BOUNDED PROJECTIONS AND THE GROWTH OF HARMONIC CONJUGATES IN THE UNIT DISC

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This paper is dedicated by the first author to the memory of the second author. David Leroy Williams died unexpectedly on March 9, 1980; he was 42 years old. He received his Ph.D. in 1967 at the University of Michigan, under the direction of the first author. A fine mathematician, a fine friend.

1. Introduction. Let u be a harmonic function in the open unit disc Δ and as usual denote $M_\infty(u,r)=\sup\{|u(re^{i\theta})|:-\pi<\theta\le\pi\}$ for $r<1$. If u is bounded, elementary estimates on the conjugate Poisson kernel show that the harmonic conjugate $\tilde{u}$ satisfies the growth condition $M_\infty(\tilde{u},r)=O(\log(1/(1-r)))$. Moreover the analytic function $\log(1/1-z))$, whose imaginary part is bounded in Δ, proves that this estimate is best possible, that is, $\log(1/1-r))$ cannot be replaced with a function of slower growth. On the other hand, Hardy and Littlewood [4], [5], [3, p. 83] showed that if $M_\infty(u,r)=O((1/(1-r)^\alpha)$, $\alpha>0$, then $M_\infty(\tilde{u},r)$ satisfies the same growth condition. We fill the gap between these two results.

More precisely, for $x\ge 0$ let $\psi(x)$ be a positive increasing function for which there exists $\alpha>0$ such that $\psi(x)=O(x^\alpha)$, $x\longrightarrow\infty$. Assuming some mild regularity conditions on ψ, we show in Section 3 that if $M_\infty(u,r)=O(\psi(1/(1-r)))$, then $M_\infty(\tilde{u},r)=O(\tilde{\psi}(1/1-r)))$ where $\tilde{\psi}(x)=\int_{1/2}^{x}t^{-1}\psi(t)\,dt$. We also show that this estimate is best possible by constructing a harmonic function u on Δ such that $M_\infty(u,r)=O(\psi(1/(1-r)))$ and $M_\infty(\tilde{u},r)\ge\tilde{\psi}(1/(1-r))$, $r\in[0,1)$.

To interpret these results, one needs to know certain facts about the ratio $\tilde{\psi}/\psi$. We shall give a detailed discussion in Section 2. Here we make some brief observations. First, if $\psi(x)$ grows like x^α then so does $\tilde{\psi}(x)$, and one obtains the Theorem of Hardy and Littlewood. However, if ψ grows more slowly than any positive power of x, then, generally speaking, $\tilde{\psi}$ grows faster than ψ. For example, if $\psi(x)=\log(x+2)$, then $\tilde{\psi}(x)$ grows like $[\log(x+2)]^2$. If $\psi(x)\equiv 1$, then $\tilde{\psi}(x)$ grows like $\log x$; thus we recapture the bounded case mentioned above.

The above discussion remains valid if we replace $M_\infty(u,r)$ by

$$M_1(u,r)=\frac{1}{2\pi}\int_{-\pi}^{\pi}|u(re^{i\theta})|\,d\theta$$

throughout. Of course, for $M_p(u,r)=[(2\pi)^{-1}\int|u(re^{i\theta})|^p\,d\theta]^{1/p}$, $1<p<\infty$, the well-known theorem of M. Riesz [3, p. 54] says that $M_p(u,r)=O(\psi(1/(1-r)))$ implies $M_p(\tilde{u},r)=O(\psi(1/(1-r)))$. Therefore, in this paper we shall be concerned only with the means $M_\infty(u,r)$ and $M_1(u,r)$. However the referee has pointed out to us that Theorem 1 remains valid for a rather general class of norms, namely for the norm in any "homogeneous Banach space" in the sense of Y. Katznelson. This is discussed briefly in Section 7 at the end of the paper, where the relevant definitions and references are given.

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In Section 4 we interpret the results of Section 3 in terms of the "analytic projection" operator on certain Banach spaces of harmonic functions.

It happens that our questions and results on the growth of conjugate harmonic functions are closely related to the question of the existence of bounded projections from certain L^1 spaces onto the subspace of analytic functions. This connection is discussed in Section 5. Let η denote a finite positive Borel measure on $[0, 1)$ and let $L^1(\eta)$ denote the space of complex Borel measurable functions on the open unit disc Δ for which

$$\|f\|_\eta = \frac{1}{2\pi}\int_0^1\int_{-\pi}^{\pi} |f(re^{i\theta})|\, d\theta\, d\eta < \infty.$$

Also, define $A^1(\eta)$ to be the subspace of functions in $L^1(\eta)$ which are analytic in Δ.

For which η does there exist a bounded projection from $L^1(\eta)$ onto $A^1(\eta)$? While we cannot completely answer this question, we do obtain some interesting results. First, let us eliminate the trivial case where $A^1(\eta)$ is not closed in $L^1(\eta)$. This occurs if and only if η is supported on a subinterval $[0, \rho)$ where $0<\rho<1$. Thus, we assume η is not supported on such a subinterval. In terms of the Hausdorff moments of η, this is equivalent to assuming the sequence $\eta(n) = \int_0^1 r^n d\eta(r)$ approaches zero more slowly than any exponential sequence ρ^n, $0<\rho<1$. Roughly speaking, our techniques allow us to study the cases where the moments of η approach zero more slowly than $n^{-\alpha}$ for some $\alpha>0$. More precisely, we assume $c/\psi(n) \le \eta(n) \le C/\psi(n)$ where ψ is one of the functions described above and c and C are constants; we show that if $\tilde{\psi}/\psi$ is not bounded, then there is no bounded projection from $L^1(\eta)$ onto $A^1(\eta)$. This is related to a result of Joel Shapiro; see the discussion following the proof of Theorem 3′ in Section 5. We also show that our result is "best possible" (Theorem 3′).

A growth condition of the form $M_\infty(u, r) = O(\psi(1/(1-r)))$ naturally describes a Banach space of harmonic functions which we denote by $h_\infty(\psi)$. In [10] and [11] the authors studied the problem of finding a measure η with the property that $h_\infty(\psi)$ is isomorphic to the dual of $h^1(\eta) = \{u \in L^1(\eta) : u \text{ harmonic}\}$. This duality problem is closely related to the problems discussed in this paper; in fact, solving the duality problem yields alternate proofs of some of the results of this paper. On the other hand, the results of this paper provide further insights into the work in [10] and [11] and answer some questions raised there. Therefore, in the last section of this paper we review briefly some of this earlier work and discuss its relation to the main theorems of the present paper.

Finally, we remark that there are analogies between some of our results on the mean M_∞ for harmonic conjugates and some results of N. K. Bari and C. B. Stečkin [1]. Bari and Stečkin studied functions on the unit circle with a modulus of continuity $\omega(f; t) = O(\phi(t))$, and they gave necessary and sufficient conditions on ϕ in order that $\omega(\tilde{f}, t) = O(\phi(t))$ also. It is well-known that fractional integrals can be used to map certain spaces of analytic functions satisfying a growth condition on the unit disc onto spaces of functions satisfying a continuity condition on the unit circle. See, for example, Chapter 5 of [3]. However, we have not been able to apply this technique in sufficient generality to obtain a direct connection between our results and those of Bari and Stečkin.

2. Some lemmas on monotone functions. In describing the spaces of harmonic functions studied in this paper, we use monotone functions of at most polynomial growth. We shall need to impose certain mild regularity conditions on these monotone functions, and in this section we define the regularity conditions and establish some consequences and equivalent formulations of these conditions. A portion of the results of this section are closely related to some of the results of N. K. Bari and C. B. Stečkin in Section 2 of [1]; however, it is easier to prove our results directly than to derive them from the results in [1].

Sergei Bernstein introduced the notion of almost increasing and almost decreasing functions (see the paragraph preceding Theorem 3 in [2]). A real function f is *almost increasing* if there exists a positive constant c such that $x_1 < x_2$ implies $f(x_1) \leq c f(x_2)$. An *almost decreasing* function is defined similarly.

Throughout this paper ψ will denote a positive increasing function defined for real $x \geq 0$. For each such function ψ we define another function by

$$\tilde{\psi}(x) = \int_{1/2}^{x} t^{-1}\psi(t)\,dt, \quad (x \geq 1/2).$$

Consider the following conditions:
(U) *There exists $a > 0$ such that $\psi(x)/x^a$ is almost decreasing for $x \geq 1/2$.*
(L) *There exists $\epsilon > 0$ such that $\psi(x)/x^\epsilon$ is almost increasing for $x \geq 1/2$.*
Condition (U) contains the restriction to polynomial growth plus a regularity condition. On the other hand, (L) implies that $\psi(x)$ grows faster than some positive power of x.

Through this paper c and C denote positive constants, not necessarily the same at each occurrence.

The symbol Δ^1 denotes the usual first difference operator, that is: $(\Delta^1\psi)(n) = \psi(n) - \psi(n+1)$.

LEMMA 1. *Let ψ be a positive increasing function for $x \geq 0$. If ψ satisfies* (U), *then the following are also true.*

(i) *There exists $c > 0$ such that for all $x \geq 0$, $\psi(2x) \leq c\psi(x)$.*

(ii) *There exists $c > 0$ such that for all $x \geq 1$, $\psi(x) \leq c\tilde{\psi}(x)$.*

(iii) *There exist C, $c > 0$ such that for all $r \in [0,1)$,*

$$c\psi(1/(1-r))/(1-r) \leq \sum_0^\infty \psi(n) r^n \leq C\psi(1/(1-r))/(1-r).$$

(iv) *There exist C, $c > 0$ such that for all $r \in [0,1)$,*

$$c\psi(1/(1-r)) \leq \sum_0^\infty [-(\Delta^1\psi)(n-1)] r^n \leq C\psi(1/(1-r)), \quad (\psi(-1) = 0).$$

(v) *There exist C, $c > 0$ such that for all $r \in [0,1)$,*

$$c\tilde{\psi}(1/(1-r)) \leq \sum_0^\infty (n+1)^{-1}\psi(n) r^n \leq C\tilde{\psi}(1/(1-r)).$$

Proof. (i). For $x \geq 1/2$ condition (U) implies that

$$\psi(2x)/\psi(x) = [\psi(2x)/(2x)^a]2^a[x^a/\psi(x)] \leq c.$$

For $0 \leq x \leq 1/2$, $\psi(2x)/\psi(x) \leq \psi(1)/\psi(0)$.

