A NONZERO-DIVISOR CHARACTERIZATION OF BUCHSBAUM MODULES

Wolfgang Vogel

1. INTRODUCTION AND RESULTS

M. Hochster asked the following question in a discussion at the University of Michigan:

Let $A = R/\mathfrak{a}$ be a local ring where R is regular and $\mathfrak{a}$ is an ideal of R. Suppose that

(i) $A_\mathfrak{p}$ is a Cohen-Macaulay ring for all $\mathfrak{p} \in \operatorname{Spec} A \setminus \mathfrak{m}$ where $\mathfrak{m}$ is the maximal ideal of A.

(ii) there exists a nonzero-divisor x of A such that $A/(x)$ is a Buchsbaum ring.

Is it true that then A is a Buchsbaum ring?

Analyzing this question we get a nonzero-divisor characterization of Buchsbaum modules (see theorem) and the Examples 1 and 2 of this note. First we will give the Theorem of this paper.

THEOREM. *Let A be a local ring with maximal ideal* $\mathfrak{m}$ *(A is noetherian, commutative with unit). Let M be a finitly generated and unitary A-module of dimension* $d \geq 1$. *Suppose that* $\operatorname{depth}(M) \geq 1$ *then the following statements are equivalent:*

(i) *M is a Buchsbaum module.*

(ii) *There exists a nonzero-divisor* $x \in \mathfrak{m}$ *for M such that the following conditions are true:*

(a) $M/(x)M$ *is a Buchsbaum module.*

(b) $x \cdot H^i_\mathfrak{m}(M) = 0$ for all $i = 0, \ldots, d-1$.

(ii′) *For every nonzero-divisor* $x \in \mathfrak{m}$ *for M the conditions* (a), (b) *of* (ii) *are true.*

(iii) *There exists a nonzero-divisor* $x \in \mathfrak{m}$ *for M such that the following conditions are true:*

(c) $M/(x)M$ *is a Buchsbaum module.*

(d) $\mathfrak{m} \cdot H^i_\mathfrak{m}(M/(x^2)M) = 0$ *for all* $i = 0, \ldots, d-2$.

(iii′) *For every nonzero-divisor* $x \in \mathfrak{m}$ *for M the conditions* (c), (d) *of* (iii) *are true.*

Received July 13, 1978. Revision received August 14, 1979.
The author was supported by the National Science Foundation at Brandeis University.
Michigan Math. J. 28 (1981).

(iv) *There exists a nonzero-divisor $x \in \mathfrak{m}^2$ for M such that $M/(x)M$ is a Buchsbaum module.*

(iv′) *For every nonzero-divisor $x \in \mathfrak{m}^2$ for M the $A/(x)$-module $M/(x)M$ is a Buchsbaum module.*

The following example 1 shows that the answer to the above question is negative.

Example 1. Take $A = K[[x_1, x_2, x_3, x_4]]/(x_1^2, x_2) \cap (x_3, x_4)$ where K is an arbitrary field. Then we get the following claims:

(i) $A_\mathfrak{p}$ is a Cohen-Macaulay ring for all $\mathfrak{p} \in \operatorname{Spec} A \setminus m$.

(ii) $A/(x_1 + x_3)A$ is a Buchsbaum ring.

(iii) A is not a Buchsbaum ring.

(iv) $\mathfrak{m} \cdot H^1_\mathfrak{m}(A) \neq 0$.

The following example 2 sheds some light on the case of depth $(A) = 0$.

Example 2. Take $A = K[[x_1, x_2, x_3, x_4]]/\mathfrak{a}$ where

$$\mathfrak{a} = (x_1, x_2) \cap (x_3, x_4) \cap (x_1^2, x_2, x_3^2, x_4).$$

We get the following claims:

(i) $A/(x_1 + x_3)A$ is not a Buchsbaum ring; that is A is not a Buchsbaum ring.

