DIVISION IN $H^\infty + C$

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1. INTRODUCTION AND PRELIMINARIES

Our aim is to gain insight into the algebraic structure of $H^\infty + C$ by studying division in that algebra, especially for H^∞ functions. Our notations are standard: H^∞ denotes the space of boundary functions for bounded holomorphic functions in the open unit disk, D, and C denotes the space of continuous, complex valued functions on ∂D. It is well known that $H^\infty + C$ is a closed subalgebra of L^∞ (of Lebesgue measure on ∂D). Its most immediate properties are discussed in [8].

Division in the algebra H^∞ itself is well understood [3]. The facts are, briefly, as follows. Each nonzero function in H^∞ has a unique factorization as the product of an outer function and an inner function. One function divides another in H^∞ if and only if the same is true for the corresponding outer and inner factors. For outer functions the situation is especially simple: an outer function divides another one in H^∞ if and only if it divides it in L^∞. Each inner function has a unique factorization as the product of a Blaschke product, which is associated with the zero set of the function, and a singular function, with which is associated a certain singular measure on ∂D. One inner function divides another one in H^∞ if and only if the same is true of the corresponding Blaschke and singular factors. For Blaschke products, divisibility in H^∞ amounts to containment of zero sets, and for singular functions, it amounts to a relation of domination between the associated singular measures.

The situation in $H^\infty + C$ is more complicated. For example, a pair of infinite Blaschke products can be mutually prime in H^∞ and yet codivisible in $H^\infty + C$. One can produce such a pair in an elementary way by starting with any Blaschke product and suitably perturbing its zeros to obtain the second Blaschke product. Also, there is an example in [7] of a Blaschke product and a singular function which are codivisible in $H^\infty + C$. It is still unknown whether two distinct singular functions can be codivisible in $H^\infty + C$. In fact, it has been unknown up to now whether one singular function can divide another one in $H^\infty + C$ without dividing it in H^∞. In Section 2 we present an example of two singular functions, φ and ψ, such that ψ does not divide φ in H^∞, yet ψ^n divides φ in $H^\infty + C$ for every positive integer n.

The method used to produce the example in Section 2 can be modified so as to yield a general criterion for deciding when all positive powers of one inner function divide another one in $H^\infty + C$. This result is presented in Section 3 along with a similar result obtainable by the same method.

It is well known that the set of Blaschke products is a residual subset of the set of all inner functions. In Section 4 we present a related result on the

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abundance of Blaschke products: every inner function is codivisible in $H^\infty + C$ with a Blaschke product. This is a consequence of a recent basic theorem of T. Wolff.

We close in Section 6 by listing a few open questions.

We mention now a few conventions and immediate observations. We shall identify a function in H^∞ or $H^\infty + C$ with its holomorphic or harmonic extension to D as well as with its Gel'fand transform on the maximal ideal space of H^∞ or $H^\infty + C$. The latter maximal ideal spaces will be denoted by $M(H^\infty)$ and $M(H^\infty + C)$. The disk D can be identified in the obvious way with an open subset of $M(H^\infty)$ [3]. The corona theorem of L. Carleson [2] says that D is dense in $M(H^\infty)$. The space $M(H^\infty + C)$ consists of $M(H^\infty)$ with D deleted [8]. One can extend a function in $H^\infty + C$ continuously to $M(H^\infty)$ by taking its Gel'fand transform on $M(H^\infty + C)$ and its harmonic extension in D.

If ψ divides φ in $H^\infty + C$ then, on $M(H^\infty + C)$, the function $|\varphi|$ is bounded by a constant times $|\psi|$. The second theorem in Section 3 gives a partial converse to this implication for the case where ψ is an inner function. An example in Section 3 shows that the unrestricted converse is false.

If ψ is an inner function and if all positive powers of ψ divide φ in $H^\infty + C$, then $|\varphi| \leqq |\psi|^n$ on $M(H^\infty + C)$ for every positive integer n. The latter condition implies that $\varphi = 0$ everywhere on $M(H^\infty + C)$ that $|\psi| < 1$. The first theorem in Section 3 gives the reverse implication; in other words, the necessary condition just derived is also sufficient. The condition can be restated without reference to the maximal ideal space as follows: If (z_n) is a sequence in D such that $|z_n| \to 1$ and $\sup |\psi(z_n)| < 1$, then $\varphi(z_n) \to 0$.

