra B, then there exists a sequence in $F(A,B)$ converging to ϕ in $\|.\|_{\text{cb}}$.*

Proof. Let π be the quotient map of A onto A/J and choose a finite set $\{u_i\}_{i=1}^n$ in A such that $\{\pi(u_i)\}_{i=1}^n$ is a basis for A/J. Using the set $\{f_i\}_{i=1}^n$ in A^* such that $\pi(x)=\sum_{i=1}^n f_i(x)\pi(u_i)$ for each x in A, we put $\psi_1(x)=\sum_{i=1}^n f_i(x)u_i$ and $\psi_2(x)=x-\psi_1(x)\in J$ for each x in A. Both ψ_1 and ψ_2 are completely bounded. Since the restriction $\phi\,|\,J$ is compact, there exists, by Proposition 7, a sequence in $F(A,B)$ converging to $\phi\circ\psi_2$ in $\|.\|_{\text{cb}}$. The map $\phi\circ\psi_1$ is of finite rank. Then there exists a sequence in $F(A,B)$ converging to $\phi=\phi\circ\psi_1+\phi\circ\psi_2$ in $\|.\|_{\text{cb}}$.

5. The projective C^*-tensor product. For C^*-algebras A and B, the least cross norm λ on their algebraic tensor product $A\odot B$ is defined by

$$\lambda\left(\sum_{i=1}^n x_i\otimes y_i\right)=\sup\left\{\left|\sum_{i=1}^n f(x_i)g(y_i)\right|: f\in A^*,\ g\in B^*,\ \|f\|=\|g\|=1\right\}.$$

The greatest cross norm γ on $A\odot B$ is defined by $\gamma(u)=\inf\sum_{i=1}^n\|x_i\|\|y_i\|$, where the inf is taken over all representations of u (see, for example, [9: Chapter IV, Section 2]). It is known that $\lambda\leqslant\min\leqslant\max\leqslant\gamma$ [9: Chapter IV, Sections 2, 4].

We can identify $(B\otimes M_m)^*$ with $B^*\otimes M_m$. For a bounded linear map ϕ of A into B^*, we say that ϕ is completely positive if each $\phi\otimes\text{id}_m$ of $A\otimes M_m$ into $B^*\otimes M_m$ is positive.

Let $B(A,B^*)$ and $B(B,A^*)$ denote the space of bounded linear maps of A into B^* and the space of bounded linear maps of B into A^*, respectively.

PROPOSITION 10. *With the above notation, let Φ denote the natural linear map of $A\otimes_\gamma B$ to $A\otimes_{\max}B$. The following statements are equivalent:*

(1) *Φ is an isomorphism between Banach spaces $A \otimes_\gamma B$ and $A \otimes_{\max} B$;*
(2) *$B(A, B^*)$ is the span of completely positive maps;*
(3) *$B(B, A^*)$ is the span of completely positive maps;*
(4) *either A or B is finite dimensional.*

Proof. (1)$\Leftrightarrow$(2), (1)$\Leftrightarrow$(3). The conjugate space $(A \otimes_\gamma B)^*$ can be identified, by a well-known process (see [6: Chapter IV, Theorem 2.3]), with $B(A, B^*)$ (or $B(B, A^*)$). The conjugate space $(A \otimes_{\max} B)^*$ can be identified, by [6: Lemma 3.2], with the span of completely positive maps of A into B^* (or the span of completely positive maps of B into A^*). The adjoint map Φ^* is an isomorphism between $(A \otimes_\gamma B)^*$ and $\Phi^*((A \otimes_{\max} B)^*)$. Then it is easy to check that (1)$\Leftrightarrow$(2) and (1)$\Leftrightarrow$(3).

(1)$\Leftrightarrow$(4). Since (4)$\Rightarrow$(1) is obvious, we assume that (1) holds. Let ϕ be a bounded linear map of A into itself. There exists the bounded linear map $\phi \otimes_\gamma \mathrm{id}_B : A \otimes_\gamma B \to A \otimes_\gamma B$. By (1) we can define the map

$$\phi \otimes_{\max} \mathrm{id}_B : A \otimes_{\max} B \to A \otimes_{\max} B.$$

For f in A^*, g in B^* and a in $A \otimes_{\max} B$, we have $(f \otimes_{\max} g) \circ (\phi \otimes_{\max} \mathrm{id}_B)(a) = (f \circ \phi) \otimes_{\max} g(a)$. An element a in $A \otimes_{\max} B$ belongs to the kernel of the natural map $\Psi : A \otimes_{\max} B \to A \otimes_{\min} B$ if and only if $f \otimes_{\max} g(a) = 0$ for all f in A^* and g in B^*. Hence if $a \in \ker \Psi$, then $\phi \otimes_{\max} \mathrm{id}_B(a) = 0$. We can then define the bounded linear map $\phi \otimes \mathrm{id}_B : A \otimes_{\min} B \to A \otimes_{\min} B$.

