ON TUBULAR NEIGHBORHOODS OF FIXED POINTS OF LOCALLY SMOOTH ACTIONS

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1. Introduction. G. E. Bredon, in his book [1], defines a locally smooth action of a compact Lie group on a topological space. Locally smooth actions of compact Lie groups on manifolds form a class of actions lying between that of topological actions and that of smooth actions.

An action of a compact Lie group G on a topological space M is said to be locally smooth if there exists a linear tube about each orbit. A linear tube is defined as follows: Let H be a closed subgroup of G and $G(x)=\{g\cdot x \mid g\in G\}$ be an orbit of type G/H, and let V be a Euclidean space on which H acts orthogonally. Then a linear tube about $G(x)$ in M is a G-equivariant imbedding $\phi: G\times_H V\to M$ such that $\phi(G\times_H V)$ is an open neighborhood of $G(x)$ in M, where $G\times_H V$ is the twisted product of G and V. The twisted product is defined to be the orbit space of the action H on $G\times V$ given by $h\cdot(g,v)=(g\cdot h^{-1}, h\cdot v)$ for $h\in H$, and $(g,v)\in G\times V$. If G acts locally smoothly on M, then M is a topological manifold and any connected component of the fixed point set $F(G,M)=\{x\in M \mid g\cdot x=x$ for all $g\in G\}$ is a topological submanifold of M. It is well known that there is a linear tube about each orbit of any smooth action. Thus a smooth action of a compact Lie group is locally smooth [1]. The following facts about locally smooth actions are shown in [1: pp. 179–185].

If a compact Lie group G acts locally smoothly on the m-manifold M with the orbit space M/G connected, then there exists a maximum orbit type G/H for G in M, i.e., H is conjugate to a subgroup of the isotropy group $G_x=\{g\in G \mid g\cdot x=x\}$ of G at each $x\in M$. The orbits of this type are called the principal orbits. The union $M_{(H)}$ of the orbits of maximum orbit type G/H is open and dense in M and its image $M_{(H)}/G$ in M/G is also connected. Let V be a linear slice at x ($x\in V\subset M$, $G_x(V)=V$). Then the orbit $G(x)$ is principal if and only if G_x acts trivially on V and $G\times_{G_x} V=(G/G_x)\times V$. Let $G(x)$ be an exceptional orbit (i.e., $\dim G(x)=\dim$ of a principal orbit, but they are not equivalent), and let V be a linear slice at the point x. If $H\subset G_x$ is a principal isotropy group for G_x on V, then H is just the ineffective part of G_x on V. Therefore G_x/H is a finite group acting effectively on the slice V. If $F(G_x, V)$ has codimension one in V, i.e., $G(x)$ is a special exceptional orbit, then G_x/H has order two and acts by reflection across the hyperplane $F(G_x, V)$ of V.

The purpose of this paper is to study an invariant tubular neighborhood of the fixed point set $F(G,M)$ of a locally smooth action of a compact Lie group G on a manifold M. An open (or closed) invariant tubular neighborhood of $F(G,M)$ in M is a normal bundle $p: E\to F(G,M)$ such that E is a G-invariant open (or closed) neighborhood of $F(G,M)$ in M, and the action of G on each fiber is

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equivalent to an orthogonal action on the Euclidean k-space $\mathbf{R}^k$, where $k = \dim M - \dim F(G,M)$. We note that if a compact Lie group G acts smoothly on a manifold M and if F is any connected component of the fixed point set $F(G,M)$, then there exists an open (and a closed) invariant tubular neighborhood of F in M [1]. However, F. Raymond found that if the action is only topological, then F does not necessarily have an invariant closed tubular neighborhood in M [9]. In case there is an invariant closed tubular neighborhood of the fixed point set in a manifold M, by deleting the interior of the tubular neighborhood we obtain an action on a manifold with boundary having no fixed points.

We prove that an invariant tubular neighborhood can be found for many cases when the codimension of $F(G,M)$ in M is bigger by 1, 2 or 3 than the dimension of a principal orbit. We also find an example of a locally smooth action of a compact Lie group on a manifold for which the fixed point set does not have an invariant tubular neighborhood. We use the fact that the 4-sphere S^4 can be imbedded in a 7-manifold M^7 having no topological closed tube [5]. Finally, we thank F. Raymond for his suggestion of the tubular neighborhood problem to us.

