WHITEHEAD TORSION AND THE SMITH CONJECTURE

Richard Hartley

The purpose of this paper is to demonstrate the close connection between two apparently unrelated conjectures in topology. The first of these conjectures concerns the dimension of elements of the Whitehead group, Wh(G) of a group, G. Cohen [1] defined the *dimension* of an element, x of Wh(G) and showed that the dimension of any element other than the identity is either 2 or 3. Taken together, the papers of Cohen [1] and Rothaus [8] demonstrate the existence of elements of dimension 3 in Wh(G) where G is a dihedral group D_p for p a prime $\geqslant 5$. Cohen was led to conjecture that elements of dimension 2 do not exist in any Wh(G). This conjecture remains unresolved for any group for which Wh(G) is nontrivial. In this paper, we are particularly interested in the case where G is a cyclic group.

The second conjecture we are interested in is the well known generalised Smith conjecture which is concerned with periodic transformations of the n-dimensional sphere. In its usual form, the generalised Smith conjecture states that if a periodic transformation of the n-sphere has an $(n-2)$-sphere as its fixed point set, then the fixed point set is unknotted. In this form, the conjecture is false in dimensions $n \geqslant 4$ [4, 5, 10] (though true for dimension $n=3$ [13]). I will propose an alternative conjecture (2.1) which should hold for all n and will show that the conjecture that there is no element of dimension 2 in Wh(G) of a cyclic group is equivalent to the modified Smith conjecture (2.1).

1. Whitehead torsion. For the reader's convenience, we define the Whitehead group of a group, G. More details may be found in [2].

The elements of the Whitehead group, Wh(G) of a group G are the equivalence classes of invertible matrices over the group ring, ZG where two matrices are equivalent if one may be obtained from the other by a sequence of the following operations:

1. Left multiply a row of the matrix by $\pm g$ for some $g \in G$.
2. Right multiply a column by $\pm g$.
3. Add a row to another row.
4. Add a column to another column.
5. Replace a matrix A by $\begin{pmatrix} A & \\ & 1 \end{pmatrix}$ (Bordering operation).
6. The inverse operation of 5.

Matrix multiplication induces a well-defined multiplication on equivalence classes, such that if X and Y are matrices of the same dimension, then $[X]\cdot[Y]=[XY]$. It may be verified that, under this multiplication, Wh(G) is an abelian group for any G. In the particular case where G is a cyclic group, Z_n, every element of Wh(G) may be represented by a 1×1 matrix. Consequently, the determinant

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map gives an isomorphism of $\mathrm{Wh}(G)$ onto $U(G)/T(G)$ where $U(G)$ denotes the group of units of ZG and $T(G)$ denotes the group of *trivial units,* $\{\pm g: g \in G\}$. $\mathrm{Wh}(Z_n)$ is a trivial group if $n=1,2,3,4,6$ and otherwise it is a non-trivial free-abelian group.

We now consider pairs of CW-complexes (K,L) with $\pi_1(L) \cong G$, $L \subset K$ and $K \searrow L$ (K is contractible onto L). There is a canonical assignment to every such pair, (K,L) of an element $\tau(K,L)$ of the Whitehead group, $\mathrm{Wh}(G)$. The element, $\tau(K,L)$ is known as the torsion of the pair. For details of this assignment, see [2]. The *dimension* of an element, x, of $\mathrm{Wh}(G)$ is defined to be the minimum dimension of $K-L$ taken over the set of all pairs, (K,L) where $\pi_1(L) \cong G$, $K \searrow L$ and $\tau(K,L)=x$. The following conjecture is due to Cohen [1].

(1.1). *If x is a non-trivial element of* $\mathrm{Wh}(G)$ *for any group, G, then the dimension of x is equal to* 3.