(ii). For $x \geq 1$, (U) implies that

$$\tilde{\psi}(x) = \int_{1/2}^{x} [\psi(t)/t^a]t^{a-1}\,dt \geq c[\psi(x)/x^a] \int_{1/2}^{x} t^{a-1}\,dt \geq c\psi(x).$$

(iii). This is the conclusion of Lemma 1 of [11] where a slightly stronger hypothesis than (U) was used; however, the same proof works with hypothesis (U). Actually, only the right hand inequality makes use of (U); the left hand inequality depends only on the assumption that ψ is increasing.

(iv). Summing by parts, $\Sigma_0^p[-(\Delta^1\psi)(n-1)]r^n = \Sigma_0^{p-1}\psi(n)(r^n - r^{n+1}) + \psi(p)r^p$. Because (U) limits ψ to polynomial growth, for each $r \in [0,1)$, $\lim \psi(p)r^p = 0$. Thus $\Sigma_0^\infty[-(\Delta^1\psi)(n-1)]r^n = (1-r)\Sigma_0^\infty \psi(n)r^n$, and (iv) follows from (iii).

(v). This follows by integrating (iii) from 0 to r. □

LEMMA 2. *Let ψ be a positive increasing function for $x \geq 0$. The function ψ satisfies* (L) *if and only if there exists $c > 0$ such that for all $x \geq 1$, $\tilde{\psi}(x) \leq c\psi(x)$.*

Proof. If ψ satisfies (L), then for $x \geq 1$

$$\tilde{\psi}(x) = \int_{1/2}^{x} [\psi(t)/t^\epsilon]t^{\epsilon-1}\,dt \leq c[\psi(x)/x^\epsilon] \int_{1/2}^{x} t^{\epsilon-1}\,dt \leq c\psi(x).$$

Conversely, suppose $\tilde{\psi}(x) \leq c\psi(x)$ for $x \geq 1$. Letting $A = e^{2c}$,

$$\psi(x)\log A = \psi(x) \int_{x}^{Ax} t^{-1}\,dt \leq \int_{x}^{Ax} t^{-1}\psi(t)\,dt \leq \tilde{\psi}(Ax)$$
$$\leq c\psi(Ax) = (1/2)(\log A)\psi(Ax),$$

that is, $\psi(Ax) \geq 2\psi(x)$ for $x \geq 1$. Now fix x and x_1 such that $1 \leq x < x_1 < \infty$, and let p be the integer such that $A^p x \leq x_1 < A^{p+1}x$. Then for any $\epsilon > 0$,

$$\psi(x)/x^\epsilon \leq \psi(Ax)/(2x^\epsilon) \leq \cdots \leq \psi(A^p x)/(2^p x^\epsilon) \leq \psi(x_1)/(2^p x^\epsilon)$$
$$\leq [\psi(x_1)/x_1^\epsilon][2^{-p}x_1^\epsilon/x^\epsilon].$$

Also, $x_1/x < A^{p+1}$; hence $\psi(x)/x^\epsilon \leq A^\epsilon(A^\epsilon/2)^p\psi(x_1)/x_1^\epsilon$. Choosing $\epsilon > 0$ so that $A^\epsilon = 2$ gives (L) for $x \geq 1$, and therefore for $x \geq 1/2$. □

Later we shall use the following concept.

DEFINITION. A positive increasing function ψ defined for $x \geq 0$ is *normal* if it satisfies both (U) and (L).

In the authors paper [10] the word "normal" was used in a slightly different sense: instead of "almost increasing" and "almost decreasing" (in the definitions of (U) and (L)) we required the functions in question to be "increasing" and "decreasing", respectively. The theorems in [10] remain valid with this change.

One could establish other relations between the above properties of a positive increasing ψ. For example, (i) of Lemma 1 implies (U). Also, Lemma 1 (ii) and Lemma 2 together show that if ψ is normal, then there exist C, $c>0$ such that $c\psi(x) \le \tilde{\psi}(x) \le C\psi(x)$ for $x \ge 1$; the converse is also true.

3. Growth of harmonic conjugates. The following theorem is a generalization of the theorem of Hardy and Littlewood [4], [5], [3; p. 83] mentioned in the introduction.

THEOREM 1. *Let ψ be a positive increasing function for $x \ge 0$ which satisfies* (U), *and let u be harmonic in the unit disc.*

(i) *If $M_\infty(u,r) = O(\psi(1/(1-r)))$, then $M_\infty(\tilde{u},r) = O(\tilde{\psi}(1/(1-r)))$.*

(ii) *If $M_1(u,r) = O(\psi(1/(1-r)))$, then $M_1(\tilde{u},r) = O(\tilde{\psi}(1/(1-r)))$.*

Proof. We shall prove (ii); the proof of (i) is similar. First, let

$$\lambda(re^{i\theta}) = i\sum_{-\infty}^{-1}(|n|+1)r^{|n|}e^{in\theta} - i\sum_{1}^{\infty}(n+1)r^n e^{in\theta}$$

and note that $M_1(\lambda, r) \doteq O(1/(1-r))$. Now using the Fourier expansion $u(re^{i\theta}) = \sum \hat{u}(n) r^{|n|} e^{in\theta}$, $re^{i\theta} \in \Delta$, we define

$$(\lambda * u)(re^{i\theta}) = i\sum_{-\infty}^{-1}(|n|+1)\hat{u}(n)r^{|n|}e^{in\theta} - i\sum_{1}^{\infty}(n+1)\hat{u}(n)r^n e^{in\theta}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} u(\sqrt{r}e^{i(\theta-t)})\,\lambda(\sqrt{r}e^{it})\,dt.$$

by Fubini's Theorem,

$$M_1(\lambda * u, r) \le M_1(\lambda, \sqrt{r})M_1(u,\sqrt{r}) \le c\psi(1/(1-\sqrt{r}))/(1-\sqrt{r}).$$

Thus by Lemma 1(i), $M_1(\lambda * u, r) \le c\psi(1/(1-r))/(1-r)$. From the definitions of λ and of the conjugate function, $\tilde{u}(re^{i\theta}) = \int_0^1 (\lambda * u)(r\rho e^{i\theta})\,d\rho$, and by Fubini's Theorem,

$$M_1(\tilde{u}, r) \le \int_0^1 M_1(\lambda * u, r\rho)\,d\rho \le c\int_0^1 (1-r\rho)^{-1}\psi(1/(1-r\rho))\,d\rho$$

$$= cr^{-1}\int_1^{(1-r)^{-1}} t^{-1}\psi(t)\,dt \le c\tilde{\psi}(1/(1-r)). \qquad \square$$

If ψ is normal, that is, ψ satisfies both (U) and (L), then Lemmas 1(ii) and 2 together imply that ψ and $\tilde{\psi}$ have the same rate of growth. Thus, when ψ is normal, the above Theorem says that if u is harmonic in the unit disc and $M_\infty(u,r) = O(\psi(1/(1-r)))$, then $M_\infty(u,r) = O(\psi(1/(1-r)))$. Likewise if $M_1(u,r) = O(\psi(1/(1-r)))$ then $M_1(u,r) = O(\psi(1/(1-r)))$.

If ψ satisfies (U) but not (L), then $\tilde{\psi}$ grows at a faster rate than ψ, and one must ask if the above theorem is best possible, i.e., is there a harmonic function u such that $M_\infty(u,r) = O(\psi(1/(1-r)))$ and $M_\infty(\tilde{u},r) \ge \tilde{\psi}(1/(1-r))$. If we assume a bit more regularity on the growth of ψ, we can prove that this is the case.

THEOREM 1′. *Let ψ be a positive increasing function for $x \geq 0$ which satisfies (U). If ψ is either convex or concave (or, more generally, if condition (*) of Lemma 3 below holds), then there exist real-valued functions u and v harmonic in the unit disc such that*

(i) $M_\infty(u,r) = O(\psi(1/(1-r)))$ *and* $M_\infty(\tilde{u},r) \geq c\tilde{\psi}(1/(1-r))$.

(ii) $M_1(v,r) = O(\psi(1/(1-r)))$ *and* $M_1(\tilde{v},r) \geq c\tilde{\psi}(1/(1-r))$.

Our construction of the examples for this theorem is interesting in that we are able to construct harmonic functions satisfying rather general growth conditions by specifying the Fourier coefficients of the functions. To accomplish this we require a mild regularity condition on ψ. The next lemma is the essential ingredient in our construction. The following function will play a special role in our calculations. If ψ is a positive increasing function for $x \geq 0$, then let

$$k(re^{i\theta}) = \sum_{-\infty}^{\infty} \psi(|n|) r^{|n|} e^{in\theta} = \psi(0) + 2 \sum_{1}^{\infty} \psi(n) r^n \cos n\theta.$$

If ψ grows less rapidly than any exponential (in particular, if ψ satisfies (U)), then the series converges for $r < 1$ and so k is harmonic in the open disc Δ. Also, as usual, let $\Delta^2\psi = \Delta^1(\Delta^1\psi)$; thus

$$(\Delta^2\psi)(n) = \psi(n) - 2\psi(n+1) + \psi(n+2).$$

LEMMA 3. *Let ψ be a positive increasing function for $x \geq 0$ which satisfies* (U). *Then*

$$(*) \qquad M_1(k,r) = O(\psi(1/(1-r))) \quad (0 \leq r < 1)$$

if any one of the following conditions is satisfied:

(i) *ψ is convex,*

(ii) *ψ is concave,*

(iii) $|(\Delta^2\psi)(n)| \leq -c\dfrac{(\Delta^1\psi)(n)}{n}, \quad n = 1, 2, \ldots.$

Further, if ψ_1, ψ_2 are two functions (positive, increasing and satisfying (U)) *for which (*) holds, then (*) also holds for $\psi = \psi_1\psi_2$.*

