A REMARK ON QUASI-CONFORMAL MAPPINGS AND BMO-FUNCTIONS

Kari Astala

Let $G \subset \mathbf{R}^n$ $(n \geqslant 2)$ be a domain and let $u: G \to \mathbf{R}$ be a locally integrable function. We say that u has *bounded mean oscillation in G*, and denote $u \in \mathrm{BMO}(G)$, if

$$\|u\|_{*,G} \equiv \sup_{B \subset G} \left[\frac{1}{m(B)} \int_B |u(x) - u_B| \, dx \right] < \infty.$$

Here the supremum is taken over all balls $B \subset G$; $m(B)$ stands for the Lebesgue-measure of B and u_B for the mean value of u over B, i.e.

$$u_B = \frac{1}{m(B)} \int_B u(x) \, dx.$$

H. M. Reimann [5] has established a close connection between quasi-conformal mappings and the spaces $\mathrm{BMO}(G)$ by proving the following theorems:

1. THEOREM ([5: Theorem 4]; see also [4: p. 58]). *If $f: G \to G'$ is a K-quasi-conformal homeomorphism, then*

$$\frac{1}{C} \|u\|_{*,G'} \leqslant \|u \circ f\|_{*,G} \leqslant C \|u\|_{*,G'} \tag{1}$$

for all functions $u \in \mathrm{BMO}(G')$. The constant C in (1) *depends only on K and the dimension n.*

2. THEOREM ([5: Theorem 3]). *If an orientation preserving homeomorphism $f: G \to G'$ has the properties*

(a) *f is differentiable* a.e. *and $f \in$* ACL,

(b) *the mapping $u \to u \circ f$ is a bijective isomorphism of the spaces* $\mathrm{BMO}(G')$ *and* $\mathrm{BMO}(G)$ *for which $\|u \circ f\|_{*,G} \leqslant C \|u\|_{*,G'}$,*

then f is quasi-conformal.

For definitions of quasi-conformal and ACL mappings see [8].

The purpose of this note is to show that by localizing the problem the analytic assumptions (a) in Theorem 2 can be dropped. More precisely, we shall prove

3. THEOREM. *Let $f: G \to G'$ be an orientation preserving homeomorphism. If there exists a constant C such that*

$$\frac{1}{C} \|u\|_{*,D'} \leqslant \|u \circ f\|_{*,D} \leqslant C \|u\|_{*,D'} \tag{2}$$

Received February 1, 1983. Revision received March 17, 1983.
The research was done while the author was visiting The University of Michigan. The work was partially supported by the Finnish Cultural Foundation.
Michigan Math. J. 30 (1983).

holds for all subdomains $D \subset G$ *and for all* $u \in \mathrm{BMO}(D')$, $D' = fD$, *then* f *is quasi-conformal.*

NOTE. (2) implicitly assumes that $u \circ f \in L^1_{\mathrm{loc}}(D)$ whenever $u \in \mathrm{BMO}(D')$. However, we shall need (2) only for continuous functions u and for such u no integrability or measurability assumptions are needed.

Since quasi-conformality is a local property, Theorems 1 and 3 yield immediately

4. COROLLARY. *A homeomorphism* $f\colon G \to G'$ *is quasi-conformal if and only if* (2) *holds for some constant* C.

It remains open whether the non-localized version of Theorem 3 is true.

For Theorem 3 we need to recall the following notions and results.

We say that a (proper) subdomain D of $\mathbf{R}^n$ is *uniform* if there exist constants a and b such that each pair of points $x_1, x_2 \in D$ can be joined by a rectifiable arc $\gamma \subset D$ for which

$$\text{(3)}\qquad \begin{cases} l(\gamma) \leqslant a|x_1 - x_2| \\ \min\limits_{j=1,2} l(\gamma(x_j, x)) \leqslant b d(x, \partial D) \quad \text{for all } x \in \gamma. \end{cases}$$

Here $l(\gamma)$ denotes the length of γ, $\gamma(x_j, x)$ the part of γ between x_j and x, and $d(x, \partial D)$ the distance from x to the boundary ∂D of D.

