RANDOM SERIES WHICH ARE BMO OR BLOCH

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1. INTRODUCTION

Anderson, Clunie, and Pommerenke [1] have proved that if

$$\sum_{n=0}^{\infty} |a_n|^2 \log(n+1) < \infty \tag{1.1}$$

then the random series $F_\omega(z) = \sum_{n=0}^{\infty} a_n \omega_n z^n$ is a.s. a Bloch function, $F_\omega \in \beta$. (Here (ω_n) denotes the Steinhaus sequence.) As well, they have proved that if $\eta_n \geq 0$ and $\lim_n \eta_n = 0$, then there is a sequence (a_n) for which

$$\sum_{n=0}^{\infty} |a_n|^2 \eta_n \log(n+1) < \infty$$

but so that $F_\omega \notin \beta$ a.s.

In an unpublished manuscript, Pommerenke has shown that if 1.1) holds then F_ω is a.s. in the space of functions of vanishing mean oscillation, $F_\omega \in$ VMOA. Theorem 3.2 provides a different proof of this.

David Stegenga [8] has shown that there is a sequence (a_n) for which $\sum |a_n|^2 < \infty$ but so that $F_\omega \in$ BMOA for no choice of ω. Theorem 3.5 is a modification of his ideas and sharpens the results of [1]. Since BMOA $\subset \beta$ [1, Sections 2.2,2.3] it also extends Stegenga's result.

Section 2 contains some preparatory material and descriptions of the spaces involved. Section 3 contains the main results and Section 4 contains some closing remarks.

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2. PREPARATORY MATERIAL

Throughout, the unit disc will be denoted by Δ and its boundary by $\mathbf{T}$. The Lebesgue and Hardy spaces on $\mathbf{T}$ will be denoted, respectively, by L^p and H^p, $1 \leq p \leq \infty$. For facts concerning these spaces, see [2].

A function F, analytic in Δ, is said to be a Bloch function, $F \in \beta$, if

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$$F'(z) = O\left(\frac{1}{1-|z|}\right) \qquad |z| < 1.$$

Facts about β may be found in [1] and in the bibliography there.

A function $F \in H^1$ is said to be one of bounded mean oscillation, $F \in$ BMOA, if its boundary function, (also denoted by F) satisfies

$$\|F\|_{\mathrm{BMO}} = \sup \frac{1}{|I|} \int_I |F - F_I|\, dx < \infty,$$

where I ranges over subintervals of $\mathbf{T}$ and $F_I = 1/|I| \int_I F dx$. If

$$\lim_{|I|\to 0} \frac{1}{|I|} \int_I |F - F_I|\, dx = 0$$

then F is said to be of vanishing mean oscillation, $F \in$ VMOA.

The relevant facts concerning these spaces which will be needed here are the following:

2A) The dual of H^1 is BMOA, and the functional norms are comparable to

$$\left| \frac{1}{2\pi} \int_0^{2\pi} F dx \right| + \|F\|_{\mathrm{BMO}} \qquad [3].$$

2B) VMOA is the closure in BMO of the analytic functions on Δ with continuous boundary values [7].

The Steinhaus variables are constructed by placing a uniform measure of mass 1 on $\mathbf{T}$, and then forming the product measure P_Ω on $\Omega = \mathbf{T}^{\mathbf{N}}$. The sequence of projections from Ω into the nth coordinates is called the Steinhaus sequence. The Steinhaus variables are clearly independent. The Rademacher variables are constructed in a similar manner [4, p.4]. The results of this paper, inasmuch as they rely on the Salem-Zygmund theorems [4, Chapter VI] hold equally well for the Rademacher sequence, but the proofs will only be given for the Steinhaus sequence.

If (a_n) is a sequence of complex constants we formally write

$$F_\omega(z) = \sum_{n=0}^{\infty} a_n \omega_n z^n, \qquad \omega \in \Omega.$$

The statement "$F_\omega \in$ BMOA a.s." means that there is a set $E \subset \Omega$ so that $P_\Omega E = 1$ and so that if $\omega \in E$ there is a $G \in$ BMOA so that $(\hat{G}(n)) = (a_n \omega_n)$. Other statements of the same sort are to be taken in the same sense.