Proof. The statement that (iii) implies (*) is precisely Lemma 2 of [11]. The proof that (i) or (ii) implies (*) proceeds, in part, along the same lines. First summing by parts,

$$\psi(0) + 2\sum_{1}^{q} \psi(n) r^n \cos n\theta = \sum_{0}^{q-1} (\Delta^1(\psi(n)r^n)) D_n(\theta) + \psi(q) r^q D_q(\theta)$$

where D_n is the Dirichlet kernel. On letting $q \to \infty$, the last term on the right vanishes because of (U) and we have $k(re^{i\theta}) = \sum_0^\infty (\Delta^1(\psi(n)r^n)) D_n(\theta)$. Another summation by parts gives $k(re^{i\theta}) = \sum_0^\infty (\Delta^2(\psi(n)r^n))(n+1)K_n(\theta)$ where K_n is the Fejer kernel. Using the identity

$$\Delta^2(\psi(n)r^n) = [(1-r)^2\psi(n) + 2r(1-r)(\Delta^1\psi)(n) + r^2(\Delta^2\psi)(n)]r^n,$$

we obtain the estimate

$$|k(re^{i\theta})| \leq (1-r)^2 \sum_0^\infty (n+1)\psi(n)r^n K_n(\theta) + 2(1-r) \sum_0^\infty (n+1)|(-\Delta^1\psi)(n)|r^{n+1}K_n(\theta) + \sum_0^\infty (n+1)|(\Delta^2\psi)(n)|r^{n+2}K_n(\theta).$$

Since $\int_{-\pi}^{\pi} K_n(\theta)\,d\theta = 2\pi$,

$$M_1(k,r) \leq (1-r)^2 \sum_0^\infty (n+1)\psi(n)r^n + 2(1-r)\sum_0^\infty (n+1)(-(\Delta^1\psi)(n))r^{n+1} + \sum_0^\infty (n+1)|(\Delta^2\psi)(n)|r^{n+2}. \tag{1}$$

We now show that each of the three terms in (1) is $O(\psi(1/(1-r)))$. Applying Lemma 1(iii) to the function $(x+1)\psi(x)$ we have

$$\sum_0^\infty (n+1)\psi(n)r^n \leq c\psi(1/(1-r))/(1-r)^2. \tag{2}$$

For the second term, summing by parts and using (2), gives

$$\begin{aligned}\sum_0^\infty (n+1)(-(\Delta^1\psi)(n))r^{n+1} &= \sum_0^\infty [\psi(n+1)-\psi(0)][(n+1)r^{n+1}-(n+2)r^{n+2}] \\ &= \sum_0^\infty [\psi(n+1)-\psi(0)][(n+1)(1-r)-r]r^{n+1} \qquad (3)\\ &\leq (1-r)\sum_0^\infty (n+1)\psi(n+1)r^{n+1} \leq c\psi(1/(1-r))/(1-r).\end{aligned}$$

If ψ is either convex or concave, then in the third term of (1), $(\Delta^2\psi)(n)$ has constant sign. Thus, summing by parts,

$$\begin{aligned}\sum_0^\infty (n+1)|(\Delta^2\psi)(n)|r^{n+2} &= \left|\sum_0^\infty (n+1)[(\Delta^2\psi)(n)]r^{n+2}\right| \\ &= \left|\sum [(\Delta^1\psi)(0)-(\Delta^1\psi)(n+1)][(n+1)r^{n+2} - (n+2)r^{n+3}]\right|\end{aligned}$$

Hence

$$\sum_0^\infty (n+1)|(\Delta^2\psi)(n)|r^{n+2} \leq \left|\sum_0^\infty (\Delta^1\psi)(n+1)[(n+1)r^{n+2}-(n+2)r^{n+3}]\right| + \left|\sum_0^\infty (\Delta^1\psi)(0)[(n+1)r^{n+2}-(n+2)r^{n+3}]\right|$$

$$= \left| \sum_0^\infty (\Delta^1\psi)(n+1)[(n+1)(1-r)r^{n+2} - r^{n+3}]\right| + |(\Delta^1\psi)(0))r^2|$$

$$\leq (1-r)\sum_0^\infty (n+1)(-(\Delta^1\psi)(n+1))r^{n+2} + \sum_0^\infty (-(\Delta^1\psi)(n+1))r^{n+3} + (-(\Delta^1\psi)(0))r^2.$$

By (3) and Lemma 1(iv) this is $O(\psi(1/(1-r)))$. This concludes the proof when ψ is either convex or concave.

Now suppose $\psi = \psi_1\psi_2$ where ψ_1 and ψ_2 are positive increasing and satisfy (U) and (*). Then clearly ψ is positive increasing and satisfies (U). To see that ψ also satisfies (*), let

$$k_1(re^{i\theta}) = \sum_{n=-\infty}^{\infty} \psi_1(|n|)r^{|n|}e^{in\theta}, \quad k_2(re^{i\theta}) = \sum_{n=-\infty}^{\infty} \psi_2(|n|)r^{|n|}e^{in\theta},$$

and observe that

$$k(re^{i\theta}) = (k_1 * k_2)(re^{i\theta}) = \sum_{-\infty}^{\infty} \psi_1(|n|)\psi_2(|n|)r^{|n|}e^{in\theta}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} k_1(\sqrt{r}e^{i(\theta-t)})k_2(\sqrt{r}e^{it})\,dt.$$

By Fubini's Theorem, Lemma 1(i), and the assumption that (*) holds for ψ_1 and ψ_2, we have

$$M_1(k,r) \leq M_1(k_1,\sqrt{r})M_1(k_2,\sqrt{r}) \leq c\psi_1(1/(1-r))\psi_2(1/(1-r)) \leq c\psi(1/(1-r)). \qquad \square$$

REMARK. The hypothesis (iii) of Lemma 3 is more natural than it may first appear. For a discussion see p. 265 of [11]. Generally speaking, the obvious functions which satisfy (U) also satisfy (iii). In addition, if ψ_1 and ψ_2 are positive increasing and satisfy (U) and (iii), then so does $\psi_1\psi_2$. To prove Lemma 3 under the hypothesis (iii), follow the above argument, but use (iii) to estimate the third term of (1). For details see p. 267 of [11].

Proof of Theorem 1′. Throughout the proof the symbol Σ stands for $\Sigma_{n=1}^\infty$; at one point the summation is taken only over odd n, this is indicated separately.

(i) Let $f(z) = \Sigma n^{-1}\psi(n)z^n$, $z \in \Delta$, and let $u = \mathrm{Im}(f)$. Then $\tilde{u} = -\mathrm{Re}(f)$. Thus $u(re^{i\theta}) = \Sigma n^{-1}\psi(n)r^n \sin n\theta$, $\tilde{u}(re^{i\theta}) = -\Sigma n^{-1}\psi(n)r^n\cos n\theta$ By Lemma 1(v), $M_\infty(\tilde{u},r) = \Sigma n^{-1}\psi(n)r^n \geq c\tilde{\psi}(1/(1-r))$. On the other hand

$$u_\theta(re^{i\theta}) = \Sigma\psi(n)r^n\cos n\theta,$$

and by Lemma 3, $M_1(u_\theta, r) \leq c\psi(1/(1-r))$. Since $u(r) = 0$ for all $r \in [0,1)$, we see that $u(re^{i\theta}) = \int_0^\theta u_t(re^{it})\,dt$ and

$$|u(re^{i\theta})| \leq \int_0^\theta |u_t(re^{it})|\,dt \leq 2\pi M_1(u_\theta, r) \leq c\psi(1/(1-r)).$$

Therefore $M_\infty(u, r) = O(\psi(1/(1-r)))$.

(ii) Let $v(re^{i\theta}) = \sum \psi(n) r^n \cos n\theta$. By Lemma 3, $M_1(v, r) = O(\psi(1/(1-r)))$. Now $\tilde{v}(re^{i\theta}) = \sum \psi(n) r^n \sin n\theta$ and so

$$M_1(\tilde{v}, r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\sum \psi(n) r^n \sin n\theta| \, d\theta \geq \frac{1}{2\pi} \int_0^{\pi} |\sum \psi(n) r^n \sin n\theta| \, d\theta$$

$$\geq \left| \frac{1}{2\pi} \int_0^{\pi} [\sum \psi(n) r^n \sin n\theta] \, d\theta \right| = \frac{1}{\pi} \sum_{n \text{ odd}} n^{-1} \psi(n) r^n.$$

From Lemma 1(i) and the fact that ψ is increasing we have, for

$$n \geq 1: \; c\psi(n) \geq \psi(2n) \geq \psi(n+1).$$

Hence (recall that the constant c need not be the same at each occurrence)

$$n^{-1}\psi(n) \geq c(n+1)^{-1}\psi(n+1).$$

Thus $M_1(\tilde{v}, r) \geq c\sum n^{-1}\psi(n) r^n$; hence $M_1(\tilde{v}, r) \geq c\tilde{\psi}(1/(1-r))$, by Lemma 1(v). □

REMARK. With v as in the above proof, the analytic function

$$g(z) = v(z) + i\tilde{v}(z) = \sum \psi(n) z^n$$

must satisfy the growth condition $M_1(g, r) \geq c\tilde{\psi}(1/(1-r))$ since $|g(re^{i\theta})| \geq |\tilde{v}(re^{i\theta})|$. One can also prove this directly by appealing to Hardy's inequality [3, p. 48]. Indeed, applying Hardy's inequality to $g_r(z) = g(rz)$, $0<r<1$, we have

$$\sum_1^\infty (n+1)^{-1}\psi(n) r^n = \sum_1^\infty (n+1)^{-1}\hat{g}_r(n) \leq \pi M_1(g, r).$$

Thus by Lemma 1(v), $M_1(g, r) \geq c\tilde{\psi}(1/(1-r))$.

Theorems 1 and 1′ together yield precise growth estimates on harmonic conjugates and it is interesting to look at some examples. We have discussed the normal case above. At the other end of our growth range we have $\psi(x) \equiv 1$ with $\tilde{\psi}(x) = \log 2x$; as mentioned in the introduction, this case is well-known. Let us consider an intermediate case, say $\psi(x) = [\log(x+2)]^\alpha$ where $\alpha>0$. Then $\tilde{\psi}(x)$ grows like $[\log(x+2)]^{\alpha+1}$, i.e., taking the harmonic conjugate increases the growth by a factor of $\log(1-r)^{-1}$ in these cases also. On the other hand if we let

$$\psi(x) = \exp(\log(x+1))^\alpha, \quad \alpha \in [0, 1],$$

then $\tilde{\psi}(x)$ grows like $(\log(x+1))^{1-\alpha}\psi(x)$; therefore, taking the harmonic conjugate may increase the growth only by a factor of $[\log(1-r)^{-1}]^{1-\alpha}$.