(ii) $A/(x_2 + x_4)^n \cdot A$ is a Buchsbaum ring for all $n \geq 1$; that is the elements $(x_2 + x_4)^n$ and $(x_1 + x_3)^m$ are a weak A-sequence for all integers $n, m \geq 1$.

(iii) $\mathfrak{m} \cdot H^0_\mathfrak{m}(A) = \mathfrak{m} \cdot H^1_\mathfrak{m}(A) = 0$.

To motivate our problem below we give the following simple example 3.

Example 3. Take $A = K[[x, y]]/(x) \cap (x^2, y^2)$. Then the element y of A is a weak A-sequence but y^n is not a weak A-sequence for all $n \geq 2$.

Having the examples 2 and 3 we want to pose the following problem.

Problem. Let $a_1, \ldots, a_d \in \mathfrak{m}$ be a system of parameters for an A-module M such that the residue classes of the a_i mod $\mathfrak{m}^2$ be linearly independent over $A/\mathfrak{m}$. Let π be any permutation of $1, \ldots, d$. Suppose that the elements $a^{n_1}_{\pi(1)}, \ldots, a^{n_d}_{\pi(d)}$ are a weak M-sequence for all integers $n_1, \ldots, n_d > 0$. Is it true that then M is a Buchsbaum module?

In connection with this problem Dr. Ngo Viet Trung (Hanoi) pointed out a statement which follows from the following lemma in case $d = 2$. We denote by $U(\mathfrak{a})$ the intersection of the primary ideals $\mathfrak{q}$ belonging to the ideal $\mathfrak{a}$ of A such that $\dim A/\mathfrak{q} = \dim A/\mathfrak{a}$.

LEMMA. *Let A be a local ring with maximal ideal $\mathfrak{m}$ of dimension $d > 1$. Let $a_1, \ldots, a_k$ be a part of a system of parameters of A with an integer $0 < k \leq d$. Assume*

(i) *$A/U(0)$ is a Buchsbaum ring.*

(ii) $\mathfrak{m} \cdot U(0) = (0)$.

(iii) $\mathfrak{m}\cdot U(a_1,\ldots,a_i) \subseteq (a_1,\ldots,a_i)$ *for all integers* $i = 1, \ldots, k-1$.

(iv) $(a_1,\ldots,a_{k-1}) \cap U(0) = (0)$.

Then the elements $a_1^{n_1}, \ldots, a_k^{n_k}$ *are a weak* A*-sequence for all integers* $n_1, \ldots, n_k \geq 1$. We note that the assumption (iv) follows from (ii) and (iii) in case $k = d = 2$.

Proof. The assumption (ii) and theorem 5 in [8] show that we have to prove: $\mathfrak{m}\cdot U(a_1^{n_1},\ldots,a_i^{n_i}) \subseteq (a_1^{n_1},\ldots,a_i^{n_i})$ for all integers $i = 1, \ldots k-1$ and $n_1, \ldots, n_i \geq 1$. Let $a \in U(a_1^{n_1},\ldots,a_i^{n_i})$ and $r \in \mathfrak{m}$ be arbitrary elements. The assumption (i) yields that there are elements $u \in U(0)$ and $b \in (a_1^{n_1},\ldots,a_i^{n_i})$ such that $r\cdot a = b + u$. It follows from (iii) that

$$a \in U(a_1^{n_1},\ldots,a_i^{n_i}) \subseteq U(a_1,\ldots,a_i) \subseteq (a_1,\ldots,a_i) : m \subseteq (a_1,\ldots,a_i) : (r),$$

i.e., $r \cdot a \in (a_1,\ldots,a_i)$. Therefore we get from (iv):

$$r\cdot a - b = u \in (a_1,\ldots,a_i) \cap U(0) = (0).$$

Having this lemma it is not difficult to give examples which show that above assumption on the system of parameters can not be omitted in our problem. For instance, take the local ring for any field K:

$$A = K[\![x_1,\ldots,x_4]\!]/(x_1,x_2) \cap (x_3,x_4) \cap (x_1^2,x_2^2,x_3^3,x_4^4,x_1x_3,x_2x_4)$$

and the system of parameters $x_1^2 + x_3^2$, $x_2^2 + x_4^2$. Such examples thus follow from Stückrad's Thesis "Zur Theorie der Buchsbaum-Moduln," University of Leipzig, 1979.

