SURFACES BOUNDING THE UNLINK

Charles Livingston

Let F be a surface embedded in S^3 with boundary the unlink. A small isotopy results in an embedding $(F, \partial F) \to (B^4, S^3)$. In this paper such maps will be classified up to isotopy. In particular, it will be shown that the isotopy class of the map is determined by the homeomorphism type of F and an even integer which is easily calculated from the original embedding. Following the proof of this result we will discuss an application to 4-manifolds. The final section of the paper consists of some remarks concerning what is known in the case that ∂F is knotted. We will show that examples constructed by Trotter [10] of distinct surfaces having as boundary the same pretzel knots are isotopic when considered as surfaces in (B^4, S^3), answering question 1.20C of [5] to the negative.

Much of the motivation of this work comes from [1], and I would like to thank Rob Kirby for suggesting the problem to me. Some of the results first appeared in the author's dissertation [6].

1. Preliminaries. Throughout this paper we will work in the C^∞ category. Although most of the results will be stated for the pair (B^4, S^3), they hold as well for the pair $(\mathbf{R}^4_+, \mathbf{R}^3)$. We will move freely between these spaces without further comment.

For any surface F we let $\text{genus}(F) = \frac{1}{2}\text{rank}(H_1(\bar{F}, Z/2Z))$ where $\bar{F}$ is the surface obtained from F by capping off each boundary component with a disk. In all that follows, $N(X)$ denotes a closed regular neighborhood of X in whatever space X is embedded. By a standard surface we mean a surface embedded in S^3 which is constructed from a 2-sphere embedded in S^3 by puncturing the 2-sphere, adding trivial handles, (which may or may not be orientable) and, in the nonorientable case, adding trivial once twisted bands along a single boundary component. Finally, if F_1 and F_2 are embedded surfaces in S^3, we say they are isotopic in B^4 if after pushing F_1 and F_2 into B^4 the maps $(F_1, \partial F_1) \to (B^4, S^3)$ and $(F_2, \partial F_2) \to (B^4, S^3)$ are isotopic through maps keeping boundaries in S^3.

The proof of the main theorem is constructive. An explicit isotopy carrying the initial embedding to a standard one is described. The isotopy consists of a sequence of isotopies, each of which pushes the surface into B^4 and then pulls it out again. At certain times trivial handles are constructed, which are then ignored. To understand why this construction is valid, consider an initial isotopy which pushes F straight into B^4 into a parallel copy of S^3. ∂F bounds a vertical collar to that pushed in copy of F. All the isotopies constructed can be thought of as occurring at this lower S^3 and below, and then extended to the collar. It is easily checked that the trivial handles and bands by the side do not interfere with the construction in any way, and that after each step in the construction they are still trivial.

Received June 5, 1981.
Michigan Math. J. 29 (1982).

2. Basic isotopies. Isotopies used to simplify an embedding of a surface will be constructed as a sequence of more simple isotopies, each of which reduces the complexity of the embedding. The notion of surfaces obtained by "surgering a surface along an arc" will simplify the description of these isotopies. In the first part of this section this notion is defined. Following that definition, Propositions A, B, and C are presented. These propositions describe the basic isotopies which will be used throughout the construction.

Let F denote an embedded surface in S^3 and α an arc meeting F only at its endpoints, where it has transverse intersection. A regular neighborhood of α, $N(\alpha)$, can be chosen so that $N(\alpha)$ is diffeomorphic to $B^2 \times I$, with $\{0\} \times I = \alpha$ and $B^2 \times I \cap F = B^2 \times \{0\} \cup B^2 \times \{1\}$.

DEFINITION. The surface $F' = (F - ((B^2 \times \{0\}) \cup (B^2 \times \{1\}))) \cup S^1 \times I$ is obtained from F by surgering F along α.

