BOUNDARY BEHAVIOR OF PROPER HOLOMORPHIC MAPPINGS

E. Bedford, S. Bell, and D. Catlin

1. Introduction. It has recently been proved in [4] and in [6] that if $f: D_1 \to D_2$ is a proper holomorphic mapping between smooth bounded pseudoconvex domains in $\mathbf{C}^n$, and if the Bergman projection associated to D_1 is globally regular, then f extends smoothly to $\bar{D}_1$. The purpose of this note is to indicate how this result extends to the more general setting where D_1 and D_2 are relatively compact domains inside Stein manifolds.

If D is a relatively compact domain in a Stein manifold M, the space $L^2_{n,0}(D)$ is defined to be the set of $(n,0)$ forms ω such that

$$\|\omega\|^2 = (\sqrt{-1}\,)^{n^2} \int_D \omega \wedge \bar{\omega}$$

is finite. The space $L^2_{n,0}(D)$ is a Hilbert space with inner product given by

$$(\omega, \eta) = (\sqrt{-1}\,)^{n^2} \int_D \omega \wedge \bar{\eta}.$$

The Bergman projection P associated to D is the orthogonal projection of $L^2_{n,0}(D)$ onto $H_{n,0}(D)$, the closed subspace of $L^2_{n,0}(D)$ consisting of holomorphic (i.e., $\bar{\partial}$-closed) $(n,0)$ forms. We shall say that a smoothly bounded domain D satisfies *condition R* if the Bergman projection associated to D maps $C^\infty_{n,0}(\bar{D})$ into $C^\infty_{n,0}(\bar{D})$. The main result of this paper can now be stated.

THEOREM 1. *Suppose $f: D_1 \to D_2$ is a proper holomorphic mapping between relatively compact, smoothly bounded pseudoconvex domains D_1 and D_2 in n-dimensional Stein manifolds M_1 and M_2, respectively. If D_1 and D_2 satisfy condition R, then f extends smoothly to $\bar{D}_1$.*

REMARKS. A) A domain D is known to satisfy condition R, for example, whenever its associated $\bar{\partial}$-Neumann problem on $(n,0)$ forms is globally regular. For a detailed discussion of the regularity properties of the $\bar{\partial}$-Neumann problem and their relation to the Bergman projection, see J. J. Kohn's papers [7, 8].

B) There is an apparently stronger version of Theorem 1 that can be proved.

THEOREM 2. *Suppose $f: D_1 \to D_2$ is a proper holomorphic mapping between smoothly bounded pseudoconvex domains D_1 and D_2 that are relatively compact inside Stein manifolds of dimension n. If D_1 satisfies condition R, then f extends smoothly to $\bar{D}_1$.*

We shall not prove Theorem 2 here. Our proof of Theorem 1 reveals the basic changes that must be made in the arguments of [4] and [6] to adapt them to

Received September 3, 1982.
Michigan Math. J. 30 (1983).

the case in which $\mathbf{C}^n$ is replaced by a Stein manifold. Beyond this, a proof of Theorem 2 would merely involve a straightforward transcription of the arguments of [4] and [6] into invariant language. Furthermore, the following theorem shows that Theorem 2 is actually no more general than Theorem 1.

THEOREM 3. *Suppose $f: D_1 \to D_2$ is a proper holomorphic mapping between smoothly bounded pseudoconvex domains that are relatively compact inside n-dimensional Stein manifolds. If D_1 satisfies condition R, then so does D_2.*

Theorem 3 is proved for the case in which D_1 and D_2 are contained in $\mathbf{C}^n$ in [3]. Since the modifications involved in extending the proof given in [3] to the more general setting at hand are straightforward after the ideas used in the proof of Theorem 1 are understood, we shall not prove Theorem 3 here.

2. Proof of Theorem 1. Suppose $f: D_1 \to D_2$ is a proper holomorphic mapping between domains that satisfy the hypotheses of Theorem 1. Two key lemmas are at the heart of the proof of Theorem 1.

LEMMA 1. *If ω is a holomorphic $(n,0)$ form in $C^\infty_{n,0}(\bar{D}_2)$, then $f^*\omega$ is in $C^\infty_{n,0}(\bar{D}_1)$.*

LEMMA 2. *If ω is a holomorphic $(n,0)$ form in $C^\infty_{n,0}(\bar{D}_2)$ that vanishes to at most finite order at any boundary point of D_2, then $f^*\omega$ vanishes to at most finite order at any boundary point of D_1.*

The proofs of the lemmas will be given in §3. We now indicate how the lemmas imply Theorem 1.

