PROJECTIVE MODULES OVER KRULL SEMIGROUP RINGS

Leo G. Chouinard II

1. Introduction and notation. Since the remarkable proofs of Serre's conjecture by Quillen and Suslin several years ago ([8], [9]), a number of papers have appeared generalizing the results there to other coefficient rings (e.g. [2], [7]). However, pursuing the question in a different direction suggested by the work of Anderson [1], we inquire into the class of semigroups which can replace the free abelian semigroup of monomials in the polynomial ring. Our result, strangely enough, is that the essence of Horrocks' Theorem, on which Quillen's proof relies, can be reformulated for rings arising from Krull semigroups with torsion divisor class group [3]. Using this, we show that if S is such a semigroup and A is a ring such that all finitely generated $A[G]$-projectives are extended from A whenever G is a free abelian group, then every finitely generated $A[S]$-projective is extended from A. In particular, this holds if A is a Dedekind domain by the Suslin–Swan observation on the Quillen–Suslin result ([9], [10]).

All rings are commutative with unit unless otherwise indicated. If A is a ring and M is an A-module, we let $M^{\sim}$ denote the quasi-coherent sheaf over Spec(A) corresponding to M. Where not specified, we use [5] as our source of results and notation on algebraic geometry. If $S \to A$ is a morphism of rings, we say M is extended from S if there exists an S-module N such that $M \cong A \otimes_S N$. If A is a local ring with maximal ideal m, then $M^{\wedge}$ denotes the m-adic completion of M.

All semigroups are commutative, cancellative with unit. Furthermore, we assume our semigroups have torsion-free total quotient group; if S is a semigroup, we denote its total quotient group by $\langle S \rangle$. For the notion of a Krull semigroup and its divisor class group and essential valuations, we refer to [3]. $\mathbf{Z}$ denotes the group of integers, and we use $\bigoplus_{\alpha \in I} \mathbf{Z} x_\alpha$ for a free group with specified bases $\{x_\alpha \mid \alpha \in I\}$. If F is this group with basis, F_+ denotes the subsemigroup of elements of F such that all of the x_α have nonnegative coefficients. Other semigroups are written multiplicatively.

Where our proofs are similar to those in [6] and [8], they are briefly sketched, with more careful attention paid to the significant differences.

2. Preliminaries. We first prove some results on Krull semigroups which we will need later.

LEMMA 2.1. *Let S be a Krull semigroup with torsion divisor class group, $\{v_\alpha \mid \alpha \in I\}$ the set of essential valuations of S, and $\beta \in I$. Then there exists a $t \in S$ such that $v_\beta(t) > 0$ but $v_\alpha(t) = 0$ for all $\alpha \in I$ with $\alpha \neq \beta$.*

Proof. By the proof of Proposition 1 in [3], and Theorem 2 of the same paper, the map $\psi : \langle S \rangle \to F = \bigoplus_{\alpha \in I} \mathbf{Z} x_\alpha$ defined by $\psi(s) = \sum v_\alpha(s) x_\alpha$ satisfies $S = \psi^{-1}(F_+)$ and $\mathrm{Cl}(S) \cong F/\mathrm{im}(\psi)$, so since $\mathrm{Cl}(S)$ is torsion, $n x_\beta \in \mathrm{im}\, \psi$ for some $n > 0$. Pick $t \in \psi^{-1}(n x_\beta)$.