Conversely, if F is embedded in S^3 and there is an embedding of $B^2 \times I$ in S^3 with $B^2 \times I \cap F = S^1 \times I$, then F is obtained from

$$F' = (F - (S^1 \times I)) \cup ((B^2 \times \{0\}) \cup (B^2 \times \{1\}))$$

by surgering along the arc $\alpha = \{0\} \times I \subset B^2 \times I$.

PROPOSITION A. *Assume that F' and F'' are obtained by surgering a surface F along arcs α and α', and that there is an isotopy in S^3 carrying α to α', α_s, $0 \leqslant s \leqslant 1$, such that for all s, α_s satisfies the conditions necessary to perform surgery on F along α_s. Then F' and F'' are isotopic in S^3.*

PROPOSITION B. *If F' and F'' are obtained by surgering a surface F along arcs α' and α'', and if α' and α'' agree on $N(\partial\alpha')$, then F' and F'' are isotopic in B^4.*

Proof A. The proof is straightforward. The isotopy carrying α' to α'' extends to a regular neighborhood of α'. Restrict this isotopy to the boundary of that neighborhood to construct the desired isotopy of F' to F''. □

Proof B. There is an isotopy in B^4 carrying α' to α'' fixing $N(\partial\alpha')$ throughout. To construct that isotopy, start with a homotopy α_t of α' to α'' rel $N(\partial\alpha')$ in S^3. Pushing that homotopy into B^4 eliminates the possibility of any unwanted intersections with F. In B^4 the homotopy is easily changed into an isotopy. Now, as in the proof of A, extend that isotopy to one of $B^2 \times I$, and then restrict to $\partial(B^2 \times I)$. □

The setup of the final result of this section is the following. D^2 is an embedded disk in S^3. $\{F_i\}$ is a collection of disjoint embedded surfaces in S^3 meeting $\operatorname{int}(D^2)$ in a collection of circles and $\{\alpha_i\}$ is a collection of disjoint embedded arcs in S^3 meeting D^2 in a collection of points. Assume that there is an embedding β of S^1 in $S^3 - (\{F_i\} \cup \{\alpha_i\})$ meeting D^2 in exactly one point, which is not in the interior of any disk in D^2 bounded by a circle of intersection in $\{F_i\} \cap D^2$.

PROPOSITION C. *There is an isotopy in B^4 carrying $\{F_i\} \cup \{\alpha_i\}$ to an embedding $\{F_i'\} \cup \{\alpha_i'\}$, which fixes the boundaries of the F_i's and α_i's and such that $\{F_i'\} \cup \{\alpha_i'\}$ does not intersect* $\operatorname{int}(D^2)$.

Proof. Let D_1 be a disk in the interior of D^2 such that all intersections of $\{F_i\} \cup \{\alpha_i\}$ with $\mathrm{int}(D^2)$ are in D_1, and $\beta \cap D_1 = \phi$. A regular neighborhood of D_1 is diffeomorphic to $B^2 \times I$. That embedding of $B^2 \times I$ in S^3 can be isotoped in B^4, fixing $B^2 \times \{0,1\}$ to a new embedding in S^3 for which $B^2 \times (\frac{1}{4}, \frac{3}{4}) \subseteq N(\beta) - D^2$ and $B^2 \times [0, \frac{1}{4}] \cup B^2 \times [\frac{3}{4}, 1] \subseteq N(D^2) - D^2$. (This again uses the fact that two arcs in S^3 with the same endpoints are isotopic in B^4, fixing endpoints.) This isotopy carries $\{F_i\} \cup \{\alpha_i\}$ to a new collection, $\{F_i'\} \cup \{\alpha_i'\}$ with the desired properties. □

3. Isotoping surfaces to standard embeddings. We will prove that any surface in S^3 with boundary the unlink is isotopic in B^4 to a standard surface using three steps. We will first show that in the orientable case the problem can be reduced to one of surfaces of genus 0. The problem is then solved for genus 0 surfaces. Finally, we describe the necessary changes in the earlier arguments to have them carry over to the nonorientable case.