Let p_0 be a boundary point of D_1 and let $z_1, z_2, \ldots, z_n$ be holomorphic coordinates near p_0. We shall prove that f extends smoothly to bD_1 near p_0. Let $\{p_i\}$ be a sequence of points in D_1 that converges to p_0. By passing to a subsequence, if necessary, we can assume that $\{f(p_i)\}$ converges to a point q_0 in bD_2. Let $g_1, g_2, \ldots, g_n$ be n functions on D_2 that extend to be holomorphic in a neighborhood of $\bar{D}_2$ in M_2 and that form a coordinate chart near q_0. Define a holomorphic function u near p_0 via

$$u\, dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n = f^*(dg_1 \wedge dg_2 \wedge \cdots \wedge dg_n).$$

Lemmas 1 and 2 imply that u extends smoothly to bD_1 near p_0 and that u vanishes to finite order at p_0.

If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a multi-index, define $g^\alpha = \prod_{i=1}^n g_i^{\alpha_i}$. Lemma 1 implies that the form $f^*(g^\alpha\, dg_1 \wedge \cdots \wedge dg_n)$ extends smoothly to bD_1. Hence, u and $u(g^\alpha \circ f)$ extend smoothly to bD_1 near p_0 for each α, and u vanishes to at most finite order at p_0. Now, by the division theorem of [4], $g_i \circ f$ extends smoothly to bD_1 near p_0 for each i. Hence, f extends smoothly to bD_1 near p_0. Since p_0 was chosen arbitrarily, we conclude that f extends smoothly to all of bD_1. The proof of Theorem 1 has been reduced to proving the lemmas.

3. Proof of the lemmas. The proof of Lemma 1 depends on the transformation rule for the Bergman projections under proper holomorphic mappings. Let P_1 and P_2 denote the Bergman projections associated to D_1 and D_2, respectively.

If ω is an $(n,0)$ form in $L^2_{n,0}(D_2)$, then $P_1(f^*\omega)=f^*(P_2\omega)$. This fact is proved in [2] in the case that D_1 and D_2 are contained in $\mathbf{C}^n$. Since the argument is purely local, the same proof can be applied to the more general setting of Lemma 1.

If ω is a holomorphic $(n,0)$ form in $C^\infty_{n,0}(\bar{D}_2)$, then it is possible to construct an $(n,0)$ form ϕ in $C^\infty_{n,0}(\bar{D}_2)$ that vanishes to infinite order on bD_2 such that $P_2\phi=\omega$. To do this, we choose a Hermitian metric on M_2, and we let ∂^* denote the formal adjoint with respect to this metric of the operator,

$$\partial : C^\infty_{n-1,0}(\bar{D}_2)\to C^\infty_{n,0}(\bar{D}_2).$$

Since the differential operator $\partial\partial^*$ is non-characteristic to bD_2, there is an $(n,0)$ form ψ such that $\psi=0$ and $\nabla\psi=0$ on bD_2, and such that $\omega-\partial\partial^*\psi$ vanishes to infinite order on bD_2. Let $\phi=\omega-\partial\partial^*\psi$. Since $\partial\partial^*\psi$ is orthogonal to $H_{n,0}(D_2)$ via integration by parts, we see that $P_2\phi=\omega$. It can be shown, exactly as in [1, 2] that, because ϕ vanishes to infinite order on bD_2, it follows that $f^*\phi$ is in $C^\infty_{n,0}(\bar{D}_1)$. Now the identity $f^*\omega=f^*(P_2\phi)=P_1(f^*\phi)$ reveals that $f^*\omega$ is in $C^\infty_{n,0}(\bar{D}_1)$ because D_1 satisfies condition R. This completes the proof of Lemma 1. □

Special Sobolev Norms. Suppose D is a relatively compact, smoothly bounded domain in a Stein manifold M. Sobolev norms can be defined on forms in $L^2_{n,0}(D)$ in the usual way in terms of a fixed partition of unity of the manifold M subordinate to an open cover by coordinate charts. If s is a positive integer, we let $W^s(D)$ denote the Sobolev s-space of $(n,0)$ forms on D, $\langle\omega,\eta\rangle_s$ the inner product on $W^s(D)$ arising from the fixed partition, and $\|\omega\|_s$ the corresponding norm. Sobolev's lemma implies that the norms $\|\ \ \|_s$ $(s=1,2,3,\ldots)$ can be used to define the Fréchet topology of $C^\infty_{n,0}(\bar{D})$.