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LEMMA 2.2. *Let S, I, β, and t be as in Lemma* 2.1, *and suppose I is finite. Let $T=\{t^n \mid n\in\mathbf{Z}\}$, $S_0=\{s\in S \mid v_\beta(s)=0\}$, and $\tilde{S}=TS$. Then S_0 and $\tilde{S}$ are Krull semigroups with torsion divisor class group and one less essential valuation than S.*

Proof. Let $e_\beta : F=\bigoplus_{\alpha\in I}\mathbf{Z}x_\alpha \to \bar{F}=\bigoplus_{\alpha\in J}\mathbf{Z}x_\alpha$, where $J=\{\alpha\in I \mid \alpha\neq\beta\}$, be the natural projection. Then if $s\in\langle S\rangle$, $s\in\tilde{S}$ if and only if $t^n s\in S$ for some $n\in\mathbf{Z}$, if and only if $v_\alpha(s)\geq 0$ for all $\alpha\neq\beta$, if and only if $(e_\beta\circ\psi)(s)\in\bar{F}_+$ (with ψ as above) so $\tilde{S}$ is Krull. Moreover, $(e_\beta\circ\psi)(\tilde{S})$ is reduced in $\bar{F}$ since $\psi(S)$ is reduced in F, so Lemma 2.1 applied to the other valuations on S implies that condition (2) of Theorem 2 in [3] is satisfied, so the valuations resulting from the projections onto the factors of $\bar{F}$ are the essential valuations of $\tilde{S}$. Also, $\mathrm{Cl}(\tilde{S})\cong\bar{F}/((e_\beta\circ\psi)(\langle S\rangle))$ is clearly torsion.

Likewise, if $s\in\langle S_0\rangle$, then $v_\beta(s)=0$ so $s\in S_0$ if and only if $v_\alpha(s)\geq 0$ for all $\alpha\neq\beta$, if and only if $(e_\beta\circ\psi)(s)\in F_+$, so S_0 is Krull. Furthermore, all of the hypotheses of (2) in Theorem 2 of [3] again hold for $(e_\beta\circ\psi)(S_0)\subseteq\bar{F}_+$, except that this subsemigroup may not be reduced. But replacing each basis element of $\bar{F}$ by the smallest positive multiple appearing as a coefficient of it in $(e_\beta\circ\psi)(S_0)$, we resurrect the reduced condition, and the resulting class group is clearly isomorphic to a subgroup of $\mathrm{Cl}(S)$ via a map induced by the obvious inclusion from $\bar{F}$ into F.

REMARK. The above lemma can be generalized to any localization of any Krull semigroup S, or any subsemigroup of S of the form $S\cap H$, where H is a subgroup of $\langle S\rangle$. In particular, any localization of a Krull semigroup is a Krull semigroup, with essential valuations precisely the essential valuations of the original semigroup which are trivial on each element inverted. The resulting class group is then a homomorphic image of the original class group.

3. Horrocks' Theorem for Krull semigroups. Let A be a commutative Noetherian ring, and let S be a finitely generated Krull semigroup with $\mathrm{Cl}(S)$ torsion, and distinct essential valuations $v=v_1,\ldots,v_m$. We make $A[S]$ into a positively graded ring via $A[S]_n=A[S_n]=\bigoplus_{s\in S_n}As$ where $S_n=\{s\in S \mid v(s)=n\}$. Let $t\in S$ satisfy $v(t)>0$ but $v_i(t)=0$ for $i\geq 2$ by Lemma 2.1, and suppose that t has been chosen to minimize $d=v(t)$ among such elements. Let $\tilde{S}$ be as in Lemma 2.2, and note that $A[\tilde{S}]$ is a $\mathbf{Z}$-graded ring setting $\tilde{S}_n=\{s\in\tilde{S} \mid v(s)=n\}$ and $A[\tilde{S}]_n=A[\tilde{S}_n]$. We note $\tilde{S}_n=S_n$ if $n\geq 0$, and $S_-=\bigcup_{n\leq 0}\tilde{S}_n$ is also a finitely generated Krull semigroup with essential valuations $(-v_1),v_2,\ldots,v_m$, and $\mathrm{Cl}(S_-)\cong\mathrm{Cl}(S)$ (that S_- is finitely generated follows from Remark 1 at the end of Section 2 of [3]).