2. Invariant tubular neighborhoods. We assume that a compact Lie group G acts effectively and locally smoothly on a topological m-manifold M with the fixed point set $F(G,M)$. Let F be a connected component of $F(G,M)$ of dimension n, and k be the codimension of F in M, i.e., $k = m - n$. Let d denote the dimension of a principal orbit of G on M, and $\pi : M \to M/G$ be the natural orbit map taking x into its orbit $G(x)$ for each $x \in M$. By the definition of a locally smooth action, F is covered by $\{V_\lambda\}_{\lambda \in \Lambda}$, a collection of open subsets of F such that: (1) V_λ is homeomorphic to the Euclidean n-space $\mathbf{R}^n$ for each $\lambda \in \Lambda$; (2) for each $\lambda \in \Lambda$, V_λ has a neighborhood N_λ in M, which is equivalent to $V_\lambda \times \mathbf{R}^k$ and V_λ is corresponding to $V_\lambda \times \{0\}$ under this equivalence, thus the submanifold F is locally flat in M; and (3) the action of G on $V_\lambda \times \mathbf{R}^k$ induced by this equivalence is the product of the trivial action on V_λ with an orthogonal action on $\mathbf{R}^k$. Therefore, there is an induced orthogonal action of G on $\mathbf{R}^k$ with the fixed point set $F(G, \mathbf{R}^k) = \{0\}$ and the dimension of a principal orbit of G on $\mathbf{R}^k$ equals the dimension of a principal orbit of G on M [1].

THEOREM 1. *Let a compact Lie group G act effectively and locally smoothly on a topological m-manifold M with the fixed point set F(G,M). Let F be a connected component of F(G,M) of dimension n, and k be the codimension of F in M. Let d denote the dimension of a principal orbit of G on M. Then F has an open or closed invariant tubular neighborhood in M provided that any one of the following conditions is satisfied:*

(1) $n = 0$.
(2) $d = k - 1$.
(3) $k = 1$.
(4) $k = 2$ *and* $n \neq 2$.
(5) $k \geq 3$ *and* $d = k - 2$.
(6) $k > 4$ *and* $d = k - 3$ *and* $n \neq 2$ *and G is connected.*

Proof. (1) $n=0$. By the definition of a locally smooth action, a single point set F has a neighborhood $N=\mathbf{R}^m$ such that the action of G on N is equivalent to an orthogonal action on $\mathbf{R}^m$. Then the projection $p: N \to F$ is trivially an invariant tubular neighborhood of F in M.

(2) $d=k-1$. We know that for each $\lambda \in \Lambda$, the orbit space of G on N_λ is $N_\lambda/G \cong N_\lambda \times (\mathbf{R}^k/G)$. Since G acts orthogonally on $\mathbf{R}^k$ with $d=k-1$, $\mathbf{R}^k/G=\mathbf{R}_+=\{r \in \mathbf{R}, r \geqslant 0\}$. Therefore we have $N_\lambda/G \cong V_\lambda \times \mathbf{R}_+$. This implies that the orbit space M/G is a manifold with boundary $F(G, M)$ and F is locally collared in M/G. Therefore F has a collar $C \cong F \times [0,1]$ in M/G [2]. Let $i: F \times [0,1] \to C$ be the homeomorphism. Then the map $p: \pi^{-1}(C) \to F$, given by $p(\pi^{-1}(x))=f$ for all $x=i(f, r) \in C$, is an invariant tubular neighborhood of F in M.

(3) $k=1$. In this case, the group G acts orthogonally on $\mathbf{R}^1$ with the fixed point set $\{0\}$. Therefore, $G \cong \mathbf{Z}_2$ and $d=0=k-1$. Hence F has an invariant tubular neighborhood in M by case (2).