We shall also need two auxiliary norms on holomorphic $(n,0)$ forms. If s is a positive integer, we define the Sobolev negative s-norm of a holomorphic $(n,0)$ form η to be

$$\|\eta\|_{-s}=\operatorname*{Sup}_{\phi}\left|\int_D \eta\wedge\bar{\phi}\right|$$

where the supremum is taken over all $(n,0)$ forms ϕ in $C^\infty_{n,0}(\bar{D})$ with $\|\phi\|_s=1$ that are compactly supported in D. The special Sobolev s-norm of a holomorphic $(n,0)$ form ω is defined to be

$$|||\omega|||_s=\operatorname{Sup}\left\{\left|\int_D \omega\wedge\bar{\eta}\right| : \eta\in H_{n,0}(D);\ \|\eta\|_{-s}=1\right\}.$$

There are two basic facts that make these auxiliary norms useful.

FACT 1. For each positive integer s, there is a constant $c=c(s,D)$ such that

$$\left|\int_D \omega\wedge\bar{\eta}\right|\leqslant c\,|||\omega|||_s\,\|\eta\|_{-s}$$

for all ω and η in $H_{n,0}(D)$.

FACT 2. If D satisfies condition R, then, for each positive integer s, there is a positive integer $N=N(s,D)$ and a constant $C=C(s,D)$ such that $\|\omega\|_s \leqslant C|\!|\!|\omega|\!|\!|_N$ for all ω in $H_{n,0}(D)$.

Fact 1 is proved in [1] for D contained in $\mathbf{C}^n$. The proof can easily be modified to carry over to the more general setting at hand (see [5]). We shall prove only Fact 2 here.

Proof of Fact 2. Suppose ω is a form in $H_{n,0}(D)$. Let Y be a relatively compact open subset of D. The mapping $\eta \mapsto \langle \eta, \omega\rangle_{W^s(Y)}$ is a continuous linear functional on $H_{n,0}(D)$. Hence, there is a form θ in $H_{n,0}(D)$ such that $\langle \eta, \omega\rangle_{W^s(Y)} = \int_D \eta\wedge\bar\theta$ for all η in $H_{n,0}(D)$. Now the Bergman projection P associated to D is a *closed* linear mapping of $C^\infty_{n,0}(\bar D)$ *onto* the closed subspace of $C^\infty_{n,0}(\bar D)$ consisting of holomorphic $(n,0)$ forms that are smooth up to the boundary. Hence, the closed graph theorem implies that P is continuous in the Fréchet topology of $C^\infty_{n,0}(\bar D)$. Therefore, there is a constant C and a positive integer N such that $\|P\phi\|_s \leqslant C\|\phi\|_N$. We can now finish the proof of Fact 2 by observing that

$$\|\omega\|^2_{W^s(Y)} = \left|\int_D \omega\wedge\bar\theta\right| \leqslant |\!|\!|\omega|\!|\!|_N \|\theta\|_{-N}.$$

Furthermore,

$$\|\theta\|_{-N} = \operatorname{Sup}\left|\int_D \phi\wedge\bar\theta\right| = \operatorname{Sup}\left|\int_D P\phi\wedge\bar\theta\right| = \operatorname{Sup}|\langle P\phi, \omega\rangle_{W^s(Y)}|$$

$$\leqslant \operatorname{Sup}\|P\phi\|_s\|\omega\|_{W^s(Y)} \leqslant C\operatorname{Sup}\|\phi\|_N\|\omega\|_{W^s(Y)}$$

where the supremum is taken over all ϕ in $C^\infty_{n,0}(\bar D)$ with $\|\phi\|_N = 1$ that are compactly supported in D. Hence, $\|\omega\|_{W^s(Y)} \leqslant C|\!|\!|\omega|\!|\!|_N$. Since the constants C and N are independent of ω and Y, we conclude that if $|\!|\!|\omega|\!|\!|_N < \infty$, then ω is in $W^s(D)$ and $\|\omega\|_s \leqslant C|\!|\!|\omega|\!|\!|_N$.

We shall now show that Lemma 2 is a consequence of the following claim. Remmert's proper mapping theorem states that f is a branched cover of some finite order m. Let $F_1, F_2, \ldots, F_m$ denote the inverses to f defined *locally* on D_2 minus the image of the branch locus of f.

CLAIM. If h is a holomorphic function on D_1 in $C^\infty(\bar D_1)$, then any symmetric function of $h\circ F_1, h\circ F_2, \ldots, h\circ F_m$ extends to be a holomorphic function on D_2 in $C^\infty(\bar D_2)$.