This conjecture may be reduced to a purely group-theoretical conjecture as follows. Let G be a group, let F be the free group generated by $x_1,\ldots,x_n$ and $R_1,\ldots,R_n$ be elements in the normal closure of F in F^*G, denoted F^{F^*G}. That is, under the projection, θ of F^*G onto G, each R_i is sent to the identity. We say that the notation $\langle G,x_1,\ldots,x_n: R_1,\ldots,R_n\rangle$ is a presentation for the group $H \cong F^*G/\langle R_1,\ldots,R_n\rangle^{F^*G}$. Given such a presentation, let A be the matrix $(\partial R_i/\partial x_j)^\theta$ where $\partial/\partial x_j$ is the Fox free derivative. Then A is a matrix over the group ring ZG. The matrix, A may be interpreted as follows. Since $R_i \in \langle x_1,\ldots,x_n\rangle^{F^*G}$, the group H maps onto G, and there is an exact sequence

$$1 \to N \to H \to G \to 1.$$

The commutator quotient group of N, denoted N_{ab} is a ZG-module, where G acts on N by conjugation in H. Then A is a relation matrix for N_{ab} as a ZG-module. Therefore, if A is a singular matrix, then N_{ab} is a non-zero ZG-module, and so $H \neq G$. We are concerned with the case where A is non-singular. Then A represents an element of $\mathrm{Wh}(G)$. The following conjecture is equivalent to (1.1).

(1.2). *If $\langle G,x_1,\ldots,x_n: R_1,\ldots,R_n\rangle$ is a presentation for H where $R_i \in F^{F^*G}$, and $A=(\partial R_i/\partial x_j)^\theta$ is non-singular and represents a non-trivial element of* $\mathrm{Wh}(G)$, *then $H \neq G$. (That is, $N \neq \{id\}$.)*

The connection between these conjectures may be explained in the following way. (Fuller details are given in [1].) Suppose we are given a presentation as in (1.2). Let L be a CW-complex with $\pi_1(L) \cong G$ and base point b. We construct a CW-complex, K, containing L, by attaching n 1-cells at b corresponding to the generators, $x_1,\ldots,x_n$ and then attaching n 2-cells according to the relators R_i. The resulting CW-complex, K, has $\pi_1(K) \cong H$. If we suppose, contrary to (1.2) that $H=G$, then it can be shown, that for all i, $\pi_i(K,L)=\{1\}$, and so $K \searrow L$. Furthermore, $\tau(K,L)$ is represented by the matrix A which is supposed to represent a non-trivial element in $\mathrm{Wh}(G)$. However, $K-L$ has dimension 2, and so we have an element, $\tau(K,L)$ of $\mathrm{Wh}(G)$ with dimension 2.

2. The Smith conjecture. Let h be a periodic (piecewise-linear) transformation of an n-sphere for $n \geqslant 3$ which has as its set of fixed points an $(n-2)$-sphere. The generalised Smith conjecture is that this $(n-2)$-sphere is unknotted in S^n. If this situation is looked at from the point of view of the orbit space of S^n under the periodic transformation, then the generalised Smith conjecture may be stated as: *If* (S^n, S^{n-2}) *is a knot pair and the k-fold cyclic covering space of* S^n *branched over* S^{n-2} *is an n-sphere, then* S^{n-2} *is unknotted in* S^n. In this generality, however, the Smith conjecture is false, except in dimension $n=3$. Many counterexamples have been given [4, 5, 10].

The counterexamples seem to fall into two broad types, however. In one class fall examples where the knot group has deficiency less than one. (Of course, all knots in the three sphere have groups of deficiency one.) The examples of Giffen [4] and Gordon [5] have this property. In another class fall examples of knots which are fixed by a transformation of period k and whose Alexander polynomial, $\Delta(t)$, is congruent to $\pm t^j$ modulo $t^k - 1$. The examples of Sumners [10] fall into this class.

We are led to make the following conjecture.

(2.1). *If* (S^n, S^{n-2}) *is a knot pair and the k-fold branched covering of* S^n *branched over* S^{n-2} *is an n-sphere, then either*

1. $\pi_1(S^n - S^{n-2})$ *has deficiency less than one, or*
2. $\Delta(t) \equiv \pm t^j$ *modulo* $(t^k - 1)$, *where* $\Delta(t)$ *is the Alexander polynomial of the knot,* S^{n-2}.