Suppose that B is not subhomogeneous. Let m be a positive integer. By [5: Lemma 2] we have a completely positive contraction ψ of B into the $m \times m$ matrix algebra M_m and a sequence $\{\phi_n\}$ of completely positive contractions of M_m into B such that $\{\psi \circ \phi_n\}$ converges to the identity map id_m on M_m in the operator norm. Then $\lim_n (\mathrm{id}_A \otimes \psi) \circ (\phi \otimes \mathrm{id}_B) \circ (\mathrm{id}_A \otimes \phi_n)(a) = \phi \otimes \mathrm{id}_m(a)$ for all a in $A \otimes M_m$. Since $\|(\mathrm{id}_A \otimes \psi) \circ (\phi \otimes \mathrm{id}_B) \circ (\mathrm{id}_A \otimes \phi_n)\| \leqslant \|\phi \otimes \mathrm{id}_B\|$, we have $\|\phi \otimes \mathrm{id}_m\| \leqslant \|\phi \otimes \mathrm{id}_B\|$, so that $\|\phi\|_{\mathrm{cb}} \leqslant \|\phi \otimes \mathrm{id}_B\|$. Hence every bounded linear map of A into itself is completely bounded. Smith's result [8: Theorem 2.8] shows that A is subhomogeneous. Hence either A or B is subhomogeneous.

There exists, by [12: Proposition 1], the natural isomorphism between $A \otimes_{\min} B$ and $A \otimes_\lambda B$. We also have $A \otimes_{\max} B = A \otimes_{\min} B$. Let Φ_λ be the natural map $: A \otimes_\gamma B \to A \otimes_\lambda B$. Then $\Phi_\lambda(A \otimes_\gamma B) = A \otimes_\lambda B$. If both A and B are infinite dimensional, there exist infinite locally compact Hausdorff spaces S and T such that $C_0(S) \subseteq A$ and $C_0(T) \subseteq B$. Grothendieck's result [3: Proposition 33] implies that $\Phi_\lambda(A \otimes_\gamma B) \subsetneq A \otimes_\lambda B$. This is a contradiction. Hence either A or B is finite dimensional.

REFERENCES

1. J. De Cannière and U. Haagerup, *Multipliers of the Fourier algebras on some simple Lie groups and their discrete subgroups,* preprint.
2. M.-D. Choi and E. G. Effros, *Nuclear C^*-algebras and the approximation property,* Amer. J. Math. 100 (1978), 61–79.

3. A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires,* Mem. Amer. Math. Soc. 16 (1955).
4. T. Huruya, *Decompositions of completely bounded maps,* preprint.
5. T. Huruya and J. Tomiyama, *Completely bounded maps of C^*-algebras,* J. Operator Theory, to appear.
6. C. Lance, *On nuclear C^*-algebras,* J. Funct. Anal. 12 (1973), 157–176.
7. V. I. Paulsen, *Completely bounded maps on C^*-algebras and invariant operator ranges,* Proc. Amer. Math. Soc. 86 (1982), 91–96.
8. R. R. Smith, *Completely bounded maps between C^*-algebras,* preprint.
9. M. Takesaki, *Theory of operator algebras I,* Springer, New York, 1979.
10. J. Tomiyama, *Applications of Fubini type theorem to the tensor products of C^*-algebras,* Tôhoku Math. J. (2) 19 (1967), 213–226.
11. M. E. Walter, *Dual and dual space algebras,* preprint.
12. S. Wassermann, *The slice map problem for C^*-algebras,* Proc. London Math. Soc. (3), 32 (1976), 537–559.
13. G. Wittstock, *Ein operatorwertiger Hahn-Banach Satz,* J. Funct. Anal. 40 (1981), 127–150.

Faculty of Education
Niigata University
Niigata 950-21, Japan