Let p_0 be a point in bD_1 and let $\{p_i\}$ be a sequence of points in D_1 converging to p_0 such that the sequence $\{f(p_i)\}$ converges to some point q_0 in bD_2. Let $z_1, z_2, \ldots, z_n$ define holomorphic coordinates near p_0 and let $w_1, w_2, \ldots, w_n$ define coordinates near q_0. Let $\Delta(\epsilon)$ denote the polydisc of polyradius ϵ about p_0 in the $z_1, \ldots, z_n$ coordinates and let $B(\epsilon)$ denote the ball of radius ϵ about q_0 in the $w_1, \ldots, w_n$ coordinates. The claim can be used exactly as in [4] to show that the image of $\Delta(\epsilon)\cap D_1$ under f contains $B(\epsilon^{m+1})\cap D_2$ for small $\epsilon > 0$. Hence,

$$\int_{\Delta(\epsilon)\cap D_1} f^*\omega\wedge\overline{f^*\omega} \geqslant \int_{B(\epsilon^{m+1})\cap D_2} \omega\wedge\bar\omega \geqslant C\epsilon^Q$$

for some positive constants C and Q because ω vanishes to finite order to q_0. This implies that $f^*\omega$ vanishes to at most finite order at p_0.

Proof of the Claim. Because of Newton's identities (see [2]), it suffices to prove that $\sum_{k=1}^m h\circ F_k$ is in $C^\infty(\bar{D}_2)$ whenever h is a holomorphic function in $C^\infty(\bar{D}_1)$. Let s be a positive integer. Let q_0 be a boundary point of D_2 and let Ω be a holomorphic $(n, 0)$ form on D_2 that extends to be holomorphic on a neighborhood of $\bar{D}_2$ and that is non-zero at q_0. Let $H = (\sum_{k=1}^m h\circ F_k)\Omega$. The Sobolev norm $\|H\|_s$ is dominated by a constant times $|\!|\!|H|\!|\!|_N$ where $N = N(s, D_2)$ is the number given by Fact 2. If η is an $(n, 0)$ form in $H_{n,0}(D_2)$ with $\|\eta\|_{-N} = 1$, then

$$\left|\int_{D_2} H\wedge\bar{\eta}\right| = \left|\int_{D_1} hf^*\Omega\wedge\overline{f^*\eta}\right| \leqslant c\|hf^*\Omega\|_Q\|f^*\eta\|_{-Q}$$

where Q is chosen large enough so that $\|f^*\eta\|_{-Q} \leqslant (\text{constant})\|\eta\|_{-N}$. That such a Q exists is proved in [2, 5]. Hence, we have shown that $\|H\|_s \leqslant (\text{constant})\|hf^*\Omega\|_Q$. But $hf^*\Omega$ is a form in $C^\infty_{n,0}(\bar{D}_1)$ by Lemma 1. Hence, $\|H\|_s$ is finite for each s, and we conclude that $(\sum_{k=1}^m h\circ F_k)\Omega$ is in $C^\infty_{n,0}(\bar{D}_2)$ whenever h is a holomorphic function in $C^\infty(\bar{D}_1)$. Since $\Omega \neq 0$ near q_0, we deduce that $\sum_{k=1}^m h\circ F_k$ extends smoothly to bD_2 near q_0. This completes the proof of the claim, and hence, Lemma 2 is established. □

REFERENCES

1. S. Bell, *Biholomorphic mappings and the $\bar\partial$-problem,* Ann. of Math. (2) 114 (1981), 103-133.
2. ———, *Proper holomorphic mappings and the Bergman projection,* Duke Math. J. 48 (1981), 167-175.
3. ———, *Proper holomorphic mappings between non-pseudoconvex domains,* Amer. Math. J. (in press, 1982).
4. S. Bell and D. Catlin, *Boundary regularity of proper holomorphic mappings,* Duke Math. J. 49 (1982), 385-396.
5. K. Diederich and J. E. Fornaess, *A remark on a paper by S. R. Bell,* Manuscripta Math. 34 (1981), 31-44.
6. ———, *Boundary regularity of proper holomorphic mappings,* Invent. Math. 67 (1982), 363-384.
7. J. J. Kohn, *Boundary regularity of $\bar\partial$,* Recent developments in several complex variables (Proc. Conf., Princeton Univ., Princeton, N.J., 1979), pp. 243-260, Ann. of Math. Studies, 100, Princeton Univ. Press, Princeton, N.J., 1981.
8. ———, *Subellipticity of the $\bar\partial$-Neumann problem on pseudoconvex domains: sufficient conditions,* Acta Math. 142 (1979), 79-122.

E. Bedford
Mathematics Department
Indiana University
Bloomington, Indiana 47405

S. Bell, D. Catlin
Mathematics Department
Princeton University
Princeton, New Jersey 08544