For any domain D and pair of points $x_1, x_2 \in D$ set

$$k_D(x_1, x_2) = \inf_{\gamma} \int_{\gamma} d(x, \partial D)^{-1}\, ds$$

where the infimum is taken over all rectifiable arcs γ joining x_1 and x_2 in D. The mapping k_D is called the *quasi-hyperbolic metric* in D. Furthermore, set

$$j_D(x_1, x_2) = \frac{1}{2} \ln\left(\frac{|x_1 - x_2|}{d(x_1, \partial D)} + 1\right)\left(\frac{|x_1 - x_2|}{d(x_2, \partial D)} + 1\right).$$

F. Gehring and B. Osgood [3] have characterized the uniform domains with the help of k_D and j_D: A domain is uniform if and only if

$$\text{(4)}\qquad k_D(x, y) \leqslant c j_D(x, y) + d, \quad x, y \in D.$$

The uniformity constants of D depend only on the constants c and d in (4).

In the following, let $B_r(x)$ denote the ball with center x and radius r.

5. LEMMA. *Suppose* $u(x) = k_D(x, y)$, $y \in D$.

(a) *If* $B = B_r(z) \subset D$, $|u(z) - u_B| \leqslant n$

(b) $\|u\|_{*,D} \leqslant 2n$.

For the hyperbolic metric h_D of a simply connected domain $D \subset \mathbf{R}^2$ the proof of (a) and its corollary (b) is given in [2: Lemma 10.4]; the generalization to k_D and to an arbitrary domain $D \subset \mathbf{R}^n$ is immediate. See also [4: Lemma 2.4], where the original proof for $u \in \mathrm{BMO}(D)$ appears.

6. LEMMA ([4: Lemma 2.1] or [2: Lemma 10.2]). *If* $u \in \mathrm{BMO}(\mathbf{R}^n)$, $x, y \in D$ *and* $r = d(x, \partial D)$, $s = d(y, \partial D)$, *then*

$$|u_{B_r(x)} - u_{B_s(y)}| \leqslant (c_1 j_D(x,y) + d_1)\|u\|_{*,\mathbf{R}^n} \tag{5}$$

where $c_1 = e^n$ *and* $d_1 = 2e^n$.

Finally, let $B \subset \mathbf{R}^n$ be any ball and let $\mathcal{I}$ be the inversion in ∂B. If $u \in \mathrm{BMO}(B)$, then (see [6: p. 8]) the mapping

$$v(x) = \begin{cases} u(x), & x \in B \\ u \circ \mathcal{I}(x), & x \in \mathbf{R}^n \setminus B \end{cases}$$

belongs to $\mathrm{BMO}(\mathbf{R}^n)$ and $\|v\|_{*,\mathbf{R}^n} \leqslant M\|u\|_{*,B}$, $M = M(n)$.

Proof of Theorem 3. Let $x \in G$ be fixed, denote $B_r = B_r(x)$ and choose r_0 so small that

$$\operatorname{diam}(fB_r) < d(f(x), \partial G) \quad \text{for } r < r_0. \tag{6}$$

In [4] P. Jones proved that a domain $D \subset \mathbf{R}^n$ has the BMO-extension property if and only if (4) holds. We shall modify the necessity part of Jones' proof to show that fB_r is uniform for $r < r_0$.

First of all, if $D = fB_r$ and $y_1 \in D$, the mapping $u(y) = k_D(y, y_1)$ is by Lemma 5 in $\mathrm{BMO}(D)$ with $\|u\|_{*,D} \leqslant 2n$. Moreover, if $\alpha > 0$ and $u_\alpha(y) = \inf\{\alpha, k_D(y, y_1)\}$, $y \in D$, then $u_\alpha \in \mathrm{BMO}(D)$ by the lattice-property of the BMO spaces (cf. [6: p. 2]). We have also the estimate $\|u_\alpha\|_* \leqslant 2\|u\|_* \leqslant 4n$.

Next we apply (2) and get $\|u_\alpha \circ f\|_{*,B_r} \leqslant 4nC$. But B_r is a ball and thus $u_\alpha \circ f$ has a bounded extension v_α with $\|v_\alpha\|_{*,\mathbf{R}^n} \leqslant 4nCM$. Since v_α is bounded we may use (2) again to get $\|v_\alpha \circ f^{-1}\|_{*,G'} \leqslant 4nC^2M$.