(4) $k=2$ and $n \neq 2$. The Lie group G acts orthogonally on $\mathbf{R}^2$ with fixed point set $\{0\}$. Therefore, the dimension d of a principal orbit is less than or equal to one. If $d=1$, then we are done by case (2). If $d=0$, there are two cases to consider: (a) G acts freely on $\mathbf{R}^2-\{0\}$, or (b) G does not act freely on $\mathbf{R}^2-\{0\}$. If G acts freely on $\mathbf{R}^2-\{0\}$, then $G \cong G(x)$ for all $x \in \mathbf{R}^2-\{0\}$. Therefore, the compact Lie group G must be finite. Furthermore, $\mathbf{R}^2/G \cong \mathbf{R}^2$. Thus $N_\lambda/G \cong V_\lambda \times (\mathbf{R}^2/G)=V_\lambda \times \mathbf{R}^2$. Therefore, the orbit space M/G is again a manifold, and the submanifold F is locally flat, and it is of codimension 2 in the orbit space M/G. Then F has a normal microbundle in M/G provided $n \neq 2$ according to Kirby and Siebenmann [7]. We know that a microbundle is a fiber bundle by Kister [8]. Let $q: E \to F$ be this normal bundle of F in M/G. Then $p=q \cdot \pi: \pi^{-1}(E) \to F$ is an invariant tubular neighborhood of F in M. We consider now the case when G does not act freely on $\mathbf{R}^2-\{0\}$, that is, $G \not\cong \mathbf{Z}_l$ for any integer l. (In this case the condition $n \neq 2$ is not necessary.) Let x be any point in $\mathbf{R}^2-\{0\}$ such that $G_x \neq e$. Then $F(G_x, \mathbf{R}^2) \neq \{0\}$, and it is a submanifold of $\mathbf{R}^2$. Therefore, $F(G_x, \mathbf{R}^2) \cong \mathbf{R}^1$. Therefore, $G(x)$ is a special exceptional orbit in $\mathbf{R}^2$ and $G_x \cong \mathbf{Z}_2$ and acts on $\mathbf{R}^2$ by reflection across the hyperplane $F(G_x, \mathbf{R}^2)$ in $\mathbf{R}^2$. This implies that the restricted action of G on $S^1 \subset \mathbf{R}^2$ has orbit space homeomorphic to $[0,1]$, i.e., $S^1/G \cong [0,1]$. Since we may regard $\mathbf{R}^2$ as the open cone of S^1, $C^0(S^1)$, $\mathbf{R}^2/G \cong C^0([0,1]) \cong \mathbf{R}^2_+$. Therefore $N_\lambda/G \cong V_\lambda \times (\mathbf{R}^2/G)=V_\lambda \times C^0([0,1])$. This implies that M/G is a manifold with boundary $\partial(M/G)$. We note that

$$\dim(M/G)=\dim M \quad \text{and} \quad \dim F=\dim(M/G)-2=\dim(\partial(M/G))-1.$$

Thus F is a locally flat codimension one submanifold of $\partial(M/G)$. Therefore, by Brown [2], F has a collar E in $\partial(M/G)$, i.e., $E \cong F \times [0,1]$. Since $E \subset \partial(M/G)$ and $\partial(M/G)$ has a collar in M/G, E has a collar C in M/G, where $C \cong E \times [0,1]$. Then the composite map

$$p: \pi^{-1}(C) \xrightarrow{\pi} C \cong E \times [0,1] \xrightarrow{\pi_1} E \cong F \times [0,1] \xrightarrow{\pi_1} F$$

is an invariant tubular neighborhood of F in M.

(5) $k \geqslant 3$ and $d=k-2$. Since G acts orthogonally on $\mathbf{R}^k$ with fixed point set $\{0\}$, and $d=k-2$, G acts smoothly on S^{k-1} without fixed point. We consider the action of G on S^{k-1}: (a) Suppose every orbit is principal, then, by [1: p. 198], G acts freely on S^{k-1} and G is either S^1 or S^3, or the normalizer $N(S^1)$ of S^1 in S^3. Therefore $d=\dim G$. Since the only group that acts freely on the even dimensional spheres is $\mathbf{Z}_2$ [1: p. 148], $k-1$ is not an even integer. Therefore $k-1$ is an odd integer and $d=k-2$ is an even integer. This implies that $G \neq S^1$ and $G \neq S^3$. Furthermore, $N(S^1)/S^1$ is finite since S^1 is the maximal torus of S^3 [1: p. 26]. This implies that $\dim N(S^1)$ is also one and $G \neq N(S^1)$. This leads to a contradiction. (b) We know that not every orbit is principal by (a). We also know that $S^{k-1}/G \cong [0,1]$ by [1: p. 206]. Therefore $\mathbf{R}^k/G \cong C^0([0,1])$, and

$$N_\lambda/G \cong V_\lambda \times C^0([0,1]).$$

Thus $(\cup N_\lambda)_{\lambda \in \Lambda}/G$ is a manifold with boundary, and F is locally flat, and it is of codimension one in $\partial((\cup N_\lambda)/G)$. Thus, we can show that F has an invariant tubular neighborhood in M by a similar proof given in (4)(b).

