EXOTIC SPHERES AS STATIONARY SETS OF HOMOTOPY SPHERE INVOLUTIONS

Reinhard Schultz

This short paper applies the techniques developed by K. H. Dovermann in [5] to answer affirmatively the following question: Given a smooth homotopy sphere Σ^n, is there a smooth involution on another homotopy sphere T^{n+m} $(m>0)$ with Σ as its fixed point set? One may view this as a partial converse to the P. A. Smith Theorem [3, Chapter III] in the smooth category. Here is the specific result:

THEOREM. *Let Σ^n be a homotopy sphere. Then there is a homotopy sphere T^{2n} with a smooth involution having Σ as its fixed point set. The Pontrjagin-Thom invariants are related by $p(T)=p(\Sigma)^2$ for suitable framings of Σ and T.*

A similar result for p odd has been proved by P. Löffler, who also treated the case of involutions in case n is even (compare [7] and [8]). Our proof relies on Löffler's method together with Dovermann's machinery.

Proof of Theorem. We asume $n>2$ since the other cases are obvious. Consider a framing of Σ with zero Hopf invariant mod 2 (such exists by [1] and [6] plus the surjectivity of the J-homomorphism if $n=1,3,7$). It follows that $\Sigma\times\Sigma$ with the twist involution is a $\mathbf{Z}_2$-stably framed manifold in the sense of G. Segal [10]. Suppose we consider

$$U^{2n}=\Sigma^n\times\Sigma^n\#_{\mathbf{Z}_2}-S^n\times S^n,$$

where S^n has the usual (trivial) framing and the connected sum is done equivariantly along the fixed point set. Then we have an equivariant degree 1 collapsing map $f\colon U\longrightarrow S(\mathbf{R}[\mathbf{Z}_2]^n\oplus\mathbf{R})$ (S denotes the unit sphere) which is already a homotopy equivalence along the fixed point set. By frameability this is a normal map upon which surgery may be considered as in [5].

Case 1. If n is even, then the surgery obstruction is the multisignature of U in $\mathbf{R}[\mathbf{Z}_2]$ by [5, Theorem 2.1]. This is given by the ordinary signature of U and the so-called involution signature. The former vanishes by the Hirzebruch theorem and stable parallelizability. The latter is completely determined by the normal bundle of the involution's fixed point set [2, Section 6], which is trivial by construction (this is why we needed to add $-S^n\times S^n$). Therefore we can do surgery on U as desired to get T^{2n}. By construction, T and Σ^2 are framed bordant, and hence $p(T)=p(\Sigma^2)=p(\Sigma)^2$.

Case 2. If n is odd, then the surgery obstruction is given in two stages; first, there is the ordinary Kervaire invariant of the problem, and if this vanishes there is a $\mathbf{Z}_2$-valued obstruction $\mathrm{rank}_\Lambda\, K_n(f)\otimes\mathbf{Z}_2$ mod 2, where $\Lambda=\mathbf{Z}_2[\mathbf{Z}_2]$ (see [5, Theorem 4.1]). We claim both vanish in our particular situation.

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Since $p(U)=p(\Sigma)^2$ and $p(\Sigma)$ has filtration greater than or equal to 2 in the ordinary $\mathbf{Z}_2$ Adams spectral sequence for the stable homotopy of spheres, the vanishing of the ordinary Kervaire invariant follows immediately from the work of W. Browder and filtration considerations [4]. On the other hand, f is already highly connected by construction, and it follows immediately from the construction that

$$K_n(f) \otimes \mathbf{Z}_2 = H_n(U; \mathbf{Z}_2) = \Lambda \oplus \Lambda$$

(as Λ-modules). Hence the mod 2 rank obstruction also vanishes. This surgery can be done, and $p(T)=p(\Sigma)^2$ follows as in the case where n is even.

REMARK. In most instances one can show directly that the Kervaire invariant vanishes because $\Sigma \times \Sigma$ and $S^n \times S^n$ are almost diffeomorphic (compare [9]).

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Department of Mathematics
University of Chicago
Chicago, Illinois 60637

Current address:
Department of Mathematics
Purdue University
West Lafayette, Indiana 47907