We have chosen r so small that the ball B with center $f(x)$ and radius $\operatorname{diam}(D)$ is contained in $G' = fG$. Therefore the restriction of $v_\alpha \circ f^{-1}$ to B has an extension $w_\alpha \in \mathrm{BMO}(\mathbf{R}^n)$ such that $w_\alpha|_D = u_\alpha$ and

$$\|w_\alpha\|_{*,\mathbf{R}^n} \leqslant M\|v_\alpha \circ f^{-1}\|_{*,G'} \leqslant 4nC^2M^2.$$

Now we can use Lemma 6 which gives two absolute constants c_1 and d_1 such that

$$\begin{aligned} |(w_\alpha)_{B_1} - (w_\alpha)_{B_2}| &\leqslant (c_1 j_D(y_1, y_2) + d_1)\|w_\alpha\|_{*,\mathbf{R}^n} \\ &\leqslant c_2 j_D(y_1, y_2) + d_2 \end{aligned} \tag{7}$$

whenever $y_2 \in D$; here B_i denotes the ball with center y_i and radius $d(y_i, \partial D)$, $i = 1, 2$, and $c_2 = 4nC^2M^2e^n$, $d_2 = 8nC^2M^2e^n$.

Note that the right hand side of the inequality (7) does not depend on α. Hence by letting α go to infinity in (7) we find that $|u_{B_1} - u_{B_2}| \leqslant c_2 j_D(y_1, y_2) + d_2$ holds for $u(y) = k_D(y, y_1)$. Finally, since $B_i \subset D$, $i = 1, 2$, Lemma 5(a) implies

$$k_D(y_1, y_2) = |u(y_1) - u(y_2)| \leqslant c_2 j_D(y_1, y_2) + d_2 + 2n.$$

The considerations above prove that $D = fB_r$ is uniform for every $r < r_0$, with uniformity constants a, b independent of r or x (cf. (4)). Thus if $y \in B_r$ is such a point that $\operatorname{diam}(fB_r) < 3|f(x)-f(y)|$, there exists an arc $\gamma \subset fB_r$ (joining $f(x)$ and $f(y)$) with the properties (3). In particular, if $z \in \gamma$ satisfies $l(\gamma(f(x), z)) = l(\gamma(f(y), z))$, we have

$$d(z, \partial fB_r) \geqslant \frac{1}{b} l(\gamma(f(x), z)) = \frac{1}{2b} l(\gamma(f(x), f(y)))$$
$$\geqslant \frac{1}{2b} |f(x)-f(y)| > \frac{1}{6b} \operatorname{diam}(fB_r).$$

Therefore the ball with center z and radius $\operatorname{diam}(fB_r)/6b$ is contained in fB_r. Consequently

$$[\operatorname{diam}(fB_r)]^n \leqslant \frac{(6b)^n}{\Omega_n} m(fB_r), \quad r < r_0.$$

In brief, we have shown that there exists a constant H depending only on n and C such that

$$\limsup_{r \to 0} \frac{[\operatorname{diam}(fB_r)]^n}{m(fB_r)} \leqslant H \quad \text{for all } x \in G. \tag{8}$$

But by a well known argument (8) implies the quasi-conformality of f. (For details, cf. [1: Theorem 2, p. 94], or [7: Theorems 6.11 and 6.12].)

ACKNOWLEDGEMENTS. The author is grateful to Professor F. W. Gehring for suggesting the problem and for valuable discussions concerning it.

REFERENCES

1. P. Caraman, *n-dimensional quasi-conformal (qcf) mappings,* Editura Academiei Române, Bucharest, 1974.
2. F. W. Gehring, *Characteristic properties of quasidisks,* Séminaire de Mathématiques Supérieures, 84, Presses de l'Université de Montréal, Montreal, Que., 1982.
3. F. W. Gehring and B. G. Osgood, *Uniform domains and the quasi-hyperbolic metric,* J. Analyse Math. 36 (1979), 50–74.
4. P. W. Jones, *Extension theorems for* BMO, Indiana Univ. Math. J. 29 (1980), 41–66.
5. H. M. Reimann, *Functions of bounded mean oscillation and quasiconformal mappings,* Comment. Math. Helv. 49 (1974), 260–276.
6. H. M. Reimann and T. Rychener, *Funktionen beschränkter mittlerer Oszillation,* Lecture Notes in Math., 487, Springer, Berlin, 1975.
7. J. Väisälä, *On quasiconformal mappings in space,* Ann. Acad. Sci. Fenn. A I No. 298 (1961), 1–36.
8. ———, *Lectures on n-dimensional quasiconformal mappings,* Lecture Notes in Math., 229, Springer, Berlin, 1971.

Department of Mathematics
University of Jyväskylä
SF-40100 Jyväskylä 10
Finland