Besides H^∞ and $H^\infty + C$, we shall have occasion to refer to QC, which is, by definition, the largest C^*-algebra contained in $H^\infty + C$, and to $QA = QC \cap H^\infty$. The functions in QC are precisely those L^∞ functions one can obtain by adding a function in C to the conjugate of a function in C. The unimodular invertible functions in $H^\infty + C$ are precisely the unimodular functions in QC. Thus, two inner functions are codivisible in $H^\infty + C$ if and only if their quotient is in QC. A structural description of the general unimodular function in QC is given in [8]: a unimodular function is in QC if and only if it is the product of a unimodular function in C and a function of the form $e^{i\tilde{v}}$, where $\tilde{v}$ is the conjugate of the real function v in C.

A portion of the results in this paper are from the first author's doctoral dissertation (University of California, Berkeley, 1978).

2. AN EXAMPLE

Our example is most easily described in the upper half-plane, rather than in the unit disk. In this section only, H^∞ will stand for the space of boundary functions on $\mathbf{R}$ for bounded holomorphic functions in the upper half-plane, and C will stand for the space of continuous, complex valued functions on the one-point compactification of $\mathbf{R}$.

One of the inner functions in our example will be the function $\psi(z) = e^{iz}$, which is the singular function for the upper half-plane whose corresponding singular measure is the unit point mass at ∞. This function will be our "divider." To obtain the other function (the "dividee") we choose a sequence $(w_n)_{-\infty}^{\infty}$ of positive numbers such that $\lim_{|n|\to\infty} w_n = \infty$ and $\sum_{-\infty}^{\infty} w_n/(1+n^2) < \infty$. Let u be the Poisson integral of the measure $\mu = \sum_{-\infty}^{\infty} w_n \delta_n$:

$$u(x,y) = \int_{-\infty}^{\infty} \frac{y}{(x-t)^2+y^2}\, d\mu\,(t) = \sum_{-\infty}^{\infty} \frac{w_n y}{(x-n)^2+y^2}.$$

Let $\tilde{u}$ be the harmonic conjugate of u vanishing at i, and let $\varphi = e^{-(u+i\tilde{u})}$. The function φ is thus the singular function for the upper half-plane corresponding to the measure μ. We shall show that ψ divides φ in $H^\infty + C$, although, obviously, it does not do so in H^∞. The same argument will show that all positive powers of ψ divide φ in $H^\infty + C$.

The crucial property of φ is this: For any $y > 0$, $\lim_{|x|\to\infty} \varphi(x+iy) = 0$. To establish this it suffices to note that, if $n \leqq x \leqq n+1$, then $u(x,y) \geqq w_n y/(1+y^2)$.

To show that ψ divides φ in $H^\infty + C$, it will be enough to show that

$$\lim_{n\to\infty} \operatorname{dist}\left(\bar{\psi}\varphi, \left(\frac{z+i}{z-i}\right)^n H^\infty\right) = 0, \tag{1}$$

the distance being measured in $L^\infty(\mathbf{R})$. Because the function $(z+i)/(z-i)$ has unit modulus on $\mathbf{R}$, we have

$$\operatorname{dist}\left(\bar{\psi}\varphi, \left(\frac{z+i}{z-i}\right)^n H^\infty\right) = \operatorname{dist}\left(\left(\frac{z-i}{z+i}\right)^n \bar{\psi}\varphi, H^\infty\right).$$

We use a well-known duality argument to estimate the latter distance. Namely, this distance is the norm of the coset of the function $\left(\frac{z-i}{z+i}\right)^n \bar{\psi}\varphi$ in the quotient space L^∞/H^∞, and the latter space is the dual of the space H^1. Thus, the distance in question is the norm of the functional that $\left(\frac{z-i}{z+i}\right)^n \bar{\psi}\varphi$ induces on H^1. We accordingly want to estimate integrals of the form

$$\int_{-\infty}^{\infty} \left(\frac{x-i}{x+i}\right)^n \overline{\psi\,(x)}\varphi\,(x)\, h(x)\, dx,$$

where h is in H^1.