We will now consider F to be embedded in $\mathbf{R}^3 = \partial \mathbf{R}^4_+$. Let $\{\partial_i F\}$ be the set of boundary components of F, where i runs over a finite and possibly empty indexing set of integers.

Reduction of Genus in the Orientable Case.

LEMMA 3.1 (General position for knotted surfaces). *If F is an embedded orientable surface in $\mathbf{R}^3$ with boundary the unlink, then F can be isotoped in $\mathbf{R}^3$ so that the function $(x,y,t) \to t$ restricts to a Morse function h on F with each critical value of h corresponding to exactly one critical point. Furthermore it can be assumed that $\partial_i F \subseteq P_i = \{(x,y,t) \mid t = i\}$ and that $h(x)$ is a decreasing function on a collar neighborhood of ∂F, as a function of the distance from x to ∂F.*

Proof. The only difficult point here is that concerning the boundary behavior of h. All that is required to prove this is to note that a pushoff of ∂F_i along F links ∂F_i geometrically 0 times. This is because the pushoff is homologous in the complement of $\partial_i F$ to a collection of curves $\{\partial_j F \mid j \neq i\}$ each of which links $\partial_i F$ algebraically 0 times, and $\partial_i F$ is unknotted. The rest of the lemma follows from standard general position arguments. □

THEOREM 3.2. *Any orientable surface (with standard boundary) in $\mathbf{R}^3$ can be isotoped in $\mathbf{R}^4_+$ to be a genus 0 surface with trivial handles added.*

Proof. Assume that F has genus greater than 0. We reduce the genus of F using the following three steps. The proof then follows immediately from induction on genus(F).

Step 1. Put F in general position as given in Lemma 3.1 with critical and boundary values $0, 2, 4, \ldots, 2N$. Set $F_t = \{x \in F \mid h(x) \leqslant t\}$. Genus$(F_1) = 0$ and genus$(F_{2N+1}) > 0$. Let K be the largest integer so that genus$(F_{2K+1}) = 0$. We will show that $P_{2K+1} \cap F$ contains a nonseparating circle on F. If $2K+2$ were either a maximum, minimum, or boundary level, genus$(F_{2K+3}) =$ genus$(F_{2K+1}) = 0$. Hence $2K+2$ corresponds to saddle point, and F_{2K+3} is constructed from F_{2K+1} by adding a band. If that band joined two distinct components of F_{2K+1} or if it joined some boundary component to itself the genus would remain unchanged. Hence, the band joins two different boundary components of a single component of F_{2K+1}. Each of these boundary components is nonseparating, as the core of the band along with an arc in F_{2K+1} joining the two boundary components together form a closed curve intersecting each boundary component exactly once.

Step 2. After an isotopy in $\mathbf{R}^4_+$ some nonseparating circle on F bounds a disk in $\mathbf{R}^3$ missing F except at its boundary.

On the plane P_{2K+1} used above, take an innermost circle of $P_{2K+1} \cap F$ which is nonseparating on F. Denote that circle by S_0. S_0 bounds a disk D_0 on P_{2K+1}. Let β be a closed curve on F intersecting S_0 once. If $\beta \cap \operatorname{int}(D_0) \neq \phi$, using the fact that all curves in $F \cap \operatorname{int}(D_0)$ are separating, β can be surgered to a curve β' with $\beta' \cap \operatorname{int}(D_0) = \phi$. Now push β' off F so that it intersects D_0 exactly once, and apply Proposition C.

Step 3. F is obtained from a surface of lower genus by adding a trivial handle.

Using a regular neighborhood of the disk D_0 found in Step 2, we see that F is obtained from an F' by surgering along an arc. Slide the basepoints of that arc to be close together on F'. Propositions A and B can now be used to produce an isotopy of F to a surface with a trivial handle. □

COROLLARY 3.3. *Any closed surface in S^3 is isotopic in B^4 to the standard surface.*

COROLLARY 3.4. *Any Seifert surface (orientable) for the unknot is standard in B^4.*

Proof. These follow from the facts that any genus 0 closed surface in S^3 is standard (Schonflies' Theorem [8]) and that any disk in S^3 is standard. Corollary 3.3 also follows from results in [4]. □

Surfaces of Genus 0.

