A MEAN VALUE THEOREM FOR ZETA FUNCTIONS ASSOCIATED WITH POSITIVE DEFINITE INTEGRAL FORMS

Alan H. Stein and Chungming An

1. Let $x=(x_1,x_2,\ldots,x_n)\in\mathbf{R}^n$ and let $F(x)$ be a positive definite integral form on $\mathbf{R}^n$ of degree d. Let $\zeta(F,s)=\sum_{\gamma\in\mathbf{R}^n-\{0\}}F(\gamma)^{-s}$ be a zeta function associated with F. The authors have previously shown [4] that $\zeta(F,s)$ can be continued analytically in the entire complex plane except for a simple pole at $s=n/d$ and have shown that if we write $s=\sigma+it$ then

$$\zeta(F,s)\ll\frac{|t|^{n-\sigma d}}{(n-\sigma d)(n-1-\sigma d)},\quad \frac{n-1}{d}<\sigma<\frac{n}{d}. \tag{1.1}$$

In this paper the authors exhibit the following mean value theorem for $\zeta(F,s)$.

Theorem.

(1.2)

$$\int_1^T|\zeta(F,\sigma+it)|^2\,dt\ll\begin{cases}\alpha^2T^{1+(4/[d+2])(n-\sigma d)} & \text{if } \dfrac{n-(d+2)/2d}{d}\leqslant\sigma<\dfrac{n}{d}\\[2ex] T^{2(n-\sigma d)} & \text{if } \dfrac{n-1}{d}\leqslant\sigma<\dfrac{n-(d+2)/2d}{d}\end{cases}$$

where $\alpha=\min(1/|n-\sigma d|,\log T)$.

If F is quadratic, then better estimates are available; however (1.2) is an improvement for $d>2$. The authors believe that further improvements are possible and conjecture that

$$\int_1^T|\zeta(F,\sigma+it)|^2\,dt\sim T\sum a_{F(\gamma)}F(\gamma)^{-2\sigma}$$

for $\sigma>(n-\frac{1}{2})/d$ where $a_{F(\gamma)}$ is the number of solutions to $F(\gamma')=F(\gamma)$.

2. We shall need the following analogue of integration via polar coordinates.

For $x\in\mathbf{R}^n$, let $\|x\|=\max_{1\leqslant i\leqslant n}|x_i|$. Let $B=\{x\in\mathbf{R}^n:\|x\|=1\}$. Then each $x\in\mathbf{R}^n-\{0\}$ may be written uniquely as $x=ru$, where $r=\|x\|$ and $u\in B$.

We define a measure w on B as follows. Suppose $A\subset B$ is a Borel set. Let $\tilde{A}=\{ru:0<r\leqslant1,\ u\in A\}$. We define $w(A)=nm(\tilde{A})$, where m is Lebesgue measure on $\mathbf{R}^n$. It can be shown that if f is integrable on a set $X=\{x\in\mathbf{R}^n:r_1\leqslant\|x\|\leqslant r_2\}$, then

$$\int_X f\,dx=\int_B\int_{r_1\leqslant r\leqslant r_2}r^{n-1}f(ru)\,dr\,du \tag{2.1}$$

where for brevity we write du rather than $dw(u)$.

Received October 29, 1979.
Michigan Math. J. 30 (1983).

3. We now use a special case of a generalized Euler summation formula [1]. Let $q=(q_1,\dots,q_n)$ where $q_i=0$ or 1. Define $h_q(x)=\prod_{q_i=1}(x_i-[x_i]-\frac{1}{2})$. For arbitrary $0<K<T$ we may write

$$\zeta(F,s)=Z+J_1+J_2+J_3 \tag{3.1}$$

where Z and the J_i are defined as follows.

$$Z=\sum{}' F(\gamma)^{-s} \tag{3.2}$$

where $\sum'$ represents summation over all $\gamma\in Z^n-\{0\}$ such that $-K<\gamma_i\leqslant K$ for all i.

$$J_1=\int_{\|x\|\geqslant K} F(x)^{-s}\,dx. \tag{3.3}$$

$$J_2=-s\sum_{i=1}^{n}\int_{\|x\|\geqslant K}(x_i-[x_i]-\tfrac{1}{2})F(x)^{-s-1}\frac{\partial F}{\partial x_i}\,dx. \tag{3.4}$$

$$J_3=\sum{}^*(-s)^{[|q|^+]}\int_{\|x\|\geqslant K} h_q(x)F(x)^{-s-|q|^+}\frac{\partial^q F}{\partial x^q}\,dx \tag{3.5}$$

where $|q|^+=q_1+\cdots+q_n$, $(-s)^{[|q|^+]}=(-s)(-s-1)\cdots(-s-|q|^++1)$,

$$\frac{\partial^q F}{\partial x^q}=\frac{\partial^{|q|^+}F}{\partial x_1^{q_1}\cdots\partial x_n^{q_n}}$$

and $\sum^*$ represents a sum over those q where $q_i=0$ or 1 and $|q|^+\geqslant 2$.