Let Γ denote the horizontal line $\operatorname{Im} z = 1$. Because $1/\psi$ is bounded in the strip $0 < \operatorname{Im} z < 1$, one can use Cauchy's theorem to show that the preceding integral equals

$$\int_\Gamma \left(\frac{z-i}{z+i}\right)^n \varphi(z) h(z)/\psi(z)\, dz. \tag{2}$$

(The technical details, which are omitted here, are most easily accomplished under the assumption that h lies in the dense subspace $H^1 \cap C_0$ of H^1.) Now fix $\varepsilon > 0$. Because $\varphi(z) \to 0$ as $z \to \infty$ on Γ, we can choose a positive integer n such that $\left|\left(\frac{z-i}{z+i}\right)^n \varphi(z)\right| < \varepsilon$ everywhere on Γ. Because ψ has the constant modulus $1/e$ on Γ, the integral (2), for such an n, is no larger in modulus than

$$\varepsilon e \int_\Gamma |h(z)|\, |dz| \leqq \varepsilon e \|h\|_1.$$

Hence, for such an n, we have $\operatorname{dist}\left(\bar{\psi}\varphi, \left(\frac{z+i}{z-i}\right)^n H^\infty\right) \leqq \varepsilon\, e$, and (1) is established.

The preceding reasoning clearly applies in greater generality.

THEOREM. *For a function φ in H^∞ of the upper half-plane, the following conditions are equivalent:*

(i) *φ is divisible in $H^\infty + C$ by every positive power of the function $\psi(z) = e^{iz}$;*

(ii) $\lim_{|x| \to \infty} \varphi(x + iy) = 0$ *for every* $y > 0$;

(iii) $\lim_{|x| \to \infty} \varphi(x + iy) = 0$ *for some* $y > 0$.

In fact, the implication (iii) $\Rightarrow$ (i) is established by the argument above, the implication (ii) $\Rightarrow$ (iii) is trivial, and the implication (i) $\Rightarrow$ (ii) is established in Section 1.

3. A DIVISIBILITY CRITERION

The argument in the last section can be modified so as to yield a general criterion. We return to the unit disk.

THEOREM. *If ψ is an inner function and φ is a function in H^∞, the following conditions are equivalent:*

(i) *φ is divisible in $H^\infty + C$ by all powers of ψ;*

(ii) $\varphi(1 - |\psi|) = 0$ *on* $M(H^\infty + C)$;

(iii) $\lim_{|z| \to 1} \varphi(z)(1 - |\psi(z)|) = 0$.

The argument for the implication (i) $\Rightarrow$ (ii) is given in Section 1, and the equivalence of (ii) and (iii) is clear (one direction requires the corona theorem).

We shall complete the proof of the theorem by showing that (iii) implies (i). For this we use the basic construction devised by L. Carleson [2] to prove the corona theorem, in a simplified form due to D. Marshall [4]. According to Marshall's construction, for a given inner function ψ, there is a (possibly infinite) system Γ of simple closed rectifiable curves in $\bar{D}$ with the following properties:

(a) the curves in the system Γ have mutually disjoint interiors;

(b) there are numbers α and β, with $0 < \alpha < \beta < 1$, such that every point in D where $|\psi| < \alpha$ is contained in the interior of one of the curves in Γ, and $|\psi| < \beta$ at every point of D on or in the interior of one of those curves;

(c) arc length measure on $\Gamma \cap D$ is a Carleson measure.

Assume now that (iii) holds. We shall prove that ψ divides φ in $H^\infty + C$; the same argument will produce the same conclusion for any positive power of ψ. The basic idea is the same as was used in the last section, the Marshall construction being just what we need to overcome the additional technical difficulties in the present more general situation.