We have reduced the problem of showing that all orientable surfaces bounding the unlink are standard to a proof of that fact for surfaces of genus 0. Hence the orientable case is concluded with:

THEOREM 3.5. *Any genus 0 surface bounding the unlink in S^3 is isotopic in B^4 to a standard surface.*

Proof. Pick some boundary component, $\partial_0 F$, and let it bound a disk D_0 in S^3 with the property that $\operatorname{int}(D_0)$ intersects F in a collection of circles, and F intersects D_0 transversely. The construction that follows will show that F can be isotoped, using B^4, so that $F \cap D_0' = \phi$, where D_0' is another disk bounding $\partial_0 F$. Hence F is constructed from a surface with fewer boundary components by puncturing once. As genus 0 surfaces for the unknot are standard, the proof is completed by induction.

Construction.

Step 1. If any circle of intersection in $\operatorname{int}(D_0) \cap F$ is trivial on F, pick an innermost such circle. A standard cut and paste argument says that the disk on D_0 bounded by that circle can be replaced by the disk bounded on F to yield a new D_0 with fewer circles of intersection. Continue this procedure until all such circles are eliminated.

Step 2. If any circles of intersection remain, using an innermost circle of intersection on D_0 we see that F is obtained by surgering two surfaces together along an arc, where each surface has nonempty boundary. Slide the basepoints of that arc to be near the boundary of each surface.

Step 3. Eliminate any circles of intersection of D_0 with either of the surfaces which are being tubed together if those circles bound disks on the surface, as in Step 1. Consider an innermost circle of intersection on D_0, S_0. We would like to repeat Step 2, but it is now possible that the arc constructed above intersects the disk on D_0 bounded by S_0.

As S_0 separates two boundary components of a surface, there is an arc on that surface running from one of the boundary components to the other which intersects S_0 exactly once. On the boundary of a regular neighborhood of that arc there is a closed curve β which does not intersect any of the surfaces involved and which intersects the disk on D_0 bounded by S_0 in exactly one point. (Construct β by lifting the arc above to the boundary of the regular neighborhood of F in both normal directions and joining the two lifts together with small arcs going around the boundary of F.) The conditions of Proposition C are now met so we can use Proposition C to eliminate the points at which the arc meets the disk. Hence, after an isotopy we can repeat Step 2 to get three surfaces which are tubed together along arcs.

Step 4. Repeat Step 3 until D_0 does not intersect any of the surfaces being tubed together. There are two possibilities.

a. If the surface to which D_0 belongs has more than one boundary component, the procedure used in Step 3 can be repeated to eliminate the points where arcs intersect D_0. In this case, D_0 is a disk in the complement of the tubed together surfaces, and we are done.

b. If D_0 belongs to a surface with only one boundary component, then that surface is a disk. That disk is joined to another of the surfaces by an arc. Considering the surface which results from surgering the surfaces together along that arc, we are in case *a*. above, and the proof is completed. □

Nonorientable Surfaces.

We will now trace through the previous arguments indicating what modifications must be made if the surface under consideration is nonorientable. The key problem is in proving Lemma 3.1. A component of ∂F, $\partial_i F$, when pushed off itself along F might link $\partial_i F$. Hence before applying the general position result we need the following. (This lemma is a slight extension of a similar result given in [1]. Its proof is based on the same construction.)