Note that the case where S^{n-2} is unknotted in S^n is included in Condition 2. It will be shown next that Conjecture (2.1) is equivalent to Conjecture (1.2) for cyclic groups.

3. Equivalence of the conjectures. In this section we will prove the main theorem of the paper.

(3.1) THEOREM. *Conjecture* (2.1) *is equivalent to Conjecture* (1.1) *for cyclic groups.*

In order to streamline the proof, we will separate two lemmas.

(3.2) LEMMA. *Suppose* $n \geqslant 6$ *and M is a manifold of dimension n with a handle decomposition consisting of one* 0*-handle, k* 1*-handles and k* 2*-handles. If* $\pi_1(M) \cong \{1\}$, *then M is an n-ball.*

Proof. This may be deduced using the Poincaré conjecture, or else directly using handle theory as follows. One begins by exchanging the 1-handles for 3-handles (see [9], Lemma 6.15). Then by "adding" 3-handles ([9], Lemma 6.7) one can reach a position where the incidence number of the ith 2-handle and the jth 3-handle is δ_{ij}. Finally, one can use the Whitney Lemma to cancel pairs of handles ([9], Corollary 6.5). □

(3.3) LEMMA. *Let H be the knot group of a knot* S^{n-2} *in* S^n *and let m be a meridian of the knot (an element of the knot group represented by a small loop* *around the knot). If M_k is the k-fold cyclic branched cover of S^n branched over the knot, then $\pi_1(M_k)$ is isomorphic to the kernel of the surjection $H/\langle m^k\rangle \to Z_k$.*

This is the easy result of a calculation using the Reidemeister–Schreier process. □

Proof of Theorem (3.1).

FIRST PART. (2.1) *implies* (1.1) *for cyclic groups.*

Begin by assuming conjecture (1.1) is false, and hence (1.2) is false, for some cyclic group, G. Thus, there is a presentation

$$\langle g, x_1, \ldots, x_n : g^k, R_1, \ldots, R_n\rangle \cong Z_k \tag{3.4}$$

where each R_i has exponent sum zero in g and where $\det(\partial R_i/\partial x_j)^\theta$ is a non-trivial unit of ZZ_k. Here, θ is the natural map of F_{n+1} onto $Z_k = \langle g : g^k\rangle$ where F_{n+1} is the free group generated by g and $x_1, \ldots, x_n$. We construct a counter-example to (2.1).

Consider the presentation

$$\langle g, x_1, \ldots, x_n : R_1, \ldots, R_n\rangle. \tag{3.5}$$

We will show that (3.5) is a presentation for the knot-group of a knot in the 5-sphere, and that the k-fold cyclic covering space branched over the knot is again a 5-sphere. The construction is quite similar to that used by Sumners [10]. We begin with an unknotted ball pair, B^4 in B^6. Attach n 1-handles to $B^6 - B^4$ (with attaching spheres in ∂B^6) to obtain a space $X - B^4$ where

$$X = B^6 \cup H_1^1 \cup \cdots \cup H_n^1.$$

The fundamental group of this space is a free group on $n+1$ generators. There is one generator corresponding to a loop around B^4 and one generator corresponding to each handle. Identify the first-named generator with the generator g in (3.5) and the other generators with the generators $x_1, \ldots, x_n$ in (3.5). Now attach n 2-handles to $X - B^4$ according to the relators $R_1, \ldots, R_n$ to obtain a space $Y - B^4$ where $Y = B^6 \cup (H_1^1 \cup \cdots \cup H_n^1) \cup (H_1^2 \cup \cdots \cup H_n^2)$. It is clear that (3.5) is a presentation for $\pi_1(Y - B^4)$, and that a presentation for $\pi_1(Y)$ is

$$\langle x_1, \ldots, x_n : R_1', \ldots, R_n'\rangle$$

where R_i' is the relator obtained from R_i by omitting all occurrences of the generator, g. But this is a presentation of the trivial group, since it is obtained from (3.4), a presentation for Z_k, by killing the generator, g. It follows from (3.1) that Y is a 6-ball. Using the Van Kampen theorem to examine the effect of adding 1-and 2-handles to $B^6 - B^4$, one sees that $\pi_1(\partial Y - \partial B^4) \cong \pi_1(Y - B^4)$. So $(\partial Y, \partial B^4)$ is a knotted S^3 in S^5 and $\pi_1(\partial Y - \partial B^4) = \pi_1(Y - B^4) \cong (3.5)$.