Let m be a prime ideal of $A[S_0]$ containing all of the non-invertible elements of S_0, and let $\Lambda=A[\tilde{S}]\otimes_{A[S_0]}A[S_0]_m$, with the induced grading from $A[\tilde{S}]$, so that $\Lambda_+=\bigoplus_{i\geq 0}\Lambda_i=A[S]\otimes_{A[S_0]}A[S_0]_m$ and $\Lambda_-=\bigoplus_{i\leq 0}\Lambda_i=A[S_-]\otimes_{A[S_0]}A[S_0]_m$. Note $\Lambda=\Lambda_+[t^{-1}]=\Lambda_-[t]$, so we may form a scheme $Y=\mathrm{Spec}(\Lambda_+)\cup\mathrm{Spec}(\Lambda_-)$ where we identify the subspaces corresponding to $\mathrm{Spec}(\Lambda)$. (Note: Y can also be realized as $\mathrm{Proj}(\Lambda_+[T])$, with Λ_+ graded as above and T assigned degree 1.) Let $k=(A[S_0]/m)_m\cong\Lambda_0/m\Lambda_0$.

LEMMA 3.1. *Let $\mathfrak{M}=\sqrt{m\Lambda}$. Then $\mathfrak{M}$ is a prime ideal of Λ.*

Proof. Let $m_1=\bigoplus_{s\in S_0\cap m}As\subseteq m$. Then if $s\in\tilde{S}$, clearly $s\in\sqrt{m_1A[\tilde{S}]}$ if and only if $v_i(s)>0$ for some $i\geq 2$. This implies $A[\tilde{S}]/\sqrt{m_1A[\tilde{S}]}\cong A[G_0][\bar{t},\bar{t}^{-1}]$ where

$G_0 = S_0 \backslash m$ is a free group and $\bar{t}$ is the image of t. Thus $\sqrt{mA[\tilde{S}]}$ is the inverse image in $A[\tilde{S}]$ of the prime ideal $mA[G_0][\bar{t}, \bar{t}^{-1}]$, so $\sqrt{mA[\tilde{S}]}$ is prime in $A[\tilde{S}]$. Localizing, $\sqrt{m\Lambda}$ is a prime ideal of Λ. □

COROLLARY 3.2. $\Lambda/\mathfrak{M} \cong k[\bar{t}, \bar{t}^{-1}]$, *with* $\bar{t}$ *in degree* d.

LEMMA 3.3. $\sqrt{t\Lambda_+} = \bigoplus_{i \geqslant 0} \Lambda_i$ *and* $\sqrt{t^{-1}\Lambda_-} = \bigoplus_{i<0} \Lambda_i$.

Proof. The results are true with Λ_+ and Λ_- replaced by $A[S]$ and $A[S_-]$ respectively, and localize to Λ_+ and Λ_-. □

LEMMA 3.4. *The natural map* $Y \to \operatorname{Spec}(\Lambda_0)$ *is proper, and* Y *is a separated scheme.*

Proof. This is a straightforward exercise using Lemma 3.3 and the valuative criterions for separatedness and properness ([5], Chapter II, Theorems 4.3 and 4.7). □

If M is a graded Λ-module, we can use it to define a quasi-coherent sheaf $\mathcal{S}(M)$ of $\mathcal{O}_Y$-modules as follows: Let $\mathcal{S}(M)|_{\operatorname{Spec}(\Lambda_+)} = M_+^{\sim}$ and $\mathcal{S}(M)|_{\operatorname{Spec}(\Lambda_-)} = M_-^{\sim}$, where $M_+ = \bigoplus_{i \geqslant 0} M_i$ and $M_- = \bigoplus_{i \leqslant 0} M_i$, patching these together on $\operatorname{Spec}(\Lambda)$ using the natural isomorphisms $M_+[t^{-1}] \cong M_-[t]$. If $j \in \mathbf{Z}$, let $\Sigma_j M$ denote the suspension of M by degree j, i.e. $(\Sigma_j M)_n = M_{n+j}$. We are now ready to prove our version of Horrocks' Theorem ([6], Theorem 1).