4. Banach spaces of harmonic functions. The growth conditions applied to harmonic functions in the previous section naturally describe certain Banach spaces of complex-valued harmonic functions; and the results of the previous section can be stated essentially in terms of the analytic projection operator on these spaces.

Let $h(\Delta)$ denote the vector space of all complex-valued functions harmonic in the open unit disc Δ with the usual pointwise addition and scalar multiplication. Let ψ be a positive increasing function defined for $x \geq 0$ and use it to define the norms

$$\|u\|_\psi \;= \sup\{M_\infty(u,r)/\psi(1/(1-r)) : r \in [0,1)\},$$

$$\|u\|_{1,\psi} = \sup\{M_1(u,r)/\psi(1/(1-r)) : r \in [0,1)\}.$$

The spaces $h_\infty(\psi) = \{u \in h(\Delta) : \|u\|_\psi < \infty\}$ and $h^1_\infty(\psi) = \{u \in h(\Delta) : \|u\|_{1,\psi} < \infty\}$ are Banach spaces. The spaces $h_\infty(\psi)$ were studied by the authors in [10] and [11]—they used the notation $h_\infty(\phi)$ where $\phi(r) = 1/\psi(1/(1-r))$. Many basic properties of the space $h_\infty(\psi)$ are given in Section 2 of [11], and similar results can be proved for the space $h^1_\infty(\psi)$.

In the case where ψ is bounded $h_\infty(\psi)$ is isomorphic to the space of bounded harmonic functions in the supremum norm, which has been extensively studied. Similar remarks apply to $h^1_\infty(\psi)$ when ψ is bounded. We are primarily interested in the cases where $\lim_{x\to\infty} \psi(x) = \infty$; then we can define the following subspaces of $h_\infty(\psi)$ and $h^1_\infty(\psi)$, respectively.

$$h_0(\psi) = \{u \in h(\Delta) : \lim_{r\to 1^-} M_\infty(u,r)/\psi(1/(1-r)) = 0\};$$

$$h^1_0(\psi) = \{u \in h(\Delta) : \lim_{r\to 1^-} M_1(u,r)/\psi(1/(1-r)) = 0\}.$$

It was shown in Section 2 of [11] that $h_0(\psi)$ is a closed subspace of $h_\infty(\psi)$ and that the harmonic polynomials (polynomials in $e^{i\theta}$ and $e^{-i\theta}$) are dense in $h_0(\psi)$. Similarly, $h^1_0(\psi)$ is a closed subspace of $h^1_\infty(\psi)$ and the harmonic polynomials are dense in $h^1_0(\psi)$.

The analytic projection $P_a : h(\Delta) \longrightarrow h(\Delta)$ is defined by $P_a\left(\sum_{-\infty}^{\infty} a_n r^{|n|} e^{in\theta}\right) = \sum_0^\infty a_n r^n e^{in\theta}$. The range of P_a is the space $A(\Delta)$ of all functions analytic in the open unit disc. While we could deal with the conjugate operator $u \longrightarrow \tilde{u} = iu - 2iP_a u + iu(0)$, we choose to use the analytic projection for two reasons. First, being a projection P_a is slightly simpler; second, the use of P_a is consistent with our study of projections in the later sections of this paper.

As before, we assume ψ satisfies (U). Then Theorem 1 says that

$$P_a : h_\infty(\psi) \longrightarrow h_\infty(\tilde{\psi}) \quad \text{and} \quad P_a : h^1_\infty(\psi) \longrightarrow h^1_\infty(\tilde{\psi})$$

are everywhere defined. It is easily seen from the proof of Theorem 1 that P_a is bounded for these spaces; alternatively, one can show that P_a has a closed graph in these spaces and so is bounded by the closed graph theorem. Assuming ψ satisfies the hypotheses of Theorem 1′, we have shown that, if ψ^* is another positive increasing function, then in order for $P_a : h_\infty(\psi) \longrightarrow h_\infty(\psi^*)$ or $P_a : h^1_\infty(\psi) \longrightarrow h^1_\infty(\psi^*)$ to be bounded it is necessary that $\tilde{\psi}/\psi^*$ be bounded.

We now obtain similar results for the subspaces $h_0(\psi)$ and $h^1_0(\psi)$.

THEOREM 2. *Let ψ be a positive function for $x \geq 0$ which increases to $+\infty$ and satisfies* (U). *Then*

(i) $P_a : h_0(\psi) \longrightarrow h_0(\tilde{\psi})$ *is bounded;*

(ii) $P_a : h^1_0(\psi) \longrightarrow h^1_0(\tilde{\psi})$ *is bounded.*

Proof. Since $P_a: h_\infty(\psi) \longrightarrow h_\infty(\tilde{\psi})$ is bounded and maps harmonic polynomials to harmonic polynomials, (i) follows from the fact that the harmonic polynomials are dense in $h_0(\psi)$. The proof of (ii) is similar.

THEOREM 2′. *Let ψ be a positive function for $x \geq 0$ which increases to $+\infty$, satisfies* (U), *and is either convex or concave* (*or, more generally, which satisfies condition* (*) *of Lemma* 3). *Let ψ^* be a positive increasing function for $x \geq 0$.*

(i) *If $P_a: h_0(\psi) \longrightarrow h_\infty(\psi^*)$ is bounded, then $\tilde{\psi}/\psi^*$ is bounded.*

(ii) *If $P_a: h_0^1(\psi) \longrightarrow h^1(\psi^*)$ is bounded, then $\tilde{\psi}/\psi^*$ is bounded.*

Proof. (i) Let $u(re^{i\theta}) = \sum_1^\infty n^{-1}\psi(n) r^n \sin n\theta$ and recall from the proof of Theorem 1′(i) that $u \in h_\infty(\psi)$. Then $u_\rho(re^{i\theta}) = u(\rho re^{i\theta})$ belongs to $h_0(\psi)$ for all $\rho \in [0,1)$, and

$$M_\infty(P_a u_\rho, r) = |P_a u(\rho r)| = \tfrac{1}{2}\sum_1^\infty n^{-1}\psi(n)\rho^n r^n \geq c\tilde{\psi}(1/1(1-\rho r))$$

by Lemma 1(v). But $P_a: h_0(\psi) \longrightarrow h_\infty(\psi^*)$ bounded implies

$$M_\infty(P_a u_\rho, r) \leq \|P_a\| \, \|u_\rho\|_\psi \psi^*(1/(1-r)) \leq \|P_a\| \, \|u\|_\psi \psi^*(1/(1-r)).$$

Thus $\|P_a\| \, \|u\|_\psi \psi^*(1/(1-r)) \geq c\tilde{\psi}(1/(1-\rho r))$, and, letting $\rho \longrightarrow 1^-$, we see that $\tilde{\psi}/\psi^*$ must be bounded.

(ii) The proof is similar; use the function $v \in h_\infty^1(\psi)$ defined in the proof of Theorem 1′(ii) and the remark following the proof.

5. Bounded projections. Let η be a finite positive Borel measure on $[0,1)$ which is not supported in any subinterval $[0,\rho)$, $0<\rho<1$. Let $L^1(\eta)$ denote the space of Borel measurable functions on the unit disc Δ which are integrable with respect to the measure $(2\pi)^{-1} d\theta \, d\eta(r)$, and let

$$\|f\|_\eta = \frac{1}{2\pi}\int_0^1 \int_{-\pi}^{\pi} |f(re^{i\theta})| \, d\theta \, d\eta(r).$$

Also define $h^1(\eta) = L^1(\eta) \cap h(\Delta)$ and $A^1(\eta) = L^1(\eta) \cap A(\Delta)$ where, as before, $h(\Delta)$ denotes the space of all functions harmonic in Δ, and $A(\Delta)$ denotes the subspace of all analytic functions. Both $h^1(\eta)$ and $A^1(\eta)$ are closed subspaces of $L^1(\eta)$; in fact it can be shown that the above condition on the support of η is necessary and sufficient for the completeness of $h^1(\eta)$ and $A^1(\eta)$. Also, if $u \in h^1(\eta)$ and $u=0$ a.e. $(d\theta \, d\eta(r))$, then $u \equiv 0$. Finally, the harmonic polynomials are dense in $h^1(\eta)$ and the analytic polynomials are dense in $A^1(\eta)$. For proofs of these and other basic facts, see Section 2 of [11].

In this section we study the question of the existence of bounded projections from $L^1(\eta)$ onto $A^1(\eta)$. The following lemma connects this problem with the work in the previous sections. As in Section 4, P_a will denote the analytic projection, that is, the projection onto the analytic functions. Lemmas of this sort are well known: see, for example, [9] (or [6; Chapter 9, p. 154]) where Rudin used it to give a simple proof of D. J. Newman's theorem [7] on the non-existence of bounded projections from L^1 onto H^1.

LEMMA 4. *Let H be a vector subspace of $h(\Delta)$ which contains the harmonic polynomials. Let P be a projection of H onto $H \cap A(\Delta)$. Then for any harmonic polynomial u,*

$$(P_a u)(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} (T_{-\tau}PT_\tau u)(re^{i\theta})\,d\tau,$$

where $T_\tau : h(\Delta) \longrightarrow h(\Delta)$ is defined by $(T_\tau v)(re^{i\theta}) = v(re^{i(\theta+\tau)})$.