We shall show that $\bar{\psi}\varphi$ is in $H^\infty + C$ by showing that dist $(\bar{\psi}\varphi, z^{-n}H^\infty)$ tends to 0 as $n \to \infty$. As in the last section, a duality argument shows that the preceding distance equals the norm of the functional that $z^n\bar{\psi}\varphi$ induces on H_0^1 (the space of functions in H^1 vanishing at the origin). From this it is easily deduced that dist $(\bar{\psi}\varphi, \bar{z}^n H^\infty)$ equals the supremum of

$$\left| \frac{1}{2\pi} \int_{\partial D} \frac{z^n \varphi(z) h(z)}{\psi(z)} dz \right|$$

as h ranges over the family of functions in H^∞ satisfying $\|h\|_1 = 1$. Now the integrand in the above integral is bounded and holomorphic in the region between ∂D and Γ, and this enables us the transfer the integral from ∂D to Γ. The precise argument is as follows. For $0 < r < 1$, let G_r be the set of points in the disk $D_r = \{|z| < r\}$ which are exterior to every curve in Γ. The region G_r is bounded by finitely many rectifiable curves, and the function $z^n\varphi h/\psi$ is holomorphic in its closure. By Cauchy's theorem,

$$\int_{\partial G_r} \frac{z^n \varphi(z) h(z)}{\psi(z)} dz = 0. \tag{3}$$

The preceding integral can be written as the sum of an integral over $\partial G_r \cap D_r$ and an integral over $\partial G_r \cap \partial D_r$. As $r \to 1$, the first summand clearly tends to $-\int_{\Gamma\cap D} \frac{z^n \varphi(z) h(z)}{\psi(z)} dz$ (provided the curves in Γ are assigned the usual positive orientation). The second summand tends to $\int_{\partial D} \frac{z^n \varphi(z) h(z)}{\psi(z)} dz$. To verify that, let E_r be the complement of ∂G_r in ∂D_r. Then $\int_{E_r} |\psi(z)|\, |dz| \leqq \beta \operatorname{meas}(E_r)$. Because

$$\beta \operatorname{meas}(E_r) + \operatorname{meas}(\partial G_r \cap \partial D_r) \geqq \int_{\partial D_r} |\psi(z)|\,|dz| \to 2\pi$$

as $r \to 1$, we obtain $\lim_{r\to 1} \operatorname{meas}(E_r) = 0$, and the desired conclusion follows by the bounded convergence theorem. Hence, by letting $r \to 1$ in (3), we obtain

$$\int_{\partial D} \frac{z^n \varphi(z) h(z)}{\psi(z)}\, dz = \int_{\Gamma\cap D} \frac{z^n \varphi(z) h(z)}{\psi(z)}\, dz.$$

We conclude that dist $(\bar{\psi}\varphi, \bar{z}^n H^\infty)$ is the supremum of

$$\left| \frac{1}{2\pi} \int_{\Gamma\cap D} \frac{z^n \varphi(z) h(z)}{\psi(z)}\, dz \right| \tag{4}$$

as h ranges over the family of functions in H^∞ satisfying $\|h\|_1 = 1$. Now fix $\varepsilon > 0$. By condition (iii) of the theorem and property (b) of Γ, we can find an n such that $|z^n \varphi(z)| < \varepsilon$ on $\Gamma \cap D$. For such an n, the integrand in the integral (4) is bounded in modulus by $\varepsilon|h|/\alpha$ on Γ, so the quantity (4) is no larger than

$$\frac{\varepsilon}{2\pi\alpha} \int_{\Gamma\cap D} |h(z)|\,|dz|.$$

From property (c) of Γ we have $\int_{\Gamma\cap D} |h(z)|\,|dz| \leqq K\|h\|_1$ for some constant K.

We conclude that, if n is sufficiently large, then dist $(\bar{\psi}\varphi, \bar{z}^n H^\infty) \leqq K\varepsilon/2\pi\alpha$. The proof of the theorem is now complete.

We mention that there is a well-known construction [3, p. 177] which produces, for a given inner function ψ, a Blaschke product φ which is divisible in $H^\infty + C$ by all positive powers of ψ. In fact, if ψ is the Blaschke product with zeros z_n of multiplicities p_n, one simply takes for φ a Blaschke product with zeros z_n of multiplicities q_n satisfying $q_n/p_n \to \infty$. If ψ is not a Blaschke product, one replaces it in the preceding argument by $(\psi - \lambda)/(1 - \bar{\lambda}\psi)$ for a suitable λ in D. We do not know, however, whether there exists for each inner function ψ a singular function which is divisible in $H^\infty + C$ by all positive powers of ψ.