THEOREM 3.5. *If* P *is a finitely generated* Λ_+*-module, then* P *is free if and only if* $P^{\sim}$ *extends to a locally free sheaf of* $\mathcal{O}_Y$*-modules.*

Proof. "Only if" is obvious, so let $\mathcal{G}$ be a locally free sheaf of $\mathcal{O}_Y$-modules extending $P^{\sim}$. Now $\mathcal{G} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y/\mathcal{S}(\mathfrak{M})$ is a locally free sheaf on

$$Y_k = \operatorname{Spec}(\Lambda_+/\mathfrak{M} \cap \Lambda_+) \cup \operatorname{Spec}(\Lambda_-/\mathfrak{M} \cap \Lambda_-)$$

identifying the subspaces corresponding to $\operatorname{Spec}(\Lambda/\mathfrak{M})$, but Corollary 3.2 implies $Y_k \cong \mathbf{P}_k^1$, the projective line over k. Thus $\mathcal{G} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y/\mathcal{S}(\mathfrak{M}) \cong \bigoplus r_n \mathcal{O}_{\mathbf{P}_k^1}(n)$ for some nonnegative integers r_n, not all 0 (if $P \neq 0$). Replacing $\mathcal{G}$ by the sheaf with the same sections on $U_1 = \operatorname{Spec}(\Lambda_+)$ and $U_2 = \operatorname{Spec}(\Lambda_-)$, but with the isomorphism $\psi : \mathcal{G}(U_1)[t^{-1}] \to \mathcal{G}(U_2)[t]$ replaced by $t^{-j}\psi$, we get a sheaf $\mathcal{G}(j)$ which satisfies our hypotheses and has $\mathcal{G}(j) \otimes_{\mathcal{O}_Y} \mathcal{O}_Y/\mathcal{S}(\mathfrak{M}) \cong \bigoplus r_n \mathcal{O}_{\mathbf{P}_k^1}(n+j)$, so we may assume $r_0 > 0$ and $r_i = 0$ if $i < 0$. □

Note $\Gamma(Y, \mathcal{G})$ is a Λ_0-module, so it carries a topology induced by the powers of m. By Proposition 4.2.1 in Chapter III of EGA [4] applied to the map $Y \to \operatorname{Spec}(\Lambda_0)$, we get $\Gamma(Y, \mathcal{G})^{\wedge} \cong \varprojlim_i \Gamma(Y, \mathcal{G} \otimes_{\mathcal{O}_Y} \mathcal{S}(\Lambda/m^i\Lambda))$. We wish to show the following:

LEMMA 3.6. *Let* $0 \to N \to M \to M' \to 0$ *be an exact sequence of graded* Λ*-modules, such that* N *is finitely generated and annihilates some power of* $\mathfrak{M}$. *Then* $\Gamma(Y, \mathcal{G} \otimes_{\mathcal{O}_Y} \mathcal{S}(M)) \to \Gamma(Y, \mathcal{G} \otimes_{\mathcal{O}_Y} \mathcal{S}(M'))$ *is a surjection.*

Proof. By induction on the length of N as a graded Λ-module, we may assume $N \cong \Sigma_j \Lambda/\mathfrak{M}$ for some j, and it clearly does no harm to assume $0 \leqslant j < d$. If $j = 0$, $\mathcal{S}(N) \cong \mathcal{O}_Y/\mathcal{S}(\mathfrak{M})$, while if $0 < j < d$, $\mathcal{S}(N) \cong (\mathcal{O}_Y/\mathcal{S}(\mathfrak{M}))(-1)$. Thus in the first case, $\mathcal{G}\otimes_{\mathcal{O}_Y}\mathcal{S}(N)\cong\bigoplus r_n\mathcal{O}_{\mathbf{P}^1_k}(n)$, while in the second case

$$\mathcal{G}\otimes_{\mathcal{O}_Y}\mathcal{S}(N)\cong\bigoplus r_n\mathcal{O}_{\mathbf{P}^1_k}(n-1).$$