I would like to thank D. Eisenbud and M. Hochster for stimulating conservations. Also, I want to thank the National Science Foundation for support at Brandeis University.

2. NOTATIONS AND PRELIMINARY RESULTS.

Throughout this note we assume that all rings are commutative, noetherian with identity, and all modules are finitely generated and unitary. Let A be a local ring with maximal ideal $\mathfrak{m}$, and let M be an A-module of dimension $d > 0$.

Definition. A set of elements $a_1, \ldots, a_r$ in $\mathfrak{m}$ is said to be a weak M-sequence if for each $i = 1, \ldots, r$

$$\mathfrak{m}\cdot [(a_1,\ldots,a_{i-1})M : a_i/(a_1,\ldots,a_{i-1})M] = 0$$

(for $i = 0$ we set $(a_1,\ldots,a_{i-1}) = (0)$ in A). If every system of parameters of M is a weak M-sequence, we say that M is a Buchsbaum module. A local ring A is said to be a Buchsbaum ring if it is a Buchsbaum A-module. The theory of Buchsbaum modules developed from an answer in [11] to a conjecture of D. A. Buchsbaum [1], p. 228. We will give an introduction to the theory of Buchsbaum modules in [5], see e.g. also [9]. In order to prove our theorem we also use the main result from [7]. To this let $K.(x_1,\ldots,x_t;M)$ be the Koszul complex generated over

M by $x_1, \ldots, x_t$ where $\mathfrak{m} = (x_1, \ldots, x_t)$ and $t = \dim_{A/\mathfrak{m}} (\mathfrak{m}/\mathfrak{m}^2)$. It should be noted that this complex is, up to isomorphism, independent of the set of elements $x_1, \ldots, x_t$ used to generate $\mathfrak{m}$. Let $H_i(\mathfrak{m}; M)$ be the homology modules of the Koszul complex $K.(\mathfrak{m}; M)$. We put for all integers i:

$$K^i(\mathfrak{m}; M) = K_{t-i}(\mathfrak{m}; M) \qquad \text{and} \qquad H^i(\mathfrak{m}; M) = H_{t-i}(\mathfrak{m}; M).$$

We shall denote by $H^i_{\mathfrak{m}}(M)$ the local cohomology modules of M with respect to the maximal ideal $\mathfrak{m}$ of A. Then the main result of [7] yields the following cohomological criterion for Buchsbaum modules.

LEMMA 1. *Let M be an A-module of dimension $d > 0$. M is a Buchsbaum module if and only if the canonical maps $\rho^i_M : H^i(\mathfrak{m}; M) \to H^i_{\mathfrak{m}}(M)$ are surjective for all $i \neq d$.*

Furthermore, we want to apply the following lemma.

LEMMA 2. *Suppose that M is an A-module and $x \in \mathfrak{m}^2$ is such that x is not a zero-divisor on M. Assume that for some integer $i \geq 0$ we have $\mathfrak{m} \cdot H^i_{\mathfrak{m}}(M/(x)M) = 0$. Then $\mathfrak{m} \cdot H^{i+1}_{\mathfrak{m}}(M) = 0$.*

Proof. The exact sequence $0 \to M \xrightarrow{x} M \to M/(x)M \to 0$ induces an exact sequence

$$H^i_{\mathfrak{m}}(M/(x)M) \to H^{i+1}_{\mathfrak{m}}(M) \xrightarrow{x} H^{i+1}_{\mathfrak{m}}(M).$$