If we let $I=\int_1^T|\zeta(F,\sigma+it)|^2\,dt$, $I_z=\int_1^T|Z|^2\,dt$, $I_k=\int_1^T|J_k|^2\,dt$, then we observe from (3.1) that

$$I=O(I_z+I_1+I_2+I_3). \tag{3.6}$$

We shall consider each term on the right-hand side of (3.6) separately.

4. We first estimate I_z. First we observe that $|Z|=O(\sum_{\|\gamma\|\leqslant K}F(\gamma)^{-\sigma})$. The properties of F imply that

$$\sum_{\|\gamma\|\leqslant K}F(\gamma)^{-\sigma}\ll\sum_{\|\gamma\|\leqslant K}\|\gamma\|^{-\sigma d}=\sum_{m\leqslant K}m^{-\sigma d}\{m^n-(m-1)^n\}. \tag{4.1}$$

Since $m^n-(m-1)^n=O(m^{n-1})$, (4.1) yields

$$|Z|\ll\sum_{m\leqslant K}m^{n-1-\sigma d}\ll\alpha K^{n-\sigma d}. \tag{4.2}$$

We thus obtain

$$I_z=O(\alpha^2TK^{2(n-\sigma d)}). \tag{4.3}$$

It is shown in [4] that $|J_1|=O(K^{n-\sigma d}/t)$, which immediately implies

$$I_1=O(K^{2(n-\sigma d)}). \tag{4.4}$$

This is absorbed in the bound for I_z. We shall now estimate I_3 before I_2. The observations

$$(-s)^{[|q|+]} = O(t^{|q|+}), \qquad h_q(x) = O(1),$$

$$F(x)^{-s-|q|+} = O(\|x\|^{-(\sigma+|q|+)d}), \tag{4.5}$$

$$\frac{\partial^q F}{\partial x^q} = O(\|x\|^{d-|q|+})$$

imply that $|J_3|$ is bounded by a sum of terms $J^{(h)}$ defined by

$$J^{(h)} = t^h \int_{\|x\| \geqslant K} \|x\|^{-(\sigma+h)d+d-h}\, dx. \tag{4.6}$$

Using (2.1) we see that

$$J^{(h)} \ll t^h \int_{r \geqslant K} r^{-(\sigma+h)d+d-h+n-1}\, dr \tag{4.7}$$

and thus $J^{(h)} = O(t^h K^{-(\sigma+h)d+d-h+n})$. Thus

$$\int_1^T |J^{(h)}|^2\, dt = O(T^{2h+1} K^{2(-\sigma d - hd + d - h + n)}).$$

If we restrict $K > T^{1/(1+d)}$ then this is maximized when $h = 2$, yielding

$$I_3 = O(T^5 K^{2(n-\sigma d - d - 2)}). \tag{4.8}$$

5. In order to estimate I_2, we consider a typical term of J_2. Let

$$J = -s \int_{\|x\| \geqslant K} (x_1 - [x_1] - \tfrac{1}{2}) F(x)^{-s-1} \frac{\partial F}{\partial x_1}\, dx.$$

We again use (2.1). Using the notation of §2, we observe that $x_i = ru_i$, $F(x) = F(u) r^d$ and $(\partial F/\partial x_1)(x) = r^{d-1}(\partial F/\partial x_1)(u)$, so we may write

$$J = -s \int_B F(u)^{-s-1} \frac{\partial F}{\partial x_1}(u) J'\, du \tag{5.1}$$

where

$$J' = \int_{r \geqslant K} (ru_1 - [ru_1] - \tfrac{1}{2}) r^{-ds+n-2}\, dr. \tag{5.2}$$

Write $h(r) = ru_1 - [ru_1] - \frac{1}{2}$. Note that h has period $P = 1/u_1 \geqslant 1$. Choose an integer a such that $aP \geqslant K > (a-1)P$. Note that $1 \leqslant a \leqslant K+1$. Also write $z = -ds + n - 2$. Then we have

$$J' = \int_K^{aP} h(r) r^z\, dr + \sum_{l \geqslant a} \int_{lP}^{(l+1)P} h(r) r^z\, dr. \tag{5.3}$$