So we have an exact sequence

$$\begin{aligned}\Gamma(Y,\mathcal{G}\otimes_{\mathcal{O}_Y}\mathcal{S}(M))\to\Gamma(Y,\mathcal{G}\otimes_{\mathcal{O}_Y}\mathcal{S}(M'))&\to H^1(Y,\mathcal{G}\otimes_{\mathcal{O}_Y}\mathcal{S}(N))\\&=H^1(\mathbf{P}^1_k,\bigoplus r_n'\mathcal{O}_{\mathbf{P}^1_k}(n))\end{aligned}$$

where $r_n'=0$ if $n<-1$. But $H^1(\mathbf{P}^1_k,\mathcal{O}_{\mathbf{P}^1_k}(n))=0$ if $n\geqslant -1$, and the lemma follows. □

From the lemma, the composite

$$\Gamma(Y,\mathcal{G})\to\Gamma(Y,\mathcal{G})^{\wedge}\to\Gamma(Y,\mathcal{G}\otimes_{\mathcal{O}_Y}\mathcal{S}(\Lambda/m\Lambda))\to\Gamma(Y,\mathcal{G}\otimes_{\mathcal{O}_Y}\mathcal{S}(\Lambda/\mathfrak{M}))$$

is surjective, and the last object can be rewritten as

$$\Gamma(Y_k,\mathcal{O}_{\mathbf{P}^1_k})\bigoplus\Gamma(Y_k,(r_0-1)\mathcal{O}_{\mathbf{P}^1_k})\bigoplus\Gamma\Big(Y_k,\bigoplus_{n\geqslant 1}r_n\mathcal{O}_{\mathbf{P}^1_k}(n)\Big).$$

Let $\bar{s}$ be a nonzero element of the first summand, and let s be an element of $\Gamma(Y,\mathcal{G})$ which maps to $\bar{s}$. By induction on the rank of P, we need only show that s is basic. It suffices to show this at the closed points of Y, and it follows from the above that s is basic on Y_k since $\bar{s}$ is. But if y is a closed point of Y, y maps to m in $\operatorname{Spec}(\Lambda_0)$ since $Y\to\operatorname{Spec}(\Lambda_0)$ is proper. Thus $y\in Y_k$.

4. Projective modules over semigroup rings. The proof of our main theorem now proceeds along the basic lines covered by Quillen [8].

DEFINITION. Let Γ be a positively graded A-algebra with unit (not necessarily commutative), $\theta=\sum_{i=0}^n\theta_i\in\Gamma$ where $\theta_i\in\Gamma_i$, and $g\in A$. Define $\theta(g)=\sum_{i=0}^\infty g^i\theta_i$ (where $g^0=1$).

With this definition, Lemma 1 of [8] holds true with $R[T]$ replaced by Γ and $TR[T]$ replaced by $\bigoplus_{i>0}\Gamma_i$. Theorem 1 there becomes:

LEMMA 4.1. *If Δ is a positively graded Δ_0-algebra and M is a finitely presented Δ-module such that M_m is extended from $(\Delta_0)_m$ for every maximal ideal m of Δ_0, then M is extended from Δ_0.*

The only real change needed in Quillen's proof is in the second displayed equation, where if $N=(\Delta/\bigoplus_{i>0}\Delta_i)\otimes_\Delta M$, we have

$$\operatorname{Hom}_{\Delta_f}(\Delta_f\otimes_{\Delta_0}N,\Delta_f\otimes_{\Delta_0}N)=\operatorname{Hom}_\Delta(\Delta\otimes_{\Delta_0}N,\Delta\otimes_{\Delta_0}N)\otimes_{\Delta_0}(\Delta_0)_f$$

and our graded algebra for applying Quillen's Lemma 1 is $\operatorname{End}_\Delta(\Delta\otimes_{\Delta_0}N)$ instead of $(\operatorname{End}_A(N))[T]$.