(6) $k>4$, $d=k-3$, $n \neq 2$, and G is connected. In this case, G acts smoothly on S^{k-1} without fixed point and $d=k-3$. Since G is also connected, there exists a singular orbit on S^{k-1} [1: p. 216; or Conner]. Therefore, S^{k-1}/G is a closed 2-disk. Let $B=\{x \in M \mid G(x) \text{ is a singular orbit}\}$. Thus $B \neq \varnothing$, and $F \subsetneq B$. Furthermore, $N_\lambda/G \cong V_\lambda \times C^0(D^2)$ with $(N_\lambda \cap B)/G \cong C^0(\partial D^2)$. Therefore F is locally flat and is of codimension two in B/G. Hence, F has an (open) tube $g_1: C_1 \to F$ in B/G if $n \neq 2$ by [7]. Since $B/G \subset \partial(M/G)$, C_1 has a collar $g_2: C_2 \xrightarrow{\cong} C_1 \times [0,1]$ in M/G. Then the composite

$$p=g_1 \cdot \pi_1 \cdot g_2 \cdot \pi : \pi^{-1}(C_2) \xrightarrow{\pi} C_2 \xrightarrow{g_2} C_1 \times [0,1] \xrightarrow{\pi_1} C_1 \xrightarrow{g_1} F$$

is a tubular neighborhood of F in M. □

3. An example. Let M be a manifold and A a submanifold. An open (closed) tube for A in M is a bundle $p: E \to A$ such that $E \subset M$ is a neighborhood of A, p is a retraction, and the fibers are open (closed) k-cells, where $k=\dim M - \dim A$. In the smooth category, every submanifold has a closed tube, and hence an open tube. In the piecewise linear category, it is known that such a tube exists if the codimension k is sufficiently large. Using a framed non-trivial Haefliger knot, Kirsch [5] found that there exists a piecewise linear submanifold S^4 in M^7 having no topological closed tube. The construction of the pair $S^4 \subset M^7$ is as follows: A Haefliger knot is an oriented smooth submanifold $T^3 \subset S^6$ which is diffeomorphic to the three sphere S^3. A Haefliger knot is trivial if it is diffeotopic to the standard imbedding of $S^3 \subset S^6$. A framed Haefliger knot is a pair (T^3, f), where T^3 is a Haefliger knot and $f: T^3 \times D^3 \to S^6$ is a framing of its normal bundle. That is, f is a smooth imbedding such that $f(x,0)=x$ for all $x \in T^3$. It is known that there exists a non-trivial Haefliger knot and every Haefliger knot can be framed ([3], [4]). Let $T^3 \subset S^6$ be a non-trivial Haefliger knot and f be any framing of T^3. Let $M^7=M^7(T^3, f)$ be the smooth 7-manifold obtained by attaching the handle $D^4 \times D^3$ to D^7 by $f': S^3 \times D^3 \to S^6 = \partial D^7$, where f' corresponds to f

via an orientation-preserving diffeomorphism of T^3 and S^3. Let $S^4 \subset M^7$ be a union $C(T^3) \cup (D^4 \times \{0\})$, where $C(T^3) \subset D^7$ is the cone on T^3 and $D^4 \times \{0\}$ is the core of the handle $D^4 \times D^3$.

THEOREM (Hirsch [5]). *Let $T^3 \subset S^6$ be a non-trivial Haefliger knot and let f be any framing of T^3. Then the 4-sphere S^4 in $M^7(T^3, f)$ has no topological closed tube.*

In this section we construct a locally smooth action of the circle group S^1 on a manifold for which the existence of an invariant tubular neighborhood of the fixed point set violates the above theorem of Hirsch. First, we observe the following:

LEMMA 1. *Let B be any subset of $\mathbf{R}^n$ and $E = B \times D_0^k$, where D_0^k denotes the open unit disk in $\mathbf{R}^k$. Let U be any open neighborhood of $B \times \{0\}$ in E. Then $B \times \{0\}$ has a trivial normal bundle in U and each fiber over $\{b \times 0\}$ is contained in $\{b\} \times D_0^k$ for all $b \in B$.*