Integrating by parts and simplifying yields

$$J' = \frac{P^{z+1}}{z+1}\sum_{l\geqslant a} l^{z+1} - \frac{h(K)K^{z+1}}{z+1} + u_1\frac{K^{z+2}}{(z+1)(z+2)}. \tag{5.4}$$

Since $\sum_{l\geqslant a} l^{z+1} = \zeta(-z-1) - \sum_{1\leqslant l<a} l^{z+1}$, we can use (5.2) to estimate J by

$$\begin{aligned} J = &\int_B du\, F(u)^{-s-1}\frac{\partial F}{\partial x_1}(u)\frac{sP^{n-ds-1}}{n-ds-1}\sum_{l<a} l^{n-ds-1} \\ &+ O\left(|\zeta(1+ds-n)| + K^{n-\sigma d-1} + \frac{K^{n-\sigma d}}{t}\right). \end{aligned} \tag{5.5}$$

We easily evaluate

$$\int_1^T (K^{n-\sigma d-1})^2\, dt \ll TK^{2(n-\sigma d-1)}, \tag{5.6}$$

$$\int_1^T \left(\frac{K^{n-\sigma d}}{t}\right)^2 dt \ll K^{2(n-\sigma d)} \tag{5.7}$$

and

$$\int_1^T |\zeta(1+ds-n)|^2\, dt \ll \begin{cases} \beta T & \text{if } \sigma \geqslant (n-\frac{1}{2})/d \\ \beta T^{2(n-d\sigma)} & \text{if } \sigma \leqslant (n-\frac{1}{2})/d \end{cases} \tag{5.8}$$

where $\beta = \min\left(\frac{1}{|n-(1/2)-\sigma d|}, \log T\right)$.

We are left with estimating

$$J'' = \int_1^T dt \left| \int_B du\, F(u)^{-s-1}\frac{\partial F}{\partial x_1}(u)\frac{sP^{n-ds-1}}{n-ds-1}\sum_{l<a} l^{n-ds-1}\right|^2. \tag{5.9}$$

We first observe that $J'' \ll J'''$, where

$$J''' = \int_1^T dt \left\{ \int_B du \left| \sum_{l<a} l^{n-ds-1} \right| \right\}^2. \tag{5.10}$$

We expand J''' to obtain

$$J''' = \int_B du \int_B dv \int_1^T dt \sum_{\substack{l_u<a_u \\ l_v<a_v}} l_u^{n-ds-1} l_v^{n-d\bar{s}-1} \tag{5.11}$$

and hence

$$J''' = \int_B du \int_B dv \sum_{\substack{l_u<a_u \\ l_v<a_v}} (l_u l_v)^{n-d\sigma-1} \int_1^T \left(\frac{l_v}{l_u}\right)^{idt} dt. \tag{5.12}$$

Along the part of the sum where either $l_u > 2l_v$ or $l_v > 2l_u$, $\int_1^T (l_v/l_u)^{idt}\, dt = O(1)$, so that part of the integral contributes $\ll$

$$\left(\sum_{l\leqslant K} l^{n-1-d\sigma}\right)^2 \ll (\alpha K^{n-\sigma d})^2. \tag{5.13}$$

Along the part where $l_u = l_v$, $\int_1^T (l_v/l_u)^{idt}\,dt = T-1$, so that part contributes $\ll$

$$T\sum_{l\leqslant K}(l^{n-1-\sigma d})^2 \ll \begin{cases} \beta T & \text{if } (n-\frac{1}{2})/d \leqslant \sigma \leqslant n/d \\ \beta T K^{-2\sigma d+2n-1} & \text{if } (n-1)/d \leqslant \sigma \leqslant (n-\frac{1}{2})/d. \end{cases} \tag{5.14}$$