Next, we consider the k-fold branched covering space of our knotted ball pair, (Y, B^4) (rename this (B^6, B^4), a knotted ball pair.) Denote by M_k the k-fold branched covering of B^6 branched over B^4. According to (3.3), $\pi_1(M_k)$ is isomorphic to the kernel of the map from (3.4) onto Z_k, namely, the trivial group.

But, M_k has a handle decomposition consisting of a 0-handle with nk 1-handles and nk 2-handles. By (3.1), M_k is a 6-ball. Restricting the covering to the boundary of B^6, we see that the k-fold branched covering space of $S^5 = \partial B^6$ branched over the knotted $S^3 = \partial B^4$ is a 5-sphere.

Since (3.5) is a presentation for the knot group, $\pi_1(S^5 - S^3)$ has deficiency one.

To calculate the Alexander polynomial, denote by πK the group presented by (3.5) and consider the map $\phi: F_{n+1} \to Z = \langle t \rangle$ given by $g \to t$, $x_i \to 1$. The Alexander matrix is then given by $(A' | \partial R_i / \partial g)^\phi$ where A' is the block $(\partial R_i / \partial x_j)$. However, the last column is zero, since the exponent sum of g in R_i is zero. It follows that the Alexander polynomial is $\Delta(t) = \det(A'(t)^\phi)$. If ξ is the map $\xi: Z = \langle t \rangle \to Z_k = \langle g : g^k \rangle$ taking t to g, then $A'^{\phi\xi}$ is the matrix $A = (\partial R_i / \partial x_j)^\theta$. By hypothesis, $\det(A)$ is a non-trivial unit of ZG. That is, $\Delta(t)\xi = \det(A'^{\phi\xi})$ is a non-trivial unit of ZZ_k. In other words, $\Delta(t)$ is not congruent to $\pm t^j$ modulo $(t^k - 1)$. Thus the counterexample to (1.2) is complete. Note also that the example we constructed was a Z_k-null-cobordant knot in the terminology of Sumners. That is, the knot pair (S^5, S^3) bounds a ball pair (B^6, B^4) which admits a Z_k action with fixed point set B^4.

SECOND PART. (1.1) *for cyclic groups implies* (2.1).

We now show that a counterexample to (2.1) gives a counterexample to (1.1). In fact, we assume that there is a knot pair (S^n, S^{n-2}) for $n \geqslant 3$ such that $\pi_1(S^n, S^{n-2})$ has deficiency one and $\Delta(t) \bmod (t^k - 1)$ is not equal to $\pm t^j$, but that the k-fold cyclic branched covering space, M_k, is simply connected (we do not need to assume that it is a sphere). Let

$$\pi K = \langle x_0, x_1, \ldots, x_n : R_1, \ldots, R_n \rangle \tag{3.6}$$

be a presentation for the knot group. There exists a homomorphism $\phi: \pi K \to Z = \langle t \rangle$. After an appropriate adjustment to (3.6) if necessary, we may assume that $x_0 \phi = t$ and $x_i \phi = 1$ for $i \neq 0$. Then, the exponent sum of x_0 in R_i must be zero. We rename the generator, x_0 as g and note that (3.6) is the same as (3.5).