For facts about Sidon sets, see [5].

Throughout, the letter C will denote an absolute constant, not always the same at different occurrences.

3. THE MAIN RESULTS

Let K_m denote the mth Fejér kernel [10, Vol I, p. 89] and let

$$T_m = 2K_{2^{m+2}} - K_{2^{m+1}} + K_{2^{m-1}} - 2K_{2^m}, \qquad m \geq 1$$
$$T_0 = 1 + \cos x.$$

Thus $\hat{T}_m$ is a trapezoidal function on the integers, $\hat{T}_m(K) = 1$ if $|K| \in [2^m, 2^{m+1}]$, $\hat{T}_m(K) = 0$ if $|K| \notin [2^{m-1}, 2^{m+2}]$ and $\sum_{m=0}^{\infty} \hat{T}_m(K) = 1$ for each K. Moreover, $\|T_m\|_{L^1} \leq 6$ for each m.

PROPOSITION 3.1. *If $G \in H^1$ and if*

$$\sup_x \sum_{n=0}^{\infty} |T_n * G(x)|^2 < \infty$$

then $G \in$ BMOA.

If $\sum_{n=0}^{\infty} \|T_n * G\|_{L^\infty}^2 < \infty$ then $G \in$ VMOA.

Proof. By a theorem of Stein [9] $F \in H^1$ if and only if

$$I_F = \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{n=0}^{\infty} |T_n * F|^2 \right)^{1/2} dx < \infty;$$

then $I_F \leq C\|F\|_{H^1} \leq CI_F$.

Note that only the supports of $\hat{T}_{n-2}, \hat{T}_{n-1}, \hat{T}_n, \hat{T}_{n+1}, \hat{T}_{n+2}$ intersect the support of $\hat{T}_n$. Thus if F is a polynomial then

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} F\bar{G}dx &= \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{n=0}^{\infty} T_n * F \right) \bar{G}dx \\ &= \frac{1}{2\pi} \int_0^{2\pi} \sum_{n=0}^{\infty} (T_n * F)(T_{n-2} * \bar{G} + \ldots + T_{n+2} * \bar{G})\, dx, \end{aligned}$$

so that

$$\left| \frac{1}{2\pi} \int_0^{2\pi} F\bar{G}dx \right| \leq CI_F \left\| \left(\sum_{n=0}^{\infty} |T_n * G|^2 \right)^{1/2} \right\|_{L^\infty} \leq C\|F\|_{H^1}.$$

It follows from 2A) that $G \in$ BMOA.

The proof of the second part is quite similar. Let $V_n = \sum_{m=0}^{n} T_m$, and write $H_n = G - G * V_n$. Then, as before,

$$\left| \frac{1}{2\pi} \int_0^{2\pi} F\bar{H}_n\,dx \right| \le CI_F \left(\sum_{m=0}^{\infty} \|T_m * H_n\|_{L^\infty}^2 \right)^{1/2}$$
$$\le C\|F\|_{H^1} \left(\sum_{m=n-2}^{\infty} \|T_m * G\|_\infty^2 \right)^{1/2}.$$

It follows that $\lim_n \|H_n\|_{\mathrm{BMO}} = 0$ and hence from 2B) that $G \in$ VMOA.

THEOREM 3.2. *If* $\sum_{n=0}^{\infty} |a_n|^2 \log(n+1) < \infty$ *and* $F_\omega(z) = \sum_{n=0}^{\infty} a_n \omega_n z^n$ *then* $F_\omega \in$ VMOA a.s.

Proof. According to a theorem of Salem and Zygmund [4, p. 61] there is a constant C so that

$$P_\Omega\left(\|P\|_{L^\infty} \ge C\left(\log N \sum |C_n|^2 \right)^{1/2} \right) \le 1/N^2$$

whenever $P(x) = \sum_{n=0}^{N} C_n \omega_n e^{inx}$. Let $P_N(x) = \sum_{n=2^{N-1}}^{2^{N+2}-1} a_n \omega_n e^{inx}$ then there is a constant C' so that

$$P_\Omega\left(\|P_N\|_{L^\infty} \ge C' \left(\sum_{2^{N-1}}^{2^{N+2}-1} |a_n|^2 \log(n+1) \right)^{1/2} \right) \le 1/2^{2N}.$$