We now consider the rest of J'''. Along the rest,

$$\left|\int_1^T \left(\frac{l_v}{l_u}\right)^{idt} dt\right| \ll \frac{1}{|\log(l_v/l_u)|},$$

so that the rest contributes $\ll$

$$\sum_{l\leqslant K} l^{2(n-\sigma d-1)} \sum_{\substack{l/2<l'<2l \\ l'\neq l}} \frac{1}{|\log(l'/l)|} \ll \sum_{l\leqslant K} l^{2n-2\sigma d-1}\log l \tag{5.15}$$

$$\ll \alpha K^{2(n-\sigma d)}\log K.$$

Since the other terms of J_2 can be estimated similarly, we combine (5.6)-(5.8), (5.13)-(5.15) to obtain

$$I_2 \ll TK^{2(n-\sigma d-1)} + \alpha K^{2(n-\sigma d)}\log K$$

$$+\begin{cases} \beta T & \text{if } (n-\frac{1}{2})/d \leqslant \sigma \leqslant n/d \\ \beta T^{2(n-d\sigma)} + \beta T K^{2(n-\sigma d)-1} & \text{if } (n-1)/d \leqslant \sigma \leqslant (n-\frac{1}{2})/d. \end{cases} \tag{5.16}$$

We can now combine (3.6), (4.3), (4.4), (4.8) and (5.16) and use the assumption $K<T$ to obtain

$$I \ll \alpha^2 TK^{2(n-\sigma d)} + T^5 K^{2(n-\sigma d)-2d-4}$$

$$+\begin{cases} \beta T & \text{if } (n-\frac{1}{2})/d \leqslant \sigma \leqslant n/d \\ \beta T^{2(n-d\sigma)} & \text{if } (n-1)/d \leqslant \sigma \leqslant (n-\frac{1}{2})/d. \end{cases} \tag{5.17}$$

Let $K \sim T^{2/(d+2)}$ to obtain

$$I \ll \alpha^2 T^{1+(4/[d+2])(n-\sigma d)} + \begin{cases} \beta T & \text{if } (n-\frac{1}{2})/d \leqslant \sigma \leqslant n/d \\ \beta T^{2(n-\sigma d)} & \text{if } (n-1)/d \leqslant \sigma \leqslant (n-\frac{1}{2})/d. \end{cases} \tag{5.18}$$

We now observe that $\beta \ll T^{(4/[d+2])(n-\sigma d)}$ if $(n-\frac{1}{2})/d \leqslant \sigma \leqslant n/d$, so that

$$I \ll \alpha^2 T^{1+(4/[d+2])(n-\sigma d)} \quad \text{if } \frac{n-\frac{1}{2}}{d} \leqslant \sigma \leqslant \frac{n}{d}. \tag{5.19}$$

We also observe that if $\sigma > \frac{n-(d+2)/2d}{d}$ then $T^{1+(4/[d+2])(n-\sigma d)} \gg \beta T^{2(n-\sigma d)}$, while if $\sigma < \frac{n-(d+2)/2d}{d}$, then the reverse is true. Also, if $\sigma < \frac{n-(d+2)/2d}{d}$, then $\beta = O(1)$.

So

$$I \ll \begin{cases} T^{1+(4/[d+2])(n-\sigma d)} & \text{if } \dfrac{n-(d+2)/2d}{d} \leqslant \sigma \leqslant \dfrac{n-\frac{1}{2}}{d} \\ T^{2(n-\sigma d)} & \text{if } \dfrac{n-1}{d} \leqslant \sigma \leqslant \dfrac{n-(d+2)/2d}{d}. \end{cases} \tag{5.20}$$

Combining (5.19) and (5.20) completes the proof of the theorem. □

REFERENCES

1. C. An, *On a generalization of the gamma function and its application to certain Dirichlet series,* Bull. Amer. Math. Soc. 75 (1969), 562–568.
2. ———, *A generalization of Epstein zeta functions,* Michigan Math. J. 21 (1974), 45–48.
3. ———, *On the analytic continuation of a certain Dirichlet series,* J. Number Theory 6 (1974), 1–6.
4. C. An and A. H. Stein, *Representations of integers by positive definite forms,* Bull. Inst. Math. Acad. Sinica 6 (1978), 7–14.
5. ———, *Representations of integers by positive definite forms over arithmetic progressions,* Illinois J. Math. 24 (1980), 612–618.
6. R. Ayoub, *An introduction to the analytic theory of numbers,* Math. Surveys, No. 10, Amer. Math. Soc., Providence, R.I. 1963.
7. B. C. Berndt, *On the average order of a class of arithmetical functions,* I, J. Number Theory 3 (1971), 184–203.
8. K. Mahler, *Zur approximation algebraischer zahlen.* III, Acta Math. 62 (1934), 91–166.

Department of Mathematics
The University of Connecticut
Waterbury, Connecticut 06710

and

Bell Telephone Laboratories
Naperville, Illinois 60540