LEMMA 3.6. *Any surface bounding the unlink is isotopic in B^4 to a surface F', where F' is obtained from a surface F'' bounding the unlink by adding trivial once twisted bands along its boundary. The surface F'' can be arranged to have the property that each of its boundary components does not link itself when pushed off along F.*

Proof. Perform an isotopy so that ∂F lies in the plane P_0, with ∂F bounding a collection of disjoint disks in P_0 and with F meeting P_0 transversely except at isolated points on ∂F. If the linking number of $\partial_i F$ with a copy of $\partial_i F$ pushed off along F is n, then, as these two curves form a $(2,2n)$ torus link, there is an isotopy fixing ∂F such that the pushed off copy of $\partial_i F$ meets the disk in P_0 bounded by $\partial_i F$ in exactly n points. It can hence be arranged that the number of arcs of intersection of F with the disk on P_0 bounded by $\partial_i F$ is exactly half the linking number of that boundary component with itself when pushed off along F.

Cutting F along the arcs interior to the disks shows that F is constructed from a different surface bounding the unlink, F'', by adding once twisted bands along its boundary. Isotoping F'' into B^4 it is possible to slide the bands to the side, and by sliding the bands over each other to make them all trivial. F'' along with the collection of bands forms F'.

. The next modification that must be made is in the proof of Theorem 3.2, in Step 1. The band that is added to F_{2K+1} is now possible nonorientable. If such a band joined some boundary component to itself the genus of F_{2K+3} would be positive. Hence, for the proof of Step 1 to carry over to the nonorientable case we must show that no such bands can occur.

Adding a once twisted band from a boundary component of a surface to itself does not change the total number of boundary components. On the other hand, if F_{2K+3} is constructed from F_{2K+1} by adding a band along a single boundary component, then the set of circles $F \cap P_{2K+3}$ is constructed from $F \cap P_{2K+1}$ by surgering some S^1 in $F \cap P_{2K+1}$ along an arc in P_{2K+1} running from that S^1 to itself. (Surgery of a circle along an arc in the plane is completely analogous to surgery of a surface in $\mathbf{R}^3$ along an arc in $\mathbf{R}^3$.) As any S^1 in the plane is separating, that arc runs from one side of S^1 to the same side. It is evident that this operation increases the number of boundary components.

In Step 2 of the proof of Theorem 3.2 we can no longer use a pushoff of β. Instead, use the two-fold cover of β which lies in the boundary of the normal bundle to F in S^3. The rest of the proof of Theorem 3.2 carries over as before. Only note that some of the trivial handles constructed might be nonorientable.

The results concerning genus 0 surfaces do not depend on orientation. Hence with the modifications mentioned above we have shown that any surface bounding the unlink is isotopic in B^4 to one which is constructed from a standard surface by adding once twisted bands to its boundary. All that remains to be shown is that where the bands are attached does not affect the isotopy class of the surface in B^4. □

LEMMA 3.7. *Let F be a standard surface embedded in S^3 with once twisted trivial bands added along its boundary. F is isotopic in B^4 to a standard surface for which all the twisted bands are added along a single boundary component.*

Proof. Assume F is constructed from a standard surface F_0 by adding a total of m right handed twisted bands and n left handed twisted bands along the boundary components. Let F' be the surface constructed from F_0 by adding the same number of right and left handed twisted bands along a single boundary of F_0. We will now show that F and F' are isotopic in B^4.

After a small isotopy in S^3 it can be arranged that the boundaries of F and F' are identical. Now push each surface into B^4 so that the cross sections of each are identical to the level t_0 at which point the remainder of each surface appears, each having one boundary component. (These cross sections correspond to performing the obvious moves joining the boundary components of each surface.) The surfaces at the level t_0 are isotopic in S^3, with boundary fixed. Hence, on pushing the surfaces into B^4 and performing an isotopy it can be arranged that all cross sections are identical. It follows that the original embeddings were isotopic in B^4. □

4. Main classification theorem. Let us summarize what we have shown up to this point: *Any surface bounding the unlink S^3 is isotopic in B^4 to a standard surface.* The classification of surfaces bounding the unlink is hence reduced to the classification of standard surfaces up to isotopy in B^4. A final definition is needed.

DEFINITION. If F is embedded in S^3, denote by c_i the linking number of $\partial_i F$ with itself pushed off along F. Define $c(F) = \sum c_i$.