By (3.3) the kernel of the homomorphism of $\pi K / \langle g^k \rangle$ onto Z_k is $\pi_1(M_k)$ and this is trivial by assumption. A presentation for $\pi K / \langle g^k \rangle$ is

$$\langle g, x_1, \ldots, x_n : g^k, R_1, \ldots, R_n \rangle,$$

a presentation for the trivial extension of Z_k. This is the same as (3.4). It remains to check the matrix $A = (\partial R_i / \partial x_j)^\theta$ over ZG. However, just as in the first part of the proof, $\det(A) = \Delta(t)\xi$. This by assumption is not a trivial unit of ZZ_k, and so we have the required counterexample to (1.2) and hence to (1.1). □

4. Further considerations. We may draw various conclusions from the above arguments independent of the truth of the above conjectures. Firstly

(4.1). *If K is a knot in a manifold and its knot group has deficiency one and the k-fold cyclic covering space is simply connected, then there exists for each $n \geqslant 5$ a knotted S^{n-2} in S^n which has the same knot group, such that the k-fold cyclic covering space is a sphere, S^n and such that S^{n-2} in S^n is Z_k-null-cobordant.*

This is achieved by the construction of part one of the proof of (3.1).

We now consider a purely group theoretical conjecture equivalent to (1.1) for cyclic groups. We start with the presentation $\langle g, x_1, \ldots, x_n : g^k, R_1, \ldots, R_n \rangle$, where the exponent sum of g in R_i is zero. Apply the Reidemeister-Schreier method to obtain a presentation for $N = \text{kernel}(\theta)$, where θ is the homomorphism from H to Z_k taking x_i to 1. Select coset representatives $1, g, g^2, \ldots, g^{k-1}$. Then, let X_{ij} be the generator $g^j x_i (\overline{g^j x_i})^{-1} = g^j x_i g^{-j}$, and let $G_j = g^j g (\overline{g^j g})^{-1}$. Then G_j is a trivial generator for $i = 0, 1, \ldots, k-2$. From the relator g^k, we also obtain $G_{k-1} = 1$. Thus, we get a presentation $\langle X_{ij} : R_{ij} \rangle_{i=1,\ldots,n; j=0,\ldots,k-1}$ where R_{ij} is the relator obtained from $g^j R_i g^{-j}$ by rewriting. These R_{ij} obey certain symmetry properties. Namely, with subscripts considered modulo k, let $R_{ij} = W(X_{10}, \ldots, X_{1k-1}, X_{20}, \ldots, X_{2k-1}, \ldots, X_{n0}, \ldots, X_{nk-1})$. Then

$$R_{ij+1} = W(X_{11}, \ldots, X_{1k-1}, X_{10}, X_{21}, \ldots, X_{2k-1}, X_{20}, \ldots, X_{n1}, \ldots, X_{nk-1}, X_{n0}).$$

That is, R_{ij+1} is obtained from R_{ij} by simultaneous cyclic permutation of the generators $X_{i0}, \ldots, X_{ik-1}$ for each i. Z_k acts on this group, N, by taking X_{ij} to X_{ij+1} for all i and j. If we form the matrix (over ZZ_k)

$$B_{ij} = \sum_{l=0}^{k-1} g^l (\text{exponent sum of of } X_{jl} \text{ in } R_{i0})$$

then it is easily seen that B is the same matrix as $A = (\partial R_i / \partial x_j)^\theta$. Thus, for cyclic groups, (1.2) is equivalent to

(4.2). *If* $\langle X_{ij} : R_{ij} \rangle_{i=1,\ldots,n; j=0,\ldots,k-1}$ *is a presentation where the relators,* R_{ij} *obey the symmetry property stated above, then it is not a presentation for the trivial group unless the matrix* B *has determinant a trivial unit of* Z_k.

In the case $n = 1$, we have the following conjecture, the truth of which would follow from Conjecture (1.2). (See also Problem B6 of Problems by C. T. C. Wall, [12].)

(4.3). *Let* $N = \langle X_0, \ldots, X_{k-1} : R_0, \ldots, R_{k-1} \rangle$ *be a presentation with the cyclic symmetry property: If*

$$R_i = W(X_0, \ldots, X_{k-1}), \quad \textit{then} \quad R_{i+1} = W(X_1, \ldots, X_{k-1}, X_0).$$

Then the group thus presented is not trivial unless the exponent sum of X_i *in* R_0 *is* $\pm\delta_{ib}$ *for some fixed* b.