The above theorem can be improved slightly. Namely, conditions (i)–(iii) are equivalent for any φ in $H^\infty + C$ and unimodular ψ in $H^\infty + C$. A small modification of the proof that (iii) implies (i) suffices to establish this. As the construction of Marshall applies to any unimodular function in L^∞, the system Γ can be obtained just as before. The earlier argument involving contour integration cannot be applied directly to φ and ψ. However, one can uniformly approximate φ and ψ by functions in $\bigcup_0^\infty \bar{z}^n H^\infty$ and then apply the argument to the approximating functions. The details are only slightly altered from before and we shall not spell them out.

A slightly more careful application of our method produces another divisibility criterion.

THEOREM. *There is a positive integer* N *with the following property: if* φ *is a function in* $H^\infty + C$ *and* ψ *is a unimodular function in* $H^\infty + C$ *such that* $|\varphi| \leqq |\psi|$ *on* $M(H^\infty + C)$, *then* ψ *divides* φ^N *in* $H^\infty + C$.

To establish the theorem, we may assume without loss of generality that ψ is an inner function. This follows from T. Wolff's result [9] that the most general unimodular function in $H^\infty + C$ can be written as the product of an inner function and a function in QC. We shall also assume that φ is actually in H^∞, not merely in $H^\infty + C$. Technically speaking, this is a loss of generality, but the general case can be handled by the method we shall use in conjunction with an approximation of φ from $\bigcup_0^\infty \bar{z}^n H^\infty$, as alluded to in the preceding discussion.

The general scheme is the same as in the proof of the last theorem, except we now need the original construction of Carleson, rather than Marshall's variant. Fix a small positive number ε. According to Carleson's construction one can take a system Γ, as in the last proof, so that, in condition (b), $\alpha = \varepsilon$ and $\beta = \varepsilon^K$, where K is an absolute constant between 0 and 1. In addition, condition (c) can be made more precise, namely, one can take Γ so that the Carleson constant of arc length measure on $\Gamma \cap D$ is bounded by $K_1 \varepsilon^{-2}$, where K_1 is another absolute constant. We define N to be the smallest positive integer such that $NK > 3$.

By the same argument we used before, the quantity dist $(\bar{\psi}\varphi^N, \bar{z}^n H^\infty)$ equals the supremum of

$$\left| \frac{1}{2\pi} \int_\Gamma \frac{z^n \varphi(z)^N h(z)}{\psi(z)} \, dz \right| \tag{5}$$

as h ranges over the family of functions in H^∞ satisfying $\|h\|_1 = 1$. By the basic hypothesis of the theorem, there is an $r < 1$ such that $|\varphi(z)| \leqq 2\varepsilon^K$ wherever $|\psi(z)| \leqq \varepsilon^K$ and $r < |z| < 1$. Hence, if n is sufficiently large, then $|z^n \varphi(z)^N| \leqq 2^N \varepsilon^{NK}$ everywhere on Γ, so that the integrand in (5) is bounded in modulus by $\varepsilon^{NK}|h(z)|/\varepsilon$. In that case, the quantity (5) itself is no larger than

$$\frac{2^N \varepsilon^{NK-1}}{2\pi} \int_\Gamma |h(z)| \, |dz| \leqq \frac{2^N \varepsilon^{NK-1}}{2\pi} \cdot K_2 \varepsilon^{-2} \|h\|_1,$$

where K_2 is an absolute constant. We conclude that, for large n, the quantity dist $(\bar{\psi}\varphi^N, z^{-n} H^\infty)$ is bounded by a constant times ε^{NK-3}, and the proof is complete.

An example we now present shows that one cannot take $N = 1$ in the last theorem. However, it is conceivable that one can take $N = 2$. If that is so, its proof will undoubtedly require different techniques from ours.

The example depends on the recent result of Wolff [9] that the most general unimodular function f in L^∞ can be written as $f = \varphi w/\psi$, where φ and ψ are inner functions and w is in QC. We take a measurable subset E of ∂D of positive but less than full measure, and we let f be the function which equals 1 on E and -1 off E. If the above representation holds, then neither φ nor ψ divides the other in $H^\infty + C$, because f is not in $H^\infty + C$. However, $\psi^2 = \varphi^2 w^2$, so φ^2 and ψ^2 are codivisible in $H^\infty + C$, and thus $|\varphi| = |\psi|$ on $M(H^\infty + C)$—in particular, the hypotheses of the last theorem are satisfied.