Since $\mathfrak{m} \cdot H^i_m(M/(x)M) = 0$ we get from the exact sequence that

$$\mathfrak{m} \cdot [0 : H^{i+1}_{\mathfrak{m}}(M) x] = 0.$$

Therefore we obtain that $\mathfrak{m} \cdot [0 : H^{i+1}_{\mathfrak{m}}(M) \mathfrak{m}^2] = 0$ since $x \in \mathfrak{m}^2$. Since $H^{i+1}_{\mathfrak{m}}(M)$ is an Artinian A-module (see for instance [2], Prop. 2.1) it follows that each element of $H^{i+1}_{\mathfrak{m}}(M)$ is annihilated by some power of $\mathfrak{m}$ (see also [3], Theorem 3.4). Now, let $h \in H^{i+1}_{\mathfrak{m}}(M)$. For some integer $n > 0$ we thus have $\mathfrak{m}^n \cdot h = 0$. Let $n > 1$ then $\mathfrak{m}^{n-2} \cdot h \in 0 : H^{i+1}_{\mathfrak{m}}(M) \mathfrak{m}^2$, that is $\mathfrak{m}(\mathfrak{m}^{n-2} \cdot h) = 0$. Finally, it follows $\mathfrak{m} h = 0$.

3. PROOFS

Proof of the Theorem. First we remark that the statement (i) provides the statements (ii′), (iii′) and (iv′). This follows from [8], Corollary 6 and [4], Lemma 3. Clearly (ii′) implies (ii), (iii′) implies (iii) and (iv′) implies (iv). To (iii) $\Rightarrow$ (ii): Using the Lemma 2 we get that $\mathfrak{m} \cdot H^i_{\mathfrak{m}}(M) = 0$ for all $i = 1, \ldots, d-1$. Since $\operatorname{depth}(M) > 0$ we have $H^0_{\mathfrak{m}}(M) = 0$ and therefore (ii). To (iv) $\Rightarrow$ (ii): The Lemma 3 of [4] and our Lemma 2 yield (ii). In order to prove the Theorem we still have to show the following implication (ii) $\Rightarrow$ (i): We will show that the canonical maps ρ^i_M of the Lemma 1 are surjective for all $i \neq d$. Since $\operatorname{depth}(M) > 0$ we get that ρ^0_M is an isomorphism. Since $x \cdot H^i(\mathfrak{m}; M) = 0$ (see for instance [6]) and $x \cdot H^i_{\mathfrak{m}}(M) = 0$ the exact sequence $0 \to M \xrightarrow{x} M \to M/(x)M \to 0$ induces the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^{i-1}(\mathfrak{m};M) & \longrightarrow & H^{i-1}(\mathfrak{m};M/(x)M & \longrightarrow & H^{i}(\mathfrak{m};M) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \rho_M^{i-1}} & & \downarrow{\scriptstyle \rho_{M/(x)M}^{i-1}} & & \downarrow{\scriptstyle \rho_M^{i}} & & \\
0 & \longrightarrow & H_{\mathfrak{m}}^{i-1}(M) & \longrightarrow & H_{\mathfrak{m}}^{i-1}(M/(x)M) & \longrightarrow & H_{\mathfrak{m}}^{i}(M) & \longrightarrow & 0
\end{array}$$

Since $M/(x)M$ is a Buchsbaum module the Lemma 1 yields that $\rho_{M/(x)M}^{i-1}$ is surjective for all $i-1<d-1$. The commutative diagram therefore provides that ρ_M^i is surjective for all $i=1,\ldots,d-1$.

We want to mention that Ngo Viet Trung [10] also proved the equivalence of (i) and (iv) by using the techniques of [9].

Let $\mathfrak{a}$ be an ideal of A. We shall denote by $U(\mathfrak{a})$ the intersection of the primary ideals $\mathfrak{q}$ belonging to $\mathfrak{a}$ such that $\dim A/\mathfrak{q}=\dim A/\mathfrak{a}$.