LEMMA 4.2. *Let M be a finitely generated projective $A[S]$-module, with S as in Section 3. M is the restriction of a vector bundle over $X=\operatorname{Spec}(A[S])\cup\operatorname{Spec}(AS_-])$ identifying the subspaces corresponding to $\operatorname{Spec}(A[\tilde{S}])$ if and only if M is extended from $A[S_0]$.*

Proof. If M is extended from $A[S_0]$, say $M \cong A[S] \otimes_{A[S_0]} N$, then the $\mathcal{O}_X$-module induced from $N^{\sim}$ by the map $X \to \operatorname{Spec}(A[S_0]$ restricts to $M^{\sim}$ on $\operatorname{Spec}(A[S])$ (note $N \cong M/(\bigoplus_{i>0} A[S_i]M)$ is a projective $A[S_0]$-module). Conversely, if there is a vector bundle $\mathfrak{F}$ on X which restricts to M, we may use Lemma 4.1 to reduce to a localization of $A[S]$ by a prime m in $A[S_0]$. First replace S by its localization by all elements in $S_0 \backslash m$, obtaining a (possibly) larger finitely generated Krull semigroup with torsion divisor class group, by the remark after Lemma 2.2. Then apply Theorem 3.5. □

COROLLARY 4.3. *If M is a finitely generated projective $A[S]$-module, with S as above, and $A[\tilde{S}] \otimes_{A[S]} M$ is extended from $A[S_0]$, then M is extended from $A[S_0]$.*

THEOREM 4.4. *If A is a Dedekind domain, and S is a finitely generated Krull semigroup with torsion divisor class group, then every finitely generated projective $A[S]$-module is extended from A.*

Proof. Use induction on the number of essential valuations of S. If there are no essential valuations, then S is a free group and the result holds by the Suslin–Swan observation on the Quillen–Suslin result ([9] and [10]). Otherwise, let $T, \tilde{S}$ be as before. By Lemma 2.2, $\tilde{S}$ has one less essential valuation than S, so by induction $A[\tilde{S}] \otimes_{A[S]} M$ is extended from A, so M is extended from $A[S_0]$, say $M \cong A[S] \otimes_{A[S_0]} N$. Now N is a finitely generated projective $A[S_0]$-module, but S_0 is a Krull semigroup with one less essential valuation than S, so by induction N, and thus M, is extended from A.

REMARK 1. Note that the result holds as long as projective modules over $A[G]$ are extended from A if G is a free group with $\operatorname{rank}_G = \operatorname{rank}\langle S \rangle$.

REMARK 2. Using Proposition 1 and Remark 1 from [3], and arguments present in the proof of Lemma 2.2, we can show that if S is any Krull semigroup with torsion divisor class group, then S is a directed union of finitely generated Krull semigroups with torsion divisor class groups. This enables us to drop the assumption that S is finitely generated from Theorem 4.4.

We close with two questions raised by the above.

QUESTION 1. Is the assumption that the class group be torsion really necessary? Anderson [1] has conjectured that $K[S]$-projective modules should be free whenever K is a field and S is a finitely generated semigroup such that $K[S]$ is a Krull domain and only the identity of S is invertible. The simplest case not covered by the above is the ring $K[XY, YZ, ZW, WX]$ $(\subseteq K[X, Y, Z, W])$, which has class group $\cong \mathbf{Z}$.

QUESTION 2. What is the widest class of semigroups S such that $K[S]$-projectives are free when K is a field (or *PID*)? In the same paper ([1], Example 5.2), Anderson shows that a semigroup ring need not even be Krull to have this property.

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Department of Mathematics and Statistics
The University of Nebraska-Lincoln
Lincoln, Nebraska 68588