LEMMA 4.1. *If F_1 and F_2 are isotopic in B^4 then $c(F_1) = c(F_2)$.*

Proof. In the next section it will be shown that $c(F)/2$ is the signature of the 2-fold branched cover of B^4 with branching set a copy of F pushed into B^4, and hence is invariant under isotopy in B^4. □

THEOREM 4.2 (Classification of surfaces bounding the unlink).

a) *Two surfaces, F_1 and F_2, bounding the unlink are isotopic if and only if F_1 and F_2 are homeomorphic and $c(F_1) = c(F_2)$.*

b) *If F is orientable $c(F) = 0$. If F is nonorientable $c(F)/2 = 2(\text{genus}(F)) \bmod(2)$ and $|c(F)/2| \leqslant 2(\text{genus}(F))$.*

c) *For any surface F and any even integer c satisfying the conditions given in* b) *above for $c(F)$ there is an embedding of F in S^3 with boundary the unlink and with $c(F) = c$.*

Proof. By the above arguments we can assume F_1 and F_2 are standard. Furthermore, we can assume that all twisted bands twist in the same direction. If there is a pair of oppositely twisted bands, by sliding them next to each other and then one over the other they can be combined to give a trivial nonorientable handle.

a) If F_1 and F_2 are isotopic they are certainly homeomorphic, and by Lemma 4.1 $c(F_1) = c(F_2)$. Conversely, $c(F_i)/2$ is the number of twisted bands. Furthermore, if the surfaces are nonorientable it can be assumed all handles are nonorientable. (Slide one basepoint on an orientable handle around an orientation reversing path to make the handle nonorientable.) Result a) is then immediate.

b) In the proof of Lemma 3.1 it was shown that $c(F)$ is 0 for orientable surfaces. If F is nonorientable then the number of twisted bands in the standardly presented F is $|c(F)/2|$. However, the number of twisted bands is also $2(\text{genus}(F)) - 2(\text{number of trivial handles})$. With these observations the conditions given in b) follow immediately.

c) To construct such an embedding add $c/2$ twisted bands to a disk embedded in S^3. Add $(\text{genus}(F) - c/4)$ trivial handles and puncture the surface to get the correct number of boundary components. □

5. Branched covers. In their paper on branched covers [1] Akbulut and Kirby prove, as Corollary 4.2: "Let P^4 be the 4-manifold constructed by plumbing on a graph. Suppose ∂P^4 is a homotopy 3-sphere. Then P^4 is diffeomorphic to

$(\#k(S^2\times S^2)) - \mathrm{int}(B^4)$ or to $(\#k(CP^2)\#1(-CP^2)) - \mathrm{int}(B^4)$; the former occurs when the weights of the framings on the graph are all even, the latter when some are odd". (A different proof is given in [9].) The proof of this (in [1]) shows that P^4 is the double branched cover of B^4 branched over a pushed in surface bounding the unknot in S^3. Then, they complete the proof using Theorem 4.3 of the same paper: "If F [a surface for the unknot] is orientable, the Q, [the p-fold branched cover of B^4 branched over F union a 4-ball] is diffeomorphic to $\#g(p-1)(S^2\times S^2)$ for $g=$ genus F. If F is not orientable, then (for $p=2$) $Q=\#p(CP^2)\#q(-CP^2)$, for $p+q=\mathrm{rank}(H_1(F;Z_2))$".

The geometric explanation for this is contained in Theorem 4.2 of this paper. The surfaces become standard when pushed into B^4. Using Theorem 4.2 we can generalize the result to surfaces with boundary the unlink.

Let F be a surface with $\mathrm{rank}(H_1(F;Z_2)) = m$ and $\#(\partial F) = n$. If F is embedded in S^3 with boundary the unlink, denote the p-fold cover of B^4 branched over F by M_p. Denote by $(S^2\times S^2)^0$ the space $S^2\times S^2-\mathrm{int}(B^4)$.