Proof. We assume that $E \neq U$. For any two points (x, t), $(x', t') \in E \subseteq \mathbf{R}^{n+k}$, let $d((x, t), (x', t'))$ denote the distance between (x, t) and (x', t') in $\mathbf{R}^{n+k}$. We define a map $g: B \to [0, 1]$ by $g(b) = \min\{1, d((b, 0), E - U)\}$. Then g is a continuous positive real valued function on B and $f(b) \leq 1$ for all $b \in B$. Then we define a map $h: B \times D_0^k \to U$ by $h(b, t) = (b, g(b)t)$. This induces the required normal bundle of $B \times \{0\}$. □

We also observe that the imbedding $S^4 \subset M^7$, in Hirsch's theorem, is locally flat.

LEMMA 2. *The imbedding $S^4 \subset M^7(T^3, f)$ is locally flat.*

Proof. Let x be the vertex of the cone $C(T^3)$, and

$$V_1 = S^4 - \{x\}, \quad V_2 = \mathrm{Int}(C(T^3)).$$

Then S^4 is covered by $\{V_1, V_2\}$ and $V_1 \cong \mathbf{R}^4 \cong V_2$. It is clear that

$$E_1 = (C(f(T^3 \times D_0^3)) - \{x\}) \cup (D^4 \times D_0^3) = (S^4 - \{x\}) \times D_0^3 \xrightarrow{\pi_1} S^4 - \{x\}$$

is a trivial normal bundle of V_1 in M^7, where D_0^3 also denotes the open unit disk in $\mathbf{R}^3$. Since the submanifold $T^3 \subset S^6$ is topologically unknotted [6: p. 201], $(C(T^3), C(S^6)) \cong (C(T^3), D^7)$ is homeomorphic to (D^4, D^7). Therefore $V_2 = \mathrm{Int}\, C(T^3)$ has a trivial normal bundle E_2 in M^7. □

We denote any point $(x, y) \in \mathbf{R}^2 \times \mathbf{R}^2 = \mathbf{R}^4$ by $x = (a \cos\alpha, a \sin\alpha)$, $y = (b \cos\beta, b \sin\beta)$ for some real numbers a, b, α and β. Then we define an action of the circle group S^1 on $\mathbf{R}^4$ by $\theta \cdot (x, y) = (\theta \cdot x, \theta \cdot y)$, where

$$\theta \cdot x = (a \cos(\alpha + \theta), a \sin(\alpha + \theta)) \quad \text{and} \quad \theta \cdot y = (b \cos(\beta + \theta), b \sin(\beta + \theta))$$

for any element $\theta \in S^1$. Then S^1 acts orthogonally on $\mathbf{R}^4$ and the unit sphere S^3 in $\mathbf{R}^4$ is invariant under the action. Therefore, we may consider the S^1-action on S^3. Since S^1 acts freely on S^3, every orbit in S^3 is principal, and the orbit space S^3/S^1

is a connected 2-manifold without boundary. Furthermore, the induced map $\pi_*: \pi_1(S^3) \to \pi_1(S^3/S^1)$ of the natural orbit map $\pi: S^3 \to S^3/S^1$ is surjective. This implies that S^3/S^1 is simply connected, and hence $S^3/S^1 \cong S^2$. The action defined is the standard cone over the Hopf action. Therefore, this action of S^1 and S^3 is classified by the generator c of $\mathbf{Z}=H^2(S^2;\mathbf{Z})$. Since the S^1-action on $\mathbf{R}^4-\{0\}=S^3\times\mathbf{R}$ is equivalent to the product of the S^1-action on S^3 with the trivial action on $\mathbf{R}$, we see that $\mathbf{R}^4-\{0\}/S^1 \cong (S^3\times\mathbf{R})/S^1 \cong S^2\times\mathbf{R}\cong\mathbf{R}^3-\{0\}$, and see that this S^1-action on $\mathbf{R}^4-\{0\}$ is also classified by the same element $c\in\mathbf{Z}\cong H^2(\mathbf{R}^3-\{0\};\mathbf{Z})\cong H^2(S^2;\mathbf{Z})$.