Now $(\|P_{3N}\|_{L^\infty})$ is a sequence of independent random variables, and so

$$P_\Omega\left(\forall N, \|P_{3N}\|_{L^\infty} < C' \left(\sum_{2^{3N-1}}^{2^{3N+2}-1} |a_n|^2 \log(n+1) \right)^{1/2} \right)$$
$$\ge \prod \left(1 - \frac{1}{2^{6N}} \right) > 0.$$

Thus $P_\Omega\left(\sum \|P_{3N}\|_{L^\infty}^2 < \infty \right) > 0$. But this is a "tail event" and so by the law of $0-1$ [4, p. 6], $\sum \|P_{3N}\|_{L^\infty}^2 < \infty$ a.s.

Similar arguments applied to $(\|P_{3N+1}\|_{L^\infty})$ and $(\|P_{3N+2}\|_{L^\infty})$ show that

$$\sum_{n=0}^{\infty} \|P_n\|_{L^\infty}^2 < \infty \quad \text{a.s.}$$

and so, since $\|T_N * F_\omega\|_{L^\infty} \le 6\|P_N\|_{L^\infty}$ it follows from Proposition 3.1 that $F_\omega \in$ VMOA a.s.

We now proceed to prove the partial converse to Theorem 3.2 that was announced in the Introduction.

PROPOSITION 3.3. *If* $a_n \geq 0$ *and if* $F(z) = \sum_{n=0}^{\infty} a_n z^n \in \beta$ *then*

$$\sup_m \sum_{n=m}^{2m} a_n < \infty.$$

Proof. There is a constant C so that

$$(1 - |z|) \left| \sum_{n=1}^{\infty} n a_n z^n \right| \leq C \qquad |z| < 1.$$

Consequently $(1 - r) \sum_{n=1}^{\infty} n a_n r^n \leq C \quad 0 < r < 1$. But then if $1 - r = 1/m$ it follows that $\sum_{n=m}^{2m} n a_n r^n \leq Cm$ and since there is a $\delta > 0$ so that $r^n \geq \delta$ if $m \leq n \leq 2m$ the conclusion follows.

PROPOSITION 3.4. *If* $F \in \beta$, *and if* μ *is a measure on* $\mathbf{T}$, *then* $F * \mu \in \beta$.

Proof.

$$|(F * \mu)'(z)| = \left| \frac{1}{2\pi} \int_0^{2\pi} F'(ze^{-i\theta})\, d\mu \right| \leq C \frac{\|\mu\|}{1 - |z|}$$

THEOREM 3.5. *If* $\eta_n \geq 0$ *and* $\lim_n \eta_n = 0$ *then there exists a sequence* (a_n) *for which*

$$\sum_{n=1}^{\infty} \eta_n \log(n+1) |a_n|^2 < \infty$$

but so that $\sum_{n=1}^{\infty} a_n \omega_n z^n \in \beta$ *for no choice of* (ω_n).

Proof. Let $E_N = \{2^N + 2^K;\ K = 1, \ldots, N\}$. Then E_N is a Sidon set with uniform Sidon constant, i.e. there is an absolute constant C (independent of N) so that for any numbers $(e^{i\theta_j})_1^N$ there is a measure μ_N so that

a) $\hat{\mu}_N(2^N + 2^K) = e^{-i\theta_K} K = 1, \ldots, N$

b) $\|\mu_N\| \leq C$.

Without loss of generality, we may assume that

c) $M_N = \operatorname{supp} \hat{\mu}_N \subset [2^{N-1}, 2^{N+2}]$,

for if $U_N(x) = e^{i3 \cdot 2^{N-1}x} V_{N-2}(x)$ (where V_N is as in the proof of Theorem 3.2) then $U_N * \mu_N$ satisfies a) and c) as well as b), with C replaced by 3C.

Let $\eta_K^* = \sup_{j \geq 2^K} \eta_j$; then $\eta_K^* \to 0$ and so (n_K) may be chosen so that

d) $\sum \eta_{n_K}^{*\ 1/8} < \infty$

and so that

e) (M_{N_K}) is a disjoint sequence of sets.