THEOREM 5.1. *If F is orientable,*

$$M_p = (\#(m-n+1)/2)(S^2\times S^2)^0\#(p-1)(n-1)(S^2\times B^2).$$

If F is not orientable,

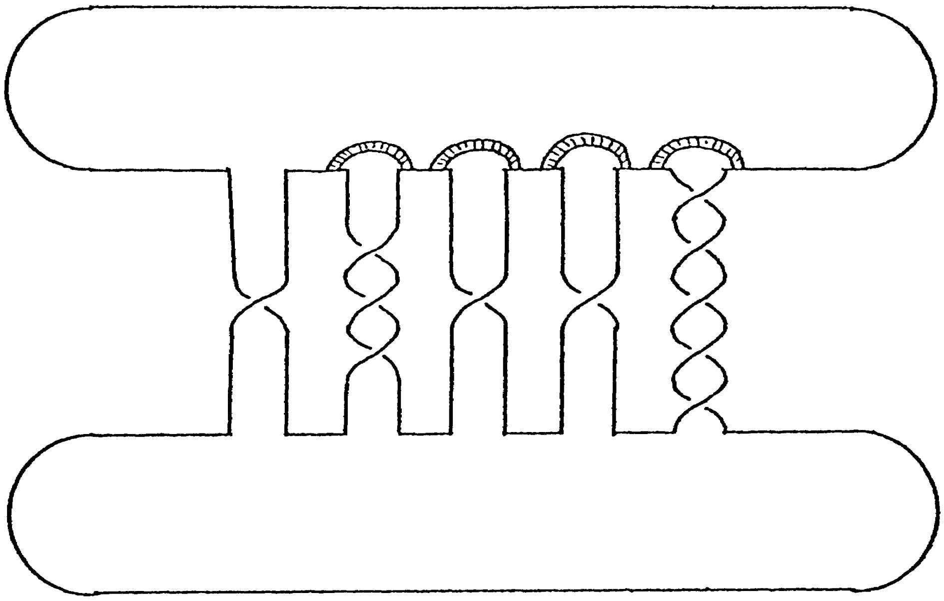
$$M_2 = ((m-n+1-(c(F)/2))/2)(CP^2\#(-CP^2))$$
$$\#(c(F)/2(CP^2))^0\#(n-1)(S^2\times B^2).$$

Proof. This result follows immediately from the techniques of [1], as it is sufficient to find the branched covers of B^4 branched over standard surfaces. □

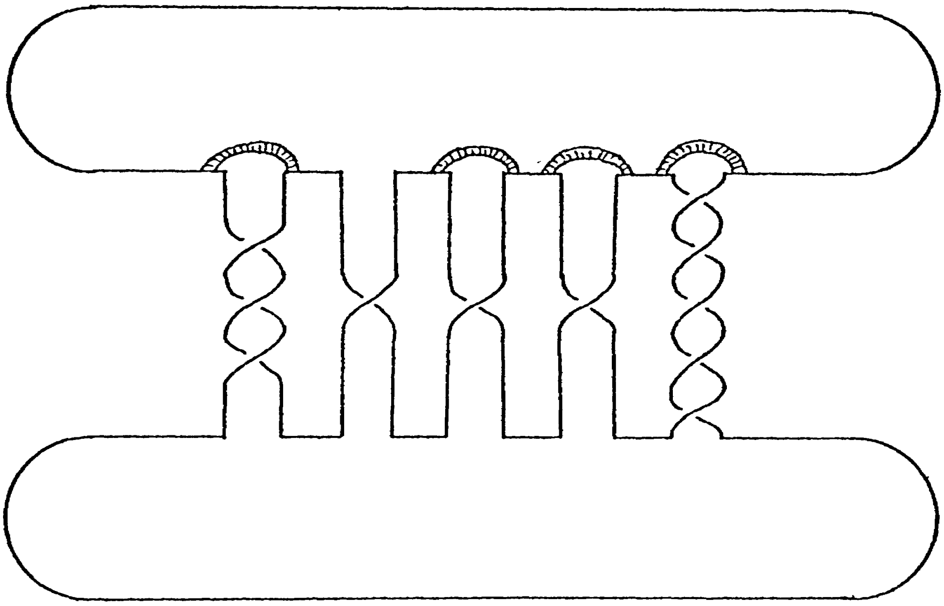
6. ∂F is knotted. Many authors have constructed examples of distinct incompressible surfaces for a given knot ([2, 3, 7, 10]). We can ask the question whether two surfaces bounding the same knot are isotopic when pushed into B^4, given that they are topologically the same. One interesting aspect of this question relates to 4-manifolds. By constructing branched covers of B^4 over pushed in Seifert surfaces for a knot one gets 4-manifolds having the same boundary which are hard to distinguish algebraically.