Proof. It is enough to verify the result for the monomials $u_n(re^{i\theta}) = r^{|n|}e^{in\theta}$ ($n=0, \pm1, \pm2, \ldots$). We have $T_\tau u_n = e^{in\tau}u_n$ for all n. If $n \geq 0$, then $Pu_n = u_n$ by assumption, and the result follows. If $n<0$, then $f_n = Pu_n$ is analytic: $f_n(z) = \sum a_k z^k$. Hence $(T_{-\tau}PT_\tau u_n)(re^{i\theta}) = e^{in\tau}\sum_0^\infty a_k r^k e^{ik\theta}e^{-ik\tau}$, where, for fixed $re^{i\theta} \in \Delta$, the series converges uniformly in τ. When one integrates with respect to τ, every term is zero (since $n<0$). This completes the proof. □

As we stated in the introduction, the techniques used in this paper allow us to consider measures η whose moments $\eta(n) = \int_0^1 r^n\,d\eta(r)$, $n=0,1,2,\ldots$, satisfy a conditon of the form

$$c/\psi(n) \leq \eta(n) \leq C/\psi(n), \quad n = 0,1,2,\ldots, \tag{4}$$

where c and C are positive constants and ψ is a positive function for $x \geq 0$ which increases to $+\infty$ and satisfies (U). Also recall from the introduction that, if η satisfies (4) where ψ satisfies (U), then $A^1(\eta)$ is a closed subspace of $L^1(\eta)$.

THEOREM 3. *Let η be a finite positive Borel measure on* $[0,1)$. *Assume there are positive constants c and C such that*

$$c/\psi(n) \leq \int_0^1 r^n\,d\eta(r) \leq C/\psi(n), \quad n = 0,1,2,\ldots, \tag{5}$$

where ψ is a positive function for $x \geq 0$, which increases to ∞, satisfies (U), *and is either convex or concave* (*or, more generally, satisfies* (*) *of Lemma* 3). *If there exists a bounded projection of $L^1(\eta)$ onto $A^1(\eta)$, then ψ satisfies* (L).

REMARK. Recall from Lemma 2 that ψ satisfies (L) if and only if $\tilde{\psi}/\psi$ is bounded.

Proof of Theorem 3. Suppose $P: L^1(\eta) \longrightarrow L^1(\eta)$ is a bounded projection with range $A^1(\eta)$. Then the restriction of P to $h^1(\eta)$ is a bounded projection from $h^1(\eta)$ onto $A^1(\eta)$. We now show that if there is any bounded projection from $h^1(\eta)$ onto $A^1(\eta)$, then the analytic projection, P_a, is also bounded. Indeed, by Lemma 4, for any harmonic polynomial u,

$$|(P_a u)(re^{i\theta})| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |(T_{-\tau}PT_\tau u)(re^{i\theta})|\,d\tau;$$

and by Fubini's Theorem

$$\|P_a u\|_\eta \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \|T_{-\tau}PT_\tau u\|_\eta\,d\tau \leq \sup_\tau \|T_{-\tau}PT_\tau u\|_\eta.$$

Since T_τ and $T_{-\tau}$ are isometries on $L^1(\eta)$, $\|P_a u\|_\eta \le \|P\| \, \|u\|_\eta$. Because the harmonic polynomials are dense in $h^1(\eta)$, we may conclude that P_a is bounded on $h^1(\eta)$.

Let $h_\rho(re^{i\theta}) = h(\rho re^{i\theta}) = \sum_1^\infty \psi(n)^2 \rho^n r^n \cos n\theta$, $\rho \in [0,1)$. We claim that $\|h_\rho\|_\eta \le c\psi(1/(1-\rho))$. By Lemma 3, $M_1(h_\rho, r) = M_1(h, \rho r) \le c\psi(1/(1-\rho r))^2$, and so to establish the claim it suffices to show that $\int_0^1 \psi(1/(1-\rho r))^2 \, d\eta(r) \le c\psi(1/(1-\rho))$. Applying Lemma 1(iv) to ψ^2, $\psi(1/(1-\rho r))^2 \le c\sum_0^\infty [-(\Delta^1\psi^2)(n-1)]\rho^n r^n$. Using this and the hypothesis on the moments of η,

$$
\begin{aligned}
\int_0^1 \psi(1/(1-\rho r))^2 \, d\eta(r) &\le c \int_0^1 \left[\sum_0^\infty [-(\Delta^1\psi^2)(n-1)]\rho^n r^n\right] d\eta(r) \\
&\le c \sum_0^\infty [-(\Delta^1\psi^2)(n-1)]\psi(n)^{-1}\rho^n.
\end{aligned} \tag{6}
$$

Now $[-(\Delta^1\psi^2)(n-1)]\psi(n)^{-1} = [\psi(n)^2 - \psi(n-1)^2]\psi(n)^{-1} \le -2(\Delta^1\psi)(n-1)$. Thus by (6) and by Lemma 1(iv),

$$
\int_0^1 \psi(1/(1-\rho r))^2 \, d\eta(r) \le c\sum_0^\infty [-(\Delta^1\psi)(n-1)]\rho^n \le c\psi(1/(1-\rho)).
$$

Therefore $\|h_\rho\|_\eta \le c\psi(1/(1-\rho))$, as claimed above.

On the other hand by the remark following the proof of Theorem 1′(ii), $M_1(P_a h_\rho, r) = M_1(P_a h, \rho r) \ge c\tilde{\psi}^2(1/(1-\rho r))$. Using Lemma 1(v) applied to ψ^2,

$$
\sum_0^\infty (n+1)^{-1}\psi(n)^2\rho^n r^n \le c\tilde{\psi}^2(1/(1-\rho r)) \le cM_1(P_a h_\rho, r).
$$

Integrating with respect to $d\eta(r)$ and using the hypothesis on the moments of η gives $\sum_0^\infty (n+1)^{-1}\psi(n)\rho^n \le c\|P_a h_\rho\|_\eta$. Thus by Lemma 1(v) applied to ψ,

$$
\tilde{\psi}(1/(1-\rho)) \le c\sum_0^\infty (n+1)^{-1}\psi(n)\rho^n \le c\|P_a h_\rho\|_\eta \le c\|P_a\| \, \|h_\rho\|_\eta.
$$

Finally, the above estimate on $\|h_\rho\|_\eta$ gives $\tilde{\psi}(1/(1-\rho)) \le c\|P_a\|\psi(1/(1-\rho))$; i.e., $\tilde{\psi}/\psi$ is bounded. □

We remark that with appropriate modifications in some of our definitions we could have included in the statement and proof of Theorem 3 the result of Newman that there is no bounded projection from the space of Lebesgue integrable functions on the unit circle onto the Hardy space H^1. This corresponds to the case $\psi \equiv 1$.

In practice the hypotheses of Theorem 3 are not difficult to verify. For example, consider the family of measures $d\nu_\alpha(r) = (1-r)^{-1}[\log(1/(1-r))]^{-\alpha}dr$, $\alpha > 1$. A calculation shows that there are positive constants c and C such that

$$
c/[\log(n+2)]^{\alpha-1} \le \int_0^1 r^n d\nu_\alpha(r) \le C/[\log(n+2)]^{\alpha-1}, \quad n = 0, 1, 2, \ldots .
$$

Thus inequality (5) in the hypothesis of Theorem 3 is satisfied with $\psi_\alpha(x) = [\log(x+2)]^{\alpha-1}$. Since $\tilde{\psi}_\alpha(x)/\psi_\alpha(x) \ge c\log(x+2)$, Theorem 3 shows that there is no bounded projection from $L^1(\nu_\alpha)$ onto $A^1(\nu_\alpha)$, for any $\alpha > 1$.

Theorem 3 is related to an unpublished result of Joel Shapiro. To state this result we let η be a finite positive Borel measure on $[0,1)$ and let $\eta(n)=\int_0^1 r^n d\eta(r)$ $(n=0,1,2,\dots)$. Shapiro shows that if

$$\sum_0^\infty \frac{1}{n+1}\frac{\eta(n)^2}{\eta(2n)}=\infty, \tag{7}$$

then there is no bounded projection from $L^1(\eta)$ onto $A^1(\eta)$ (in fact, there is no bounded projection of $L^1(\eta)$ onto any closed subspace isomorphic to $A^1(\eta)$). He also points out that if $\sum\eta(n)(n+1)^{-1}=\infty$, then (7) is satisfied. These two conditions are equivalent if $\eta(2n)\geq c\eta(n)$, which will be the case in the situation considered in Theorem 3 when ψ satisfies (U). Theorem 3 applies to some situations where Shapiro's result does not; for example, his result shows there is no bounded projection from $L^1(\nu_\alpha)$ onto $A^1(\nu_\alpha)$ only for $1<\alpha\leq 2$, whereas ours show it for all $\alpha>1$. On the other hand, his result applies in certain situations where we cannot produce a ψ which satisfies our conditions (for example, where $\psi\notin$(U)). Also, as noted above, his result shows that there is no projection onto any subspace isomorphic to $A^1(\eta)$.

Suppose η is a finite positive Borel measure on [0, 1) for which (5) in the statement of Theorem 3 holds for some ψ satisfying (U). Then we conjecture that the converse of Theorem 3 holds, that is, if ψ satisfies (L), then there exists a bounded projection from $L^1(\eta)$ onto $A^1(\eta)$. However, it appears to be quite difficult to construct bounded projections in such generality, and so we settle for showing that Theorem 3 is "best possible".

THEOREM 3′. *Let ψ be a positive increasing function for $x\geq 0$ which satisfies* (U). *If ψ satisfies* (L), *then there exists a finite positive absolutely continuous measure η on* [0, 1) *such that*

(i) *there exist positive constants c and C with* $c/\psi(n)\leq\int_0^1 r^n d\eta(r)\leq C/\psi(n)$, $n=0,1,2,\dots$,

(ii) *there exists a bounded projection from $L^1(\eta)$ onto $A^1(\eta)$.*

Proof. According to (U), there exists $a>0$ such that $\psi(x)/x^a$ is almost decreasing for $x\geq 1$. Choose any $\alpha>a-1$ and let

$$d\eta(r) = [(1-r)\psi(1/(1-r))]^{-1}dr,$$

$$K(z) = (\alpha+1)(1-z)^{-(\alpha+2)}, \quad z\in\Delta,$$

$$(Qf)(w) = \frac{1}{2\pi}\int_0^1\int_{-\pi}^{\pi} K(wre^{i\theta})f(re^{i\theta})(1-r^2)^\alpha r\,d\theta\,dr, \quad f\in L^1(\eta).$$

It is easily seen from (L) that η is a finite measure. Also

$$(1-r)^\alpha\leq c[(1-r)\psi(1/(1-r))]^{-1},$$

and so $(Qf)(w)$ is defined and analytic for $w\in\Delta$. Let $f\in A^1(\eta)$. By expanding f and K in their Taylor series, it is easily seen that $Qf=f$ (for details see Lemmas 4 and 9 and Theorem 1 of [10]). Thus to establish (ii) it remains to show that Q is bounded on $L^1(\eta)$.