Proof of the Claims of Example 1. Clearly we have (i). The statement (iii) follows from Corollary 12 of [4]. To see the statement (ii) we want to apply the Theorem 5, (i) and (ii) of [8]. We have to show that

$$(x_1,x_2,x_3,x_4)\cdot U(\mathfrak{a}+(F))\subseteq(\mathfrak{a}+(F))$$

where $\mathfrak{a}=(x_1^2x_3,x_1^2x_4,x_2x_3,x_2x_4)$ and $F=x_1+x_3$. Since

$$U(\mathfrak{a}+(F))=(x_1,x_3,x_4)\cap(x_1^2,x_3^2,x_2,x_1+x_3)=(x_1^2,x_3^2,x_1x_2,x_2x_4,x_1+x_3)$$

it is not hard to show that the above claim is true. From Theorem 3 of [4] follows that $\mathfrak{m}\cdot H_{\mathfrak{m}}^1(A)\neq 0$.

Proof of the Claims of Example 2. In order to prove the statement (i) we will show that $(x_1,x_2,x_3,x_4)\cdot U(\mathfrak{a}+(x_1+x_3))\not\subseteq(\mathfrak{a}+(x_1+x_3))$. The Theorem 5, (ii) of [8] then yields the claim. It is $U(\mathfrak{a}+(x_1+x_3))=(x_1,x_2,x_3)\cap(x_1,x_3,x_4)$. For example, it follows that $x_1^2\in(x_1,x_2,x_3,x_4)\cdot U(\mathfrak{a}+(x_1+x_3))$ but $x_1^2\notin(\mathfrak{a}+(x_1+x_3))$. The statement (ii) follows again from Theorem 5, (ii) of [8] since

$$(x_1,x_2,x_3,x_4)\cdot U(\mathfrak{a}+(x_2+x_4)^n)\subseteq(\mathfrak{a}+(x_2+x_4)^n)$$

for all $n>0$; namely it is $U(\mathfrak{a}+(x_2+x_4)^n)=(x_1,x_2,x_4^n)\cap(x_2^n,x_3,x_4)$. Applying the Lemma of 3 of [9] we get that the statement (ii) yields (iii).

REFERENCES

1. D. A. Buchsbaum, *Complexes in local ring theory.* In: Some aspects of ring theory, C.I.M.E. Rom 1965.

2. I. G. Macdonald and R. Y. Sharp, *An elementary proof of the non-vanishing of certain local cohomology modules.* Quart. J. Math. Oxford (2) 23(1972), 197–204.

3. E. Matlis, *Injective modules over Noetherian rings.* Pacific J. Math. 8 (1958), 511–528.

4. B. Renschuch, J. Stückrad and W. Vogel, *Weitere Bemerkungen zu einem Problem der Schnittheorie und über ein Mass von A. Seidenberg für die Imperfektheit.* J. Algebra 37 (1975), no. 3, 447–471.

5. P. Schenzel, J. Stückrad and W. Vogel, *Buchsbaum Moduln.* In preparation.

6. J. P. Serre, *Algèbre locale-Multiplicités.* Lecture Notes in Math., No. 11, Springer, Berlin, 1965.

7. J. Stückrad, *Über die kohomologische Charakterisierung von Buchsbaum-Moduln.* Math. Nachr. (to appear).

8. J. Stückrad and W. Vogel, *Eine Verallgemeinerung der Cohen-Macaulay Ringe und Anwendungen auf ein Problem der Multiplizitätstheorie.* J. Math. Kyoto Univ. 13 (1973), 513–528.

9. ———, *Toward a theory of Buchsbaum singularities.* Amer. J. Math. 100 (1978), no. 4, 727–746.

10. Ngo Viet Trung, Thesis, Martin-Luther-University, Halle, 1978.

11. W. Vogel, *Über eine Vermutung von D. A. Buchsbaum.* J. Algebra 25 (1973), 106–112.

Department of Mathematics
Martin-Luther-University
401 Halle, German Democratic Republic.