SOME REMARKS ON NIELSEN EXTENSIONS OF RIEMANN SURFACES

Noemi Halpern

Let S be a Riemann surface of finite type $(g;n,m)$, that is, a surface of genus g with n punctures and m holes. Assume that $6g-6+2n+3m>0$ and $m>0$. Then S can be represented as U/G where U denotes the upper half-plane and G a torsion-free Fuchsian group of the second kind. Let $\mathcal{I}$ be the set of maximal open intervals on $\mathbf{R}\cup\{\infty\}$ on which G acts discontinuously. For each $I\in\mathcal{I}$ the stabilizer of I in G is generated by a hyperbolic element whose axis $A(I)$ is the non-Euclidean line joining the endpoints of the interval. The Nielsen convex region $N(G)$ is the complement of the union of the closures of all the half-planes bounded by I and $A(I)$, $I\in\mathcal{I}$. The surface $S_0=N(G)/G$ is the Nielsen kernel of S; S is the *Nielsen extension* of S_0. Every surface has a Nielsen extension [1].

Given a surface S_0 let S_k be the Nielsen extension of S_{k-1} or equivalently let S_{k-1} be the Nielsen kernel of S_k, for $k\in\mathbf{Z}$. Define the *infinite Nielsen extension* of S_0 to be $S_\infty=S_1\cup S_2\cup\cdots$ and define the *infinite Nielsen kernel* of S_0 to be $S_{-\infty}=S_{-1}\cap S_{-2}\cap\cdots$.

The purpose of this paper is to obtain several inequalities concerning the lengths of certain geodesics of Nielsen extensions and kernels. The Poincaré metric is used throughout and the length of a boundary curve denotes the Poincaré length of the geodesic to which it is freely homotopic.

THEOREM 1. *Let S be a finite Riemann surface with a boundary curve of length l. Then the length of the corresponding boundary curve of the Nielsen kernel of S is greater than $2l$.*

Proof. Assume that $S=U/G$ has a boundary curve C corresponding to $X: z\to e^l z$. Let f be the conformal map that takes the Nielsen convex region of S onto U and which fixes 0, 1, and ∞. Then the Nielsen kernel of S is $S_{-1}=U/fGf^{-1}$. The function $f_1: z\to z^2$ maps the first quadrant, which contains the Nielsen convex region of S, onto U. Then $f=f_2\circ f_1$ for some conformal map f_2 which takes a region $U_1\subsetneqq U$ onto U.

Let C_0 be the corresponding boundary curve of S_{-1} and let l_0 be its length. Let d denote length in the Poincaré metric of U. Since U_1 is a proper subset of U, $d(C_0)$ is greater than $d(f_2^{-1}(C_0))$. C_0 is a simple curve in U whose endpoints are identified by fXf^{-1}. No other points of C_0 are equivalent under any other element of the group fGf^{-1}. Therefore the only points of $f_2^{-1}(C_0)$ equivalent under any element of $f_1Gf_1^{-1}$ are its endpoints, which are identified by $f_1Xf_1^{-1}$. So $d(f_2^{-1}(C_0))$ cannot be less than the length of the shortest geodesic joining the semicircles $x^2+y^2=1$ and $x^2+y^2=e^{2l}$, $y>0$. Therefore

$$l_0=d(C_0)>d(f_2^{-1}(C_0))\geqslant d(i,e^{2l}i)=2l. \qquad \square$$

Received February 3, 1982. Revision received June 1, 1982.
Michigan Math. J. 30 (1983).

COROLLARY. *Let S be a finite Riemann surface with a boundary curve. The length of the corresponding boundary element of the infinite Nielsen extension is zero; the "length" of the corresponding "boundary element" of the infinite Nielsen kernel is infinite.*

The corollary follows immediately from the theorem. The first statement has been proven by L. Bers [1]; the second by J. Wason [3].

THEOREM 2. *Let S be a Riemann surface with a boundary curve and with a geodesic of length x not homotopic to a boundary curve. If x_1 is the length of the corresponding geodesic of the Nielsen extension of S then $x_1 < x$. Moreover, if all the boundary curves of S have length less than l then $x_1 > kx$ where $k = 1 - (2/\pi)\tan^{-1}(2\sinh l/2)$.*

Proof. Assume that $S = U/G$ has a geodesic C of length x. Let $S_1 = U/G_1$ be its Nielsen extension with the corresponding geodesic C_1 of length x_1. Let f be the conformal map that takes the Nielsen convex region of S_1 onto U and let d denote the Poincaré metric of U. Since $S \subset S_1$ and $S \neq S_1$, $d(C)$ must be greater than $d(f^{-1}(C))$. Therefore

$$x_1 = d(C_1) < d(f^{-1}(C)) < d(C) = x$$

and the first inequality is proved.