To show Q is bounded on $L^1(\eta)$, first note that by the Lemma on p. 65 of [3], $M_1(K,r)=O((1-r)^{-(\alpha+1)})$. Then for $f\in L^1(\eta)$, Fubini's Theorem yields

$$\begin{aligned}\|Qf\|_\eta &= \frac{1}{2\pi}\int_0^1\int_{-\pi}^{\pi}|(Qf)(\rho e^{it})|\,dt\,d\eta(\rho)\\ &\leq \frac{1}{2\pi}\int_0^1\int_{-\pi}^{\pi}\int_0^1 M_1(K,\rho r)\,d\eta(\rho)\,|f(re^{i\theta})|\,(1-r^2)^\alpha r\,d\theta\,dr\\ &\leq c\int_0^1\int_{-\pi}^{\pi}\int_0^1(1-\rho r)^{-(\alpha+1)}d\eta(\rho)\,|f(re^{i\theta})|\,(1-r^2)^\alpha r\,d\theta\,dr.\end{aligned}$$

Now, if we can prove that

$$\int_0^1(1-\rho r)^{-(\alpha+1)}d\eta(\rho)\leq c[(1-r)^{\alpha+1}\psi(1/(1-r))]^{-1},\tag{8}$$

it will follow that

$$\|Qf\|_\eta\leq c\int_0^1\int_{-\pi}^{\pi}|f(re^{i\theta})|\,[(1-r)\psi(1/(1-r))]^{-1}r\,d\theta\,dr\leq c\|f\|_\eta$$

and this will complete the proof of (ii).

To prove (8) break the integral at r and apply (U) to the first half to obtain

$$\begin{aligned}\int_0^r(1-\rho r)^{-(\alpha+1)}d\eta(\rho) &= \int_0^r(1-\rho r)^{-(\alpha+1)}(1-\rho)^{-1}\psi(1/(1-\rho))^{-1}d\rho\\ &\doteq \int_0^r(1-\rho r)^{-(\alpha+1)}(1-\rho)^{a-1}[(1-\rho)^a\psi(1/(1-\rho))]^{-1}d\rho\\ &\leq c[(1-r)^a\psi(1/(1-r))]^{-1}\int_0^r(1-\rho r)^{-(\alpha+1)}(1-\rho)^{a-1}d\rho\\ &\leq c[(1-r)^a\psi(1/(1-r))]^{-1}\int_0^r(1-\rho)^{a-\alpha-2}d\rho\\ &\leq c[\,(1-r)^{\alpha+1}\psi(1/(1-r))]^{-1}.\end{aligned}$$

To estimate the other half of the integral in (8), we use the hypothesis (L), that is, there exists $\epsilon>0$ such that $\psi(x)/x^\epsilon$ is almost increasing. Clearly $a>\epsilon$ so that $\alpha>a-1>\epsilon-1$, and so

$$\begin{aligned}\int_r^1(1-\rho r)^{-(\alpha+1)}\,d\eta(\rho) &= \int_r^1(1-\rho r)^{-(\alpha+1)}(1-\rho)^{-1}\psi(1/(1-\rho))^{-1}d\rho\\ &= \int_r^1(1-\rho r)^{-(\alpha+1)}(1-\rho)^{\epsilon-1}[(1-\rho)^\epsilon\psi(1/(1-\rho))]^{-1}d\rho\\ &\leq c(1-r)^{-\epsilon}\psi(1/(1-r))^{-1}\int_r^1(1-\rho r)^{-(\alpha+1)}(1-\rho)^{\epsilon-1}d\rho\end{aligned}$$

$$\leq c(1-r)^{-\epsilon}\psi(1/(1-r))^{-1}(1-r)^{-\alpha-1}\int_r^1(1-\rho)^{\epsilon-1}\,d\rho$$

$$\leq c[(1-r)^{\alpha+1}\psi(1/(1-r))]^{-1}.$$

This completes the proof of (8) and (ii).

To prove (i) we first observe that for $\beta>-1$ and $n>0$,

$$\int_0^1 r^n(1-r)^\beta\,dr=\Gamma(\beta+1)\Gamma(n+1)/\Gamma(\beta+n+2)\leq \text{const.}\,\Gamma(\beta+1)e^\beta n^{-\beta-1}.$$

Now, to estimate $\int_0^1 r^n\,d\eta(r)$ we break the integral at $\rho=1-n^{-1}$. Making use of (U) and the above observation we have

$$\int_0^\rho r^n\,d\eta(r)=\int_0^\rho r^n[(1-r)\psi(1/(1-r))]^{-1}\,dr$$

$$=\int_0^\rho r^n(1-r)^{a-1}[(1-r)^a\psi(1/(1-r))]^{-1}\,dr$$

$$\leq c[(1-\rho)^a\psi(1/(1-\rho))]^{-1}\int_0^\rho r^n(1-r)^{a-1}\,dr$$

$$\leq c/\psi(n).$$

Using (L) and the fact that $r^n\leq 1$ we have

$$\int_\rho^1 r^n\,d\eta(r)=\int_\rho^1 r^n[(1-r)\psi(1/(1-r))]^{-1}\,dr$$

$$=\int_\rho^1 r^n(1-r)^{\epsilon-1}[(1-r)^\epsilon\psi(1/(1-r))]^{-1}\,dr$$

$$\leq c[(1-\rho)^\epsilon\psi(1/(1-\rho))]^{-1}\int_\rho^1 r^n(1-r)^{\epsilon-1}\,dr$$

$$\leq c/\psi(n).$$

Thus $\int_0^1 r^n\,d\eta(r)\leq c/\psi(n)$. To obtain the lower bound on the moments of η, we again use (U). Letting $\rho=1-n^{-1}$ for $n>1$,

$$\int_0^1 r^n\,d\eta(r)\geq\int_\rho^1 r^n\,d\eta(r)\geq c[(1-\rho)^a\psi(1/(1-\rho))]^{-1}\int_\rho^1 r^n(1-r)^{a-1}\,dr$$

$$\geq c[(1-\rho)^a\psi(1/(1-\rho))^{-1}\rho^n\int_\rho^1(1-r)^{a-1}\,dr$$

$$=c[(1-\rho)^a\psi(1/(1-\rho))]^{-1}\rho^n a^{-1}(1-\rho)^a$$

$$=c\rho^n/\psi(1/(1-\rho))=c(1-n^{-1})^n/\psi(n)\geq c/\psi(n).$$

This completes the proof of (i) and the theorem. □

Theorem 3′ is closely related to the authors' Theorem 1 of [10] which also deals with the existence of bounded projections. In the next section we shall discuss the relationship in some detail.

In view of the above theorems one might ask about the existence of bounded projections from $L^1(\eta)$ onto $h^1(\eta)$. Unlike the analytic case, in the harmonic case the condition (L), i.e., that $\tilde{\psi}/\psi$ be bounded, is not relevant. In fact, if one assumes that ψ satisfies (U) and is sufficiently regular, then one can find a measure η, with moments which decrease like $1/\psi(n)$, and a bounded projection on $L^1(\eta)$ with range equal to $h^1(\eta)$. Most of the work involved in proving this was done by the authors in [11]; we give more details at the end of the next section.

One can also consider the existence of bounded projections on the spaces $h_\infty(\psi)$ and $h_0(\psi)$ having ranges equal to

$$A_\infty(\psi) = h_\infty(\psi) \cap A(\Delta) \quad \text{and} \quad A_0(\psi) = h_0(\psi) \cap A(\Delta),$$

respectively. As before we assume ψ satisfies (U), and, since the case where ψ is bounded is well-known, we assume $\psi(x) \longrightarrow +\infty$ as $x \longrightarrow +\infty$. Suppose there exists a bounded projection P on $h_\infty(\psi)$ with range $A_\infty(\psi)$ or a bounded projection P_0 on $h_0(\psi)$ with range $A_0(\psi)$. From Lemma 4, for a harmonic polynomial u,

$$\begin{aligned} |P_a u(re^{i\theta})| / \psi(1/(1-r)) &\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \|T_{-\tau} P T_\tau u\|_\psi \, d\tau \\ &\leq \sup_\tau \|T_{-\tau} P T_\tau u\|_\psi \leq \|P\| \, \|u\|_\psi . \end{aligned}$$

The same computation is valid for P_0. Thus, if P or P_0 is bounded, then P_a is bounded on the harmonic polynomials in the norm $\|\cdot\|_\psi$. Therefore, since the harmonic polynomials are dense in $h_0(\psi)$, P_a is bounded on $h_0(\psi)$. Thus the question of the existence of bounded projections P or P_0 is answered by Theorems 2 and 2′, that is, if ψ satisfies (U) and (*) of Lemma 3, then bounded projections exists if and only if $\tilde{\psi}/\psi$ is bounded, i.e., ψ is normal.

One can make similar statements for the spaces $h^1_\infty(\psi)$, $h^1_0(\psi)$ and their subspaces of analytic functions $A^1_\infty(\psi)$ and $A^1_0(\psi)$, respectively.

For further results concerning the construction of projections related to the $\|\cdot\|_\psi$ norm, we refer the reader to Theorem 1(i) of [10] and Theorems 1(ii) and 4 of [11]. We also remark that similar constructions are possible for the norm $\|\cdot\|_{1,\psi}$.

6. The duality problem. There is a very close relation between the above ideas concerning conjugate functions and projections and a duality problem which the authors studied in [10] and [11]. We shall briefly review the duality problem, discuss its relation to the present paper, and then show how it may be used to obtain alternate proofs of some of the above results.

As above, let ψ be a positive increasing function for $x \geq 0$. Since the duality ideas we are about to discuss do not apply to the space of bounded harmonic functions, we shall assume that $\psi(x) \longrightarrow \infty$ as $x \longrightarrow \infty$. Also, we shall assume that ψ is continuous (if ψ is not continuous then there is a continuous function ψ_1 such that $h_0(\psi) = h_0(\psi_1)$ and $h_\infty(\psi) = h_\infty(\psi_1)$, with equivalent norms).