Write $\xi_K = \eta^*_{n_K}$, and let α_K be so that

f) $\sum \xi_K \alpha_K^2 < \infty$

but

g) $(\xi_K^{1/8} \alpha_K)$ is unbounded.

Let

$$\begin{cases} a_j = \alpha_K / n_K & \text{on } E_{n_K} \\ \quad = 0 & \text{off } \cup E_{n_K}. \end{cases}$$

Then

$$\sum \eta_K |a_K|^2 \log(K+1) = \sum \sum_{2^{n_p}+1}^{2^{n_p+1}} \eta_K |a_K|^2 \log(K+1)$$
$$\leq C \sum \xi_p n_p \left(\frac{\alpha_p}{n_p}\right)^2 n_p = C \sum \xi_p \alpha_p^2 < \infty \qquad \text{by f).}$$

But given $(\omega_n) = (e^{i\varphi_n})$, let $F(z) = \sum_{n=0}^{\infty} a_n \omega_n z^n$, and let μ_{n_p} satisfy a), b), c) with $\hat{\mu}_{n_p}(j) = e^{-i\varphi_j}$, $j \in E_{n_p}$. Then $\mu = \sum \xi_p^{1/8} \mu_{n_p}$ is a measure, by b), d), but if $j \in E_{n_p}$ then it follows from e) that

$$b_j = \widehat{\mu * F}(j) = a_j e^{i\varphi_j} \xi_p^{1/8} e^{-i\varphi_j} = \frac{\alpha_p}{n_p} \xi_p^{1/8},$$

and $b_j = 0$ off $\cup E_{n_p}$ by the definition of (a_j). So $\sum_{2^{n_p}}^{2^{n_p+1}} b_j = \alpha_p \xi_p^{1/8}$, which by g) is unbounded. Since $b_j \geq 0$ then by Proposition 3.3 $\mu * F \notin \beta$ and hence by Proposition 3.4, $F \notin \beta$.

4. CLOSING REMARKS

Billard [4, p. 47] has proved that if

$$F_\omega(x) = \sum_{n=0}^{\infty} a_n \omega_n z^n \in H^\infty \quad \text{a.s.}$$

then F_ω is a.s. a continuous function on $\mathbf{T}$. Since $H^\infty \subset$ BMOA one might ask if it might not be the case that F_ω is a.s. continuous when $F_\omega \in$ BMOA a.s. That this is not true may be seen as follows.

PROPOSITION 4.1. *There is a sequence* (a_n) *so that* $F_\omega \in$ VMOA a.s. *but* $F_\omega \in H^\infty$ *for no choice of* ω.

Proof. Paley and Zygmund [6, p. 350] have shown that if

$$a_n = \begin{cases} \dfrac{1}{m \log \ (m+1)} & n = 2^m \\ 0 & \text{otherwise} \end{cases}$$

then $F_\omega \in H^\infty$ for no choice of ω. But $\sum_{n=0}^{\infty} |a_n|^2 \log(n+1) < \infty$, so $F_\omega \in$ VMOA a.s.

The next result is in the same vein.

PROPOSITION 4.2. *There is a sequence* (a_n) *so that* $\sum_{n=0}^{\infty} |a_n|^2 \log(n+1) = +\infty$ *but* $F_\omega \in$ BMOA *for each* ω.

Proof. Let

$$a_K = \begin{cases} 1/n & K = 2^n \\ 0 & \text{otherwise} \end{cases}$$

A theorem of Paley [2, p. 104] states that there is a constant C so that

$$\sum_{n=0}^{\infty} |\hat{F}(2^n)|^2 \leq C \|F\|_{H^1}^2.$$

Then $\sum_{K=0}^{\infty} |a_K \hat{F}(K)| \leq C\|F\|_{H^1}$ so by 2A) $F_\omega \in$ BMOA for each choice of ω.

The result of Billard stated earlier raises another question. The spaces BMOA and VMOA stand in a very similar relationship to that of H^∞ and the continuous analytic functions on $\mathbf{T}$ [7]. Is it true that whenever $F_\omega \in$ BMOA a.s. then it follows that $F_\omega \in$ VMOA a.s.?

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