That is, the group is not trivial unless

$$P(g) = \sum_{i=0}^{k-1} g^i \cdot (\text{exponent sum of } X_i \text{ in } R_0)$$

is a trivial unit of ZZ_k where $Z_k = \langle g : g^k \rangle$. It is easily seen that $N/[N, N]$ is trivial if and only if $P(g)$ is a unit of ZZ_k.

Examples of groups with presentations of the form (4.3) are:

(i) $\langle X_1, X_2 : X_1 X_2^\alpha X_1^{-1} X_2^{1-\alpha}, X_2 X_1^\alpha X_2^{-1} X_1^{1-\alpha} \rangle$. This is well known to be a trivial group, but in this case, $P(g) = 1$, a trivial unit.

(ii) The Fibonnaci groups

$$\mathrm{Fib}_k = \langle X_1, \ldots, X_k : X_1X_2 = X_3, X_2X_3 = X_4, \ldots, X_kX_1 = X_2 \rangle.$$

However, these groups never even obey $\mathrm{Fib}_k/[\mathrm{Fib}_k, \mathrm{Fib}_k] = \{1\}$, except for $k = 1, 2$. The order of $\mathrm{Fib}_k/[\mathrm{Fib}_k, \mathrm{Fib}_k]$ grows in a manner related to the Fibonnaci sequence. Note that Fib_k is the kernel of the map $H \to Z_k$ where H is the group with presentation $\langle G, X : XGXGX^{-1}G^{-2} \rangle$. Thus, using the construction of part 1 of the proof of (3.1), there exists a null-cobordant knot, S^{n-2} in S^n for any $n \geqslant 5$ such that Fib_k is the fundamental group of the k-fold cyclic covering space of S^n branched over S^{n-2}. The same holds for all groups with presentation as in (4.3).

5. Remarks on Cohen's example. Cohen gave an example of an extension of Z_5 for which the matrix A represented a non-trivial element of the Whitehead group, and such that the extension was a proper extension. His example was (essentially) the presentation $\langle g, x : g^5, gxgx^{-1}g^{-2}x^{-1} \rangle = H$. He showed directly by a representation onto S_5 (the symmetric group) that this group is a proper extension of Z_5.

According to (3.3), $\ker(H \to Z_5)$ is the fundamental group of the 5-fold cyclic branched covering space of a knot with presentation $\langle g, x : gxgx^{-1}g^{-2}x^{-1} \rangle$. Making the substitution $x = hg^{-1}$ and eliminating x, we get a presentation $\langle g, h : ghgh^{-1}g^{-1}h^{-1} \rangle$ which is easily recognised as the group of the trefoil knot. Thus, H is an extension of $\pi_1(M_5)$ by Z_5, where M_5 is the 5-fold cyclic branched covering space of the trefoil knot. By the classical Smith conjecture, $\pi_1(M_5)$ is non-trivial. In fact, M_5 is the famous dodecahedral space, and $\pi_1(M_5)$ is the binary dodecahedral group of order 120. H is an extension of this group by Z_5—a group of order 600.

Since a cyclic covering space of a knot in S^3 cannot be trivial by the (now proven) Smith conjecture, we see how many other examples can be generated. Take a knot in S^3 with a k-fold cyclic covering space which is not a homology sphere, and such that $\Delta(t) \bmod (t^k - 1)$ is not $\pm t^j$. (For the sort of polynomials which can occur, see Fox [3], Kinoshita [6] or Hartley [7]. One can take for instance $\Delta(t) = 1 - t + t^2$, $k \equiv \pm 1$ modulo 6. There are infinitely many knots with this (or any other) polynomial: Seifert [11].) Then, proceeding as in part 2 of the proof of Theorem (3.1), we obtain an example of a presentation $\langle g, x_1, \ldots, x_n : g^k, R_1, \ldots, R_n \rangle = H$ satisfying the hypotheses of (1.2) and for which $H \neq Z_k$, according to the Smith conjecture, since $N = \pi_1(M_k) \neq \{1\}$.

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Department of Mathematics
University of Missouri - St. Louis
St. Louis, Missouri 63121