It has been shown by Rubel and Shields [8] that $h_\infty(\psi)$ is isometrically isomorphic to the second dual of $h_0(\psi)$. In [10] and [11] the authors have studied the problem of representing the intermediate space, the dual of $h_0(\psi)$, as a space of harmonic functions with an L^1 norm. We say the harmonic duality problem is solvable for ψ if there exists a finite positive Borel measure η on $[0, 1)$ (but not supported on $[0, \rho)$ for any $\rho < 1$) such that the topological isomorphisms $h^1(\eta)^* \sim h_\infty(\psi)$, $h_0(\psi)^* \sim h^1(\eta)$ hold. (See Sections 4 and 5 for the definitions of these spaces.) More precisely, each element of $h_\infty(\psi)$ may be identified with a linear functional on $h^1(\eta)$ by means of the bilinear form

$$(u, v) = \frac{1}{2\pi} \int_0^1 \int_{-\pi}^{\pi} u(re^{i\theta}) v(re^{-i\theta}) \psi(1/(1-r))^{-1} \, d\theta \, d\eta(r) \tag{9}$$

where $u \in h_\infty(\psi)$ and $v \in h^1(\eta)$, and every continuous linear functional is obtained in this manner. The linear functional norm is equivalent to the $h^1(\eta)$ norm. The relation $h_0(\psi)^* \sim h^1(\eta)$ is defined in a similar manner. We note that $h^1(\eta)^* \sim h_\infty(\psi)$ if and only if $h_0(\psi)^* \sim h^1(\eta)$; see Theorem 1 of [11].

Analogously, we say that the analytic duality problem is solvable for ψ if there exists a measure η such that $A^1(\eta)^* \sim A_\infty(\psi)$ and $A_0(\psi)^* \sim A^1(\eta)$. The action of $f \in A^1(\eta)$ on $g \in A_\infty(\psi)$ is to be given by the bilinear form (9) restricted to the analytic functions. Again, we note $A^1(\eta)^* \sim A_\infty(\psi)$ if and only if $A_0(\psi)^* \sim A^1(\eta)$ [11, Theorem 2, p. 263].

The relationship between the harmonic and analytic duality problems is closely tied to questions about harmonic conjugates and is easily discussed in terms of the analytic projection operator P_a defined in Section 4. Recall that P_a is bounded on any of our spaces if and only if the conjugate function operator is bounded. Thus letting h and A represent any of the spaces $h_0(\psi)$, $h^1(\eta)$ or $h_\infty(\psi)$, and $A_0(\psi)$, $A^1(\eta)$, or $A_\infty(\psi)$, respectively, then P_a is a bounded projection of h onto A if and only if h is self-conjugate. (We say that a space of harmonic functions h is self-conjugate if $u \in h$ implies $\tilde{u} \in h$.) As pointed out on p. 263 of [11], it is easy to see that if $h_0(\psi)^* \sim h^1(\eta)$ and $h^1(\eta)^* \sim h_\infty(\psi)$ and if P_a is bounded on any one of these three spaces, then it is bounded on all of them. In fact, in this case $(P_a u, v) = (u, P_a v)$ for all $u \in h_\infty(\psi)$, $v \in h^1(\eta)$. The following theorem connects the harmonic and analytic duality problems and reveals the strong connection between the analytic duality problem and the conjugate problem.

THEOREM. [11, Theorem 3]. *In order that $A^1(\eta)^* \sim A_\infty(\psi)$, it is necessary and sufficient that $h_\infty(\psi)$ be self-conjugate and that $h^1(\eta)^* \sim h_\infty(\psi)$.*

In [10] we solved the analytic (and hence the harmonic) duality problem for the cases where ψ is normal. Thus the above theorem implies that $h_\infty(\psi)$ is self-conjugate when ψ is normal. (Actually, the definition of normal that we use here is slightly broader than that in [10]. In the definition of normal in [10], we used increasing and decreasing rather than almost increasing and almost decreasing. However, it is easy to see that all the theorems and lemmas of [10] are valid and their proofs essentially unchanged under our present definition of normal.) If we combine the above theorem with Theorem 2′, we see that one cannot expect to be able to solve the analytic duality problem for those ψ satisfying (U) but not (L).

In [11] the authors studied the harmonic duality problem for those ψ satisfying (U). For ψ satisfying (U) plus some regularity conditions, the authors solved the harmonic duality problem [11, Theorem 4]. For details regarding these regularity conditions see the discussions on pp. 265, 266, and 271 of [11]. (Incidentally, condition III on p. 265 of that paper could be replaced by any of the conditions in Lemma 3 of the present paper.) We point out here only that these conditions deal with regularity of growth, not rate of growth, and do not exclude any obviously interesting examples.

By Hölder's inequality the bilinear form (9) is defined for any ψ and η. Moreover, in the spaces $h_0(\psi)$, $h^1(\eta)$ and $h_\infty(\psi)$, point evaluation is a continuous linear functional [11, Proposition 1]. According to Proposition 3 of [11], the function

$$k(re^{i\theta}) = \sum_{-\infty}^{\infty} \hat{k}(n) r^{|n|} e^{in\theta}, \quad \hat{k}(n)^{-1} = \int_0^1 r^{2n} \psi(1/(1-r))^{-1} \, d\eta(r),$$

is harmonic in Δ and is the reproducing kernel for the bilinear form (9). That is, setting $k_w(z) = k(wz)$ we have $u(w) = (u, k_w)$ for all $u \in h_\infty(\psi)$, and $v(w) = (k_w, v)$ for all $v \in h^1(\eta)$. Note that k_w is harmonic in the disc of radius $1/|w| > 1$ and hence belongs to both $h_0(\psi)$ and $h^1(\eta)$.

If now $h^1(\eta)^* \sim h_\infty(\psi)$, then the equivalence of norms requires that

$$\|k_w\|_\eta \le c\psi(1/(1-|w|)). \tag{10}$$

See Section 5 for the definition of $\|\cdot\|_\eta$. Conversely, the estimate (10) implies that η is a solution to the harmonic duality problem [11, Theorem 1].

If k is the reproducing kernel for harmonic functions, with respect to the bilinear form (9), then a simple computation shows that $P_a k$ is the reproducing kernel for analytic functions. Moreover, η is a solution to the analytic duality problem if and only if $\|P_a k_w\|_\eta \le c\psi(1/(1-r))$. See Theorem 2 of [11].

We shall need the following lemma relating the moments of a measure which solves the harmonic duality problem to the Fourier coefficients of the reproducing kernel.

LEMMA 5. *Let ψ be a positive continuous function for $x \ge 0$ which increases to $+\infty$ and satisfies* (U). *If the measure η is a solution to the harmonic duality problem for ψ, then there exist positive constants c and C such that $c\psi(n)/\hat{k}(n) \le \int_0^1 r^n \, d\eta(r) \le C\psi(n)/\hat{k}(n)$, $n = 0, 1, 2, \ldots$, where k is the reproducing kernel for the duality.*

Proof. Let $v_n(z) = z^n$, $n \ge 0$. Now $v_n \in h^1(\eta)$ and

$$\|v_n\|_\eta = \frac{1}{2\pi} \int_0^1 \int_{-\pi}^{\pi} |r^n e^{in\theta}| \, d\theta \, d\eta(r) = \int_0^1 r^n \, d\eta(r).$$

Since $h^1(\eta)^* \sim h_\infty(\psi)$, there exist $C, c > 0$ such that

$$c\|v_n\|_\eta \le \sup\{|(u, v_n)| : \|u\|_\psi \le 1\} \le C\|v_n\|_\eta, \quad n = 0, 1, 2, \ldots.$$

For $u \in h_\infty(\psi)$ we have

$$(u, v_n) = \frac{1}{2\pi} \int_0^1 \int_{-\pi}^{\pi} u(re^{i\theta}) r^n e^{-in\theta} \psi(1/(1-r))^{-1} \, d\theta \, d\eta(r)$$

$$= \int_0^1 \hat{u}(n) r^{2n} \psi(1/(1-r))^{-1} \, d\eta(r) = \hat{u}(n)/\hat{k}(n).$$

Thus, there exist C, $c>0$ such that $c\|v_n\|_\eta \leq \sup\{|\hat{u}(n)| : \|u\|_\psi \leq 1\}/\hat{k}(n) \leq C\|v_n\|_\eta$, $n=0,1,2,\ldots$. Hence, to prove the lemma, it suffices to show there exist C, $c>0$ such that $c\psi(n) \leq \sup\{|\hat{u}(n)| : \|u\|_\psi \leq 1\} \leq C\psi(n)$, $n=0,1,2,\ldots$. For $\|u\|_\psi \leq 1$, $|u(re^{i\theta})| \leq \psi(1/(1-r))$, and

$$|\hat{u}(n)| \leq (2\pi)^{-1} r^{-n} \int_{-\pi}^{\pi} |u(re^{i\theta}) e^{-in\theta}| \, d\theta \leq r^{-n} \psi(1/(1-r)).$$

Thus $|\hat{u}(n)| \leq \inf\{r^{-n}\psi(1/(1-r)) : r<1\}$. Taking $u(z) = v_n(z)/\|v_n\|_\psi$, we have

$$\hat{u}_n(n) = 1/\|v_n\|_\psi = 1/\sup\{r^n \psi(1/(1-r))^{-1} : r<1\}$$
$$= \inf\{r^{-n}\psi(1/(1-r)) : r<1\}.$$

Therefore, $\sup\{|\hat{u}(n)| : \|u\|_\psi \leq 1\} = \inf\{r^{-n}\psi(1/(1-r)) : r<1\}$, and it suffices to show $c\psi(n) \leq \inf\{r^{-n}\psi(1/(1-r)) : r<1\} \leq C\psi(n)$, $n=0,1,2,\ldots$. The right inequality is easy since, letting $r=1-n^{-1}$,

$$\inf r^{-n}\psi(1/(1-r)) \leq (1-n^{-1})^{-n}\psi(n) \leq 4\psi(n), \quad n=2,3,\ldots.$$

As for the left inequality, for each $n>0$ let j_n be the nonnegative integer such that $2^{j_n} \leq n < 2^{j_n+1}$. For $0 \leq j \leq j_n$ we have $2^j \leq n$ and, for $1-2^j n^{-1} \leq r \leq 1-2^{j-1}n^{-1}$,

$$r^{-n}\psi(1/(1-r)) \geq (1-2^{j-1}n^{-1})^{-n}\psi(2^{-j}n) \geq e^{2^{j-1}} c_1^j \psi(n)$$

where the last inequality uses Lemma 1(i). Since $\lim_{j\to\infty} e^{2^{j-1}} c_1^j = \infty$,

$$c = \inf\{e^{2^{j-1}} c_1^j : j \geq 0\} > 0.$$

Hence, for $1-2^{j_n}n^{-1} \leq r \leq 1-(2n)^{-1}$ we have $r^{-n}\psi(1/(1-r)) \geq c\psi(n)$. For $1-(2n)^{-1} \leq r < 1$, $r^{-n}\psi(1/(1-r)) \geq \psi(2n) \geq \psi(n)$. For $0 \leq r \leq 1-2^{j_n}n^{-1}$,

$$r^{-n}\psi(1/(1-r)) \geq (1-2^{j_n}n^{-1})^{-n}\psi(0) \geq e^{2^{j_n}}\psi(0).$$

Now by (U) there exists a constant $c_2>0$ such that $\psi(x) \leq c_2 e^{x/2}$ for all $x \geq 0$. Thus $\psi(n) \leq c_2 e^{n/2} \leq c_2 e^{2^{j_n}}$, and so, for $0 \leq r \leq 1-2^{j_n}n^{-1}$,

$$r^{-n}\psi(1/(1-r)) \geq c_2^{-1}\psi(0)\psi(n).$$

This completes the proof of the lemma.