Let N_1, N_2 denote the product spaces $V_1\times\mathbf{R}^4$ and $V_2\times\mathbf{R}^4$, respectively, where V_1 and V_2 is the open covering of $S^4\subset M^7$ which is defined in Lemma 2. Let the action of S^1 on N_i, $i=1,2$, be equivalent to the trivial action on V_i and the S^1 action defined above on $\mathbf{R}^4$. Therefore, the S^1 action on $N_i-F(S^1,N_i)\cong V_i\times(\mathbf{R}^4-\{0\})$ is also classified by the same element

$$c\in\mathbf{Z}\cong H^2((V_i\times(\mathbf{R}^4-\{0\}))/S^1;\mathbf{Z})\cong H^2(V_i\times(\mathbf{R}^3-\{0\});\mathbf{Z})$$
$$\cong H^2(\mathbf{R}^4\times(\mathbf{R}^3-\{0\});\mathbf{Z})\cong H^2(S^2;\mathbf{Z}).$$

Since $N_i/S^1\cong V_i\times\mathbf{R}^3$, we can identify the orbit space N_i/S^1 with the trivial normal bundle E_i of V_i in M^7 (see Lemma 2), i.e., $N_i/S^1\cong V_i\times\mathbf{R}^3=E_i$, $i=1,2$. Therefore, we have $(N_i-F(S^1,N_i))/S^1\cong V_i\times(\mathbf{R}^3-\{0\})=E_i-V_i$, $i=1,2$. Let $q_i: N_i\to E_i$ be the natural orbit map. Since V_1-V_2 and $V_2-V_1=\{x\}$ are two disjoint closed subsets in M^7, and M^7 is a normal space, there exist disjoint open subsets U_1 and U_2 in M^7 such that $V_1-V_2\subset U_1\subset E_1$, $V_2-U_1\subset U_2\subset E_2$. We also know that $V_1\cap V_2$ is homeomorphic to $S^3\times\mathbf{R}$. By Lemma 1, V_1-V_2 has a trivial normal bundle $E_1'\cong(V_1-V_2)\times\mathbf{R}^3$ in U_1, and V_2-V_1 has a trivial normal bundle $E_2'\cong(V_2-V_1)\times\mathbf{R}^3$ in $(V_2-U_1)\times\mathbf{R}^3$, and $V_1\cap V_2$ has a trivial normal bundle $E\cong(V_1\cap V_2)\times\mathbf{R}^3$ in $E_1\cap E_2$.

REMARK. For $i=1,2$, the S^1-action on $q_i^{-1}(E-V_1\cap V_2)$ is classified by the element $c\in\mathbf{Z}=H^2(E-V_1\cap V_2;\mathbf{Z})$, which classifies the S^1-action on S^3 since in both instances the action comes from the restrictions of the Hopf constructions.

Now let $N_1'=q_1^{-1}(E\cup E_1')$, $N_2'=q_2^{-1}(E\cup E_2')$. We know that there is an equivalence $q_1^{-1}(E-V_1\cap V_2)\to q_2^{-1}(E-V_1\cap V_2)$ by the above remark. Hence there is an equivalence $h: q_1^{-1}(E)\to q_2^{-1}(E)$ such that $q_1=q_2h$. Let X be the space obtained from the disjoint union of N_1' and N_2' identifying x with $h(x)$ for each $x\in q_1^{-1}(E)$, i.e., $X=N_1'\cup N_2'/x\sim h(x)$. Then S^1 acts effectively and locally smoothly on X with the fixed point set $F(S^1,X)=V_1\cup V_2=S^4$. We note that X is an 8-manifold and S^4 is a submanifold of X. Furthermore, the orbit space $X/S^1=E_1'\cup E\cup E_2'$ is a neighborhood of S^4 in $M^7(T^3,f)$. If $F(S^1,X)$ has a closed invariant tubular neighborhood in X, then S^4 would have a closed tube in M^7 induced by the orbit map. This contradicts the theorem of Hirsch. Thus we have the following theorem:

THEOREM 2. *There is an* 8-*manifold M on which the circle group S^1 acts effectively and locally smoothly with the fixed point set S^4 and the fixed point set has no invariant closed tubular neighborhood.*

Finally, we note that the above example is excluded in Theorem 1 for dimensional reasons.

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