The conjecture that all surfaces of a given genus bounding a fixed knot are isotopic in B^4 is unlikely. However, it is interesting to note that in many of the examples of knots with distinct Seifert surfaces, the surfaces are easily shown to be isotopic in B^4. One case in which it is more difficult to see that the surfaces are isotopic in B^4 is in the case of pretzel knots described by Trotter [10]. We will now show that for the pretzel knots having distinct Seifert surfaces described by Trotter, the surfaces are isotopic in B^4. Whether or not these surfaces are isotopic is question 1.20C of [5].

Figure 1 illustrates two surfaces, F_1 and F_2. In [10] Trotter illustrates an isotopy carrying ∂F_1 to ∂F_2. He also proves that F_1 and F_2 are not isotopic in S^3. In isotoping ∂F_1 to ∂F_2 it is difficult to keep track of the image of F_1. However, it is relatively easy to keep track of the bands on F_1 illustrated in Figure 1, and to show that the isotopy can be arranged to carry them to the illustrated bands on F_2.



F_1



F_2

FIGURE 1

After performing such an isotopy it is possible to push the surfaces into B^4 so that the cross sections are identical to some level t_0 at which level the remainder of each surface appears with boundary the unknot. (These cross sections correspond to performing band moves along the illustrated bands.) In each case the remaining part of the surface is a disk. As any two disks embedded in S^3 with the same boundary are isotopic keeping the boundary fixed, it is now clear that the original surfaces were isotopic in B^4. This procedure works in general for the examples constructed by Trotter.

REFERENCES

1. S. Akbulut and R. Kirby, *Branched covers of surfaces in 4-manifolds,* preprint.
2. W. R. Alford, *Complements of minimal spanning surfaces of knots are not unique.* Ann. of Math. (2) 91 (1970), 419–424.
3. R. J. Daigle, *More on complements of minimal spanning surfaces.* Rocky Mountain J. Math. 3 (1973), 473–482.
4. F. Hosokawa and A. Kawauchi, *Proposals for unknotted surfaces in four-spaces.* Osaka J. Math. 16 (1979), 233–448.
5. R. Kirby, *Problems in low dimensional manifold theory.* Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford Calif., 1976), part 2, pp. 85–97, XXXII, Amer. Math. Soc., Providence, R.I., 1978.
6. C. Livingston, *Knotted surfaces in 4-space,* Dissertation, Department of Math., Univ. of California, Berkeley, Calif., 1980.
7. H. C. Lyon, *Simple knots without unique minimal surfaces.* Proc. Amer. Math. Soc. 43 (1974), 449–454.
8. E. Moise, *Geometric topology in dimensions* 2 *and* 3. Springer, New York, 1977.
9. W. Newmann and S. Weinbraub, *4-manifolds constructed by plumbing,* preprint.
10. H. F. Trotter, *Some knots spanned by more than one knotted surface of minimal genus.* Knots, groups, and 3-manifolds, pp. 51–62. Ann. of Math. Studies, No. 84, Princeton Univ. Press, Princeton, N. J., 1975.

Department of Mathematics
Indiana University
Bloomington, Indiana 47405