Assuming $h_1(\eta)^* \sim h_\infty(\psi)$, or equivalently $h_0(\psi)^* \sim h^1(\psi)$, an easy computation with Fourier expansions shows $P_a k_w \in h^1(\eta)$ represents the continuous linear functional $u \longrightarrow (P_a u)(w)$ on $h_\infty(\psi)$ or $h_0(\psi)$. Thus the following theorem, which estimates the norm of these linear functionals, contains essentially the information in Theorems 1(i), 1′(i), 2(i) and 2′(i).

THEOREM 4. *Let ψ be a positive continuous function for $x \geq 0$ which increases to $+\infty$ and satisfies* (U). *If the measure η is a solution to the harmonic duality problem for ψ, and if k is the reproducing kernel for the duality, then there exist positive constants c and C such that $c\tilde{\psi}(1/(1-r)) \leq \|P_a k_r\|_\eta \leq C\tilde{\psi}(1/(1-r))$, $r \in [0, 1)$.*

Proof. The proof of Theorem 1(i) shows that if $u \in h_\infty(\psi)$, then $|(P_a u)(r)| \leq c\|u\|_\psi \tilde{\psi}(1/(1-r))$. Thus from the assumption that $h^1(\eta)^* \sim h_\infty(\psi)$, we have

$$\begin{aligned}\|P_a k_r\|_\eta &\leq c\sup\{|(u, P_a k_r)| : \|u\|_\psi \leq 1\}\\ &= c\sup\{|(P_a u)(r)| : \|u\|_\psi \leq 1\} \leq C\tilde{\psi}(1/(1-r)).\end{aligned}$$

For the other half, first apply Hardy's inequality [3, p. 48] to $P_a k_r$ on a circle of radius $\rho < 1$, to obtain $\sum_{n=0}^{\infty}(n+1)^{-1}|\hat{k}_r(n)|\rho^n \leq \pi M_1(P_a k_r, \rho)$. Now, integrating with respect to $d\eta(\rho)$, $\sum_{n=0}^{\infty}(n+1)^{-1}|\hat{k}_r(n)|\int_0^1 \rho^n\, d\eta(\rho) \leq \pi\|P_a k_r\|_\eta$. Since $\hat{k}_r(n) = r^n \hat{k}(n)$, Lemma 5 gives $\sum_{n=0}^{\infty}(n+1)^{-1}\psi(n)r^n \leq c\|P_a k_r\|_\eta$, and then Lemma 1(v) completes the proof. □

Let us assume ψ satisfies the hypothesis of Theorem 4. The right hand inequality in Theorem 4 can be used to prove the conclusion of Theorems 1(i) and 2(i); of course we used the same computation to prove the right hand inequality in Theorem 4 as we did to prove Theorem 1(i).

However, the left hand inequality in Theorem 4 gives us another proof (one that does not use explicit examples) of the conclusion of Theorem 2′(i), namely, that the $\tilde{\psi}$ function is "best possible" in Theorems 1(i) and 2(i). The proof goes as follows. Suppose $P_a : h_0(\psi) \longrightarrow h_\infty(\psi^*)$ is everywhere defined; then by the closed graph theorem it is bounded, that is, there is a constant $c > 0$ such that for $\|u\|_\psi \leq 1$, $|(P_a u)(w)| \leq c\psi^*(1/(1-|w|))$. By the duality $h_0(\psi)^* \sim h^1(\eta)$,

$$\begin{aligned}\|P_a k_r\|_\eta &\leq c\sup\{|(u, P_a k_r)| : \|u\|_\psi \leq 1\}\\ &= c\sup\{|(P_a u)(r)| : \|u\|_\psi \leq 1\};\end{aligned}$$

and, therefore, $\|P_a k_r\|_\eta \leq c\psi^*(1/(1-r))$. Now the left hand inequality of Theorem 4 implies $\tilde{\psi}/\psi^*$ is bounded.

The duality problem is also closely related to the existence of certain bounded projections. According to Theorem 1 of [11] the duality $h_0(\psi)^* \sim h^1(\eta)$ is equivalent to the boundedness of the operator $S : M(\Delta) \longrightarrow L^1(\eta)$ defined by

$$(S\nu)(w) = \frac{1}{2\pi}\int_\Delta k(w\bar{z})\psi(1/(1-|z|))^{-1}\,d\nu(z), \quad \nu \in M(\Delta),\ w \in \Delta,$$

where $M(\Delta)$ is the space of finite complex Borel measures on Δ. To see the connection between this operator and the work in Section 5, identify $L^1(\eta)$ with the subspace of $M(\Delta)$ consisting of the measures absolutely continuous with respect to $d\theta\, d\eta(r)$. Then for $f \in L^1(\eta)$,

$$Sf(w) = \frac{1}{2\pi}\int_0^1\int_{-\pi}^{\pi} k(wre^{-i\theta})\psi(1/(1-r))^{-1} f(re^{i\theta})\,d\theta\,d\eta(r);$$

and, because of the reproducing property of k, S is a projection with range $h^1(\eta)$.

Similarly, Theorem 2 of [11] says that $A_0(\psi)^* \sim A^1(\eta)$ if and only if

$$P_a Sf(w) = \frac{1}{2\pi}\int_0^1\int_{-\pi}^{\pi} P_a k(wre^{-i\theta})\psi(1/(1-r))^{-1}f(re^{i\theta})\,d\theta\,d\eta(r)$$

is a bounded projection on $L^1(\eta)$ with range $A^1(\eta)$.

The difficulty in using these results to study the question of the existence of bounded projections on $L^1(\eta)$ with range $A^1(\eta)$ is that the measure η occurs as a (nonunique) solution to the duality problem for some ψ. Consequently, we have not been able to use these results to improve the theorems of Section 5. However, we can use these results to justify our comment at the end of Section 5 that for a measure η with moments decreasing like $1/\psi(n)$ (where ψ satisfies (U)) the question of the existence of a bounded projection on $L^1(\eta)$ with range $h^1(\eta)$ does not depend on the boundedness of the ratio $\tilde{\psi}/\psi$. Indeed, as stated above, in Theorem 4 of [11] we solved the harmonic duality problem for a large class of ψ satisfying (U) and some additional regularity (but not growth) conditions. In particular, we solved the duality problem for a large class of ψ that are not normal, i.e., for which $\tilde{\psi}/\psi$ is not bounded. We actually obtained a sequence of solutions η_m, $m=2,3,4,\ldots,$ with associated reproducing kernels having coefficients $\hat{k}(n)=\psi(n)^m$. Taking the solution with $m=2$, Lemma 5 tells us $c/\psi(n) \le \int_0^1 r^n\,d\eta_2(r) \le C/\psi(n)$; and by the above quoted results from [11],

$$Sf(w) = \frac{1}{2\pi}\int_0^1\int_{-\pi}^{\pi} k(wre^{-i\theta})\psi(1/(1-r))^{-1}f(re^{i\theta})\,d\theta\,d\eta_2(r)$$

is a bounded projection of $L^1(\eta_2)$ with range $h^1(\eta_2)$.

7. Homogeneous Banach spaces. Y. Katznelson has generalized the $L^p\,(1\le p<\infty)$ spaces on the unit circle by introducing the following class of spaces, which he calls homogeneous Banach spaces (see [12] and [13], p. 14).

A homogeneous Banach space B is a vector subspace of L^1 of the unit circle, together with a norm with respect to which B is complete, and which satisfies the following conditions. Here R_w denotes the operator of rotation by w $(|w|=1)$ defined by $(R_w f)(z)=f(\bar{w}z)$. As before, Δ denotes the open unit disc.

a) If $f\in B$, $w\in\partial\Delta$, then $R_w f\in B$ and $\|R_w f\|_B=\|f\|_B$.

b) If $f\in B$, $w\in\partial\Delta$, then $\lim\|R_\zeta f-R_w f\|=0$ $(\zeta\longrightarrow w,\ \zeta\in\partial\Delta)$.

We wish to thank the referee for pointing out to us that the proof of Theorem 1 establishes the following result. If u is a harmonic function in Δ and if r is given $(0<r<1)$ then u_r denotes the function defined on $\partial\Delta$ by the formula: $u_r(e^{i\theta})=u(re^{i\theta})$. We define $M_B(u,r)$ by: $M_B(u,r)=\|u_r\|_B$.

THEOREM. *Let B be a homogeneous Banach space that contains all continuous functions. Let ψ be a function satisfying the conditions of Theorem 1, and let u be a harmonic function in the unit disc.*

If $M_B(u,r)=O(\psi(1/(1-r)))$, then $M_B(\tilde{u},r)=O(\tilde{\psi}(1/(1-r)))$.

In view of this result, Theorem 1′ of Section 3 shows that the L^1 and L^∞ norms represent the "worst possible" case.

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