As we shall see in the next section, the functions φ and ψ in the preceding example can be taken to be Blaschke products. This makes it seem unlikely that there is a general and precise criterion, expressible solely in terms of the distribution of their zeros, for the divisibility of one Blaschke product by another one in $H^\infty + C$.

The above example was originally pointed out, before Wolff established the result on which it is based, by K. Davidson and D. Leucking.

4. CODIVISIBILITY WITH BLASCHKE PRODUCTS

THEOREM. *If φ is any inner function, then there is a Blaschke product which is codivisible with φ in $H^\infty + C$.*

This theorem depends on the following basic result of Wolff [9]: If f is any function in L^∞, then there is an outer function h in QA such that hf is in QC. We take such an h with $\|h\|_\infty < 1$ for the function $f = \bar{\varphi}$. Fix a point λ in D, and let g be the outer factor and ψ the inner factor of the function $\varphi - \lambda h$. We show first that φ and ψ are codivisible in $H^\infty + C$, in other words, that $\bar{\varphi}\psi$ is in QC. Because $\bar{\varphi}\psi = (1 - \lambda h\bar{\varphi})/g$, it will be enough to show that $1/g$ is in QC. The function $|\varphi - \lambda h|$ is in QC because it equals $|1 - \lambda h\bar{\varphi}|$ and QC is a C^*-algebra. Since $|\varphi - \lambda h|$ is bounded away from 0, $\log|\varphi - \lambda h|$ is in QC (again, because QC is a C^*-algebra). Thus, we can write $\log|\varphi - \lambda h| = u + \tilde{v}$, where u and v are real functions in C. For $1/\bar{g}$ we have the representation

$$1/\bar{g} = 1/e^{u+\tilde{v}-i(\tilde{u}-v)} = e^{-2u-2iv}\, e^{u+i\tilde{u}}\, e^{-\tilde{v}+iv}.$$

In the last product, the first factor is in C, and the other two factors are in H^∞. Thus $1/\bar{g}$ is in $H^\infty + C$, so $1/g$ is in QC, as desired.

To complete the proof of the theorem, it only remains to show that one can choose λ so that the inner factor of $\varphi - \lambda h$ is a Blaschke product. Now the inner factor of $\varphi - \lambda h$ is the same as that of $\varphi/h - \lambda$, and φ/h lies in the Nevanlinna class. The desired conclusion thus follows from a result of W. Rudin [6], which assures us that the set of points λ for which the inner factor of $\varphi/h - \lambda$ is not a Blaschke product has zero logarithmic capacity. The theorem is now proved.

Suppose that, in the above discussion, φ is a singular function which omits a large subset of D, say, for example, a set having 0 as a limit point. It seems reasonable to expect that then φ/h will omit some nonzero value in D. If that could be proved, it would provide an example of two distinct singular functions which are codivisible in $H^\infty + C$. Our attempts at a proof have been unsuccessful, however.

If the inner function φ in the above theorem is such that its set of singularities on ∂D is a Carleson set, then one can take the function h in the above proof to be of class C^1 on ∂D [1] (or even of class C^∞ [5]). The proof then produces a Blaschke product ψ such that $\bar{\varphi}\psi$ is in C, not merely in QC.

5. OPEN QUESTIONS

We close by listing a few problems related to our results. Some of them are already mentioned above.

Problem 1. Can two distinct singular functions divide each other in $H^\infty + C$?

Problem 2. Does there exist, for each inner function ψ, a singular inner function which is divisible in $H^\infty + C$ by all positive powers of ψ?

Problem 3. What is the best value of the constant N in the second theorem of Section 3?

Problem 4. Find criteria in terms of the distribution of their zeros for one Blaschke product to divide another one in $H^\infty + C$. (As we pointed out at the end of Section 3, there seems to be a limit to what one can hope for along these lines.)

Problem 5. Find criteria in terms of their singular measures for one singular function to divide another one in $H^\infty + C$.

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