There exists a collar of width $\omega = \sinh^{-1}(1/(2\sinh l/2))$ around all holes of length less than l which is disjoint from the collars of all other geodesics of the surface [2]. If all the boundary curves of S have length less than l then, by Theorem 1, the same must be true of boundary curves of S_1. Therefore there is a collar region A of width ω around C_1 which lies completely within the Nielsen convex region of S_1. Assume that C_1 corresponds to the element $z \to e^{x_1}z$ of G_1. Then A is bounded by the lines $\Theta = +\Theta_0$ and $\Theta = \pi - \Theta_0$ where $\tan\Theta_0 = 2\sinh l/2$ and $A \subset f^{-1}(U)$. Let $f_1(z) = (e^{-i\Theta_0}z)^{\pi/(\pi - 2\Theta_0)}$. The function f_1 maps A conformally onto U. Since A is a proper subset of the Nielsen convex region of S_1, $d(f_1(C_1))$ is greater than $d(f(C_1))$. Therefore

$$x = d(C) < d(f(C_1)) < d(f_1(C_1)) = \frac{1}{k}d(C_1) = \frac{1}{k}x_1$$

where $k = 1 - 2\Theta_0/\pi$. □

COROLLARY. *Assume the hypothesis of Theorem 2 and let x_n (x_∞) denote the length of the corresponding geodesic of the nth (the infinite) Nielsen extension. Then the x_n's satisfy the following inequalities:*

(A) $k^n x < x_n < x$, *where k is as in the theorem.*
(B) $x_n > \{1 - (2/\pi)\tan^{-1}(2\sinh l/2^n)\}x_{n-1}$.
(C) $x_n > k_n x$ *where* $k_n = \prod_{i=1}^{n}\{1 - (2/\pi)\tan^{-1}(2\sinh l/2^i)\}$.
(D) $x_\infty > k_\infty x$ *where* $k_\infty = \prod_{i=1}^{\infty}\{1 - (2/\pi)\tan^{-1}(2\sinh l/2^i)\}$ *is a convergent infinite product.*
(E) $c_n x_n < x_\infty < x_n$ *where* $c_n = \prod_{i=n}^{\infty}\{1 - (2/\pi)\tan^{-1}(2\sinh l/2^i)\}$ *and* $c_n \to 1$ *as* $n \to \infty$.

Proof. (A) the inequality follows from applying the theorem n times.

(B) If all the boundary curves of S have length less than l then, by Theorem 1, all the boundary curves of its Nielsen extension must have length less than $\frac{1}{2}l$. Therefore all the boundary curves of its nth Nielsen extension must have length less than $(1/2^n)l$. By applying Theorem 2 to the nth Nielsen extension one obtains the inequality $x_n > \{1-(2/\pi)\tan^{-1}(2\sinh l/2^n)\}x_{n-1}$.

(C) Apply inequality (B) to each of the n Nielsen extensions. Then

$$\begin{aligned} x_n &> \{1-(2/\pi)\tan^{-1}(2\sinh l/2^n)\}x_{n-1} \\ &> \{1-(2/\pi)\tan^{-1}(2\sinh l/2^n)\}\{1-(2/\pi)\tan^{-1}(2\sinh l/2^{n-1})\}x_{n-2} \\ &> \cdots \\ &> \prod_{i=1}^{n}\{1-(2/\pi)\tan^{-1}(2\sinh l/2^i)\}x. \end{aligned}$$

(D) The inequality follows from (C). To show that

$$\prod_{i=1}^{\infty}\{1-(2/\pi)\tan^{-1}(2\sinh l/2^i)\}$$

converges it is sufficient to show that

$$0<(2/\pi)\tan^{-1}(2\sinh l/2^i)<1$$

and that

$$\sum_{i=1}^{\infty}(2/\pi)\tan^{-1}(2\sinh l/2^i)$$

converges. Since $l>0$, $2\sinh l/2^i>0$ and

$$0<\tan^{-1}(2\sinh l/2^i)<\pi/2.$$

Since $\tan^{-1}u<u$ for $u>0$,

$$\sum_{i=1}^{\infty}(2/\pi)\tan^{-1}(2\sinh l/2^i)<\sum_{i=1}^{\infty}(2/\pi)2\sinh l/2^i.$$

But this series converges by the ratio test.

$$\begin{aligned} \left|\frac{(2/\pi)2\sinh l/2^{i+1}}{(2/\pi)2\sinh l/2^i}\right| &= \left|\frac{\sinh l/2^{i+1}}{\sinh 2(l/2^{i+1})}\right| = \left|\frac{\sinh l/2^{i+1}}{2\sinh l/2^{i+1}\cosh l/2^{i+1}}\right| \\ &= \frac{1}{2\cosh l/2^{i+1}} \leqslant \frac{1}{2}. \end{aligned}$$

(E) Inequalities (B), (C), and (D) imply the inequality. Since

$$\prod_{i=1}^{\infty}\{1-(2/\pi)\tan^{-1}(2\sinh l/2^i)\}$$

converges,

$$\prod_{i=n}^{\infty}\{1-(2/\pi)\tan^{-1}(2\sinh l/2^i)\}$$

must approach 1 as $n\to\infty$.

REFERENCES

1. L. Bers, *Nielsen extensions of Riemann surfaces,* Ann. Acad. Sci. Fenn. Ser. AI Math. 2 (1976), 17–22.
2. N. Halpern, *A proof of the collar lemma,* Bull. London Math. Soc. 13 (1981), 141–144.
3. J. Wason, *On infinite Nielsen kernels.* Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978), pp. 497–503, Ann. of Math. Studies, 97, Princeton, N.J., 1981.

Department of Mathematics
Brooklyn College of the City University of New York
Brooklyn, New York 11210