A VANISHING THEOREM FOR CERTAIN RIGID CLASSES

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Let V be a flat n-dimensional vector bundle and X a linear vector field on V which preserves the canonical flat foliation of V. Off a singular set of V, the flat foliation and X determine a codimension $n-1$ foliation and this new foliation has yielded interesting non-vanishing results for exotic classes [3], [6]. But these foliations cannot detect rigid exotic classes because a rigid class is also an exotic class for the codimension n flat foliation, and classes for this foliation vanish for trivial reasons. It is then reasonable to expect that if one takes an appropriate family of more than one linear vector field preserving a flat foliation one would arrive at a new foliation, readily computible in terms of linear data, with some nonzero rigid classes; the above vanishing of rigid classes for trivial reasons no longer holds.

The purpose of this paper is to show that if we take a family of commuting linear vector fields preserving a flat foliation then all rigid classes for the new lower codimension foliation still vanish. This is a companion theorem to those of Pittie [8, the first half of Theorem 2] and Bott-Haefliger [2], all of which assert that all rigid classes for a large family of homogeneous foliation must vanish.

1. DEFINITIONS AND STATEMENT OF THEOREM

For a discussion of exotic classes, see [1]. We will call a foliation *flat* if there is a locally flat basic connection on the normal bundle to the foliation. This is equivalent to having a basic connection ∇ on the normal bundle and a covering family of local framings $\{s_\lambda\}$ of the normal bundle for which $\nabla s_\lambda = 0$. Let us call such a family a family of locally flat framings. A vector field X *preserves* a foliation F if $[X, Y]$ is tangent to F whenever Y is tangent to F. Let X preserve a flat foliation. We will say X is *linear* if there is a family $\{s_\lambda\}$ of locally flat framings for which $L_X(s_\lambda) = A_\lambda s_\lambda$ where A_λ is a constant matrix. Here L_X is the Lie derivative and $L_X(s)$ is the matrix of sections obtained by lifting s to a matrix of vector fields $\tilde{s}$, applying L_X, and projecting back to the normal bundle. This is well defined since X preserves the foliation. Finally we will say that a family of vector fields $\{X_1, \ldots, X_k\}$ is *transverse* to a codimension n foliation if at each point this family and the tangent space to the foliation span a codimension $n-k$ subspace. A family of commuting vector fields $\{X_1, \ldots, X_k\}$ which preserves a codimension n foliation and which is transverse to this foliation determines a codimension $n-k$ foliation; the tangent space to the original foliation and to $\{X_1, \ldots, X_k\}$ is clearly integrable. Our main theorem is then:

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THEOREM 2. *Let F be a codimension n flat foliation of a manifold M and let $\{X_1,\dots,X_k\}$ be a family of commuting linear vector fields which preserves F and which is transverse to F. Then all rigid exotic classes for the resulting codimension $n-k$ foliation vanish.*

Remarks. (1) There is a common situation which gives rise to such data. Let G be a connected subgroup of $SL(n)$, Γ a co-compact discrete subgroup, K a maximal compact subgroup of G. $G/K \times_\Gamma R^n = V$ is a flat vector bundle and so has a canonical flat foliation. The leaves come from $G/K \times$ point via identifications. Let $x_1, \dots, x_k$ be mutually commuting matrices in $gl(n)$ which also commute with each element of g (the Lie algebra of G). Assume $x_1, \dots, x_k$ are independent and transverse to g. The one parameter groups $x_i(t)$ give rise to linear vector fields on R^n (via the standard action). Since each x_i commutes with g, the $x_i(t)$ commute with G and so also give rise to vector fields X_i on V which are linear. Then, off a singular subset S of V, $\{X_1,\dots,X_k\}$ is a family of commuting linear vector fields which preserve F and is transverse to $F|V-S$.

More specifically, let $G = \times_n SL(2)$, $K = \times_n SO(2)$, $\Gamma = \Gamma_1 \times \dots \times \Gamma_n$ where $\Gamma_j = \pi_1(\Sigma_j)$, Σ_j a surface of higher genus. Then V will be $G/K \times_\Gamma R^{2n}$. We choose, similar to [3],

$$X_1 = \sum_{i=1}^{n} \lambda_i^1 \left(x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i} \right), \dots, X_k = \sum_{i=1}^{n} \lambda_i^k \left(x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i} \right).$$

For generic choice of $\{\lambda_i^j\}$ the singular subset S will be small, and on $G/K \times_\Gamma (V-S)$ we have a situation as described in Theorem 2.

(2) There is a similar theorem for foliations of $\bar{G} \times_\Gamma G/P$ or $\bar{G}/\bar{K} \times_\Gamma G/P$ induced by projection to G/P, where (G,P) is a parabolic pair. In [2], [8] it is shown that all rigid classes are zero. There is some intersection of this result and our main theorem. For example, take $G = \bar{G} = SL(n)$, P the subgroup fixing a ray in R^n, then we get a foliation of $G \times_\Gamma G/P = \Gamma \backslash G \times G/P$ with vanishing rigid classes. This also falls in the framework of our situation. If we take the radial vector field on R^n, we get a linear vector field on $G \times_\Gamma R^n = \Gamma\backslash G \times R^n$, and the induced foliation on $\Gamma \backslash G \times S^{n-1}$ is the same foliation. There may be more intersection of the two theorems.

In the remainder of this section we recall some notation. Let ∇^1, ∇^0 be two connections on a vector bundle E over a manifold M. From [1] we have

$$\lambda(\nabla^1, \nabla^0)(f) = \pi_*\{f(\Omega_t)\}$$

where Ω_t is the curvature of $t\nabla^1 + (1-t)\nabla^0$ on $E \times [0,1]$, f is an Ad invariant polynomial function on matrices, π_* is integration over the fiber $[0,1]$, and $\lambda(\nabla^1, \nabla^0)(f)$ is a differential form on M. From [1], p. 69 we have the map $\lambda^*: H^*(WO_n) \to H^*(M)$ defining the exotic classes of a codimension n foliation of M and in [4] we have a description of the Vey basis for $H^*(WO_n)$ and the definition of rigid classes. A class $h_{i_1} \dots h_{i_r} c_J$ (J multi, $i_1 < \dots < i_r$) in the Vey basis is rigid if $i_1 + |J| > n + 1$.

2. PROOF OF MAIN THEOREM

Let $\langle\ ,\ \rangle$ be a Riemannian metric on M chosen so that $X_1, \ldots, X_k$ are orthonormal and orthogonal to $T(F)$. Let ω_i be the one form $\langle\ , X_i\rangle$ and let D be a flat basic connection on $\nu(F)$ relative to which the X_i are linear. Define a new connection on $\nu(F)$ by

$$\nabla_Y s = \sum_{i=1}^{k} \omega_i(Y) L_{X_i}(s) + D_{Y_0} s \tag{1.1}$$

where Y_0 is the component of Y orthogonal to $\{X_1,\ldots,X_k\}$. Let E be the resulting codimension $n-k$ foliation tangent to F and $\{X_1,\ldots,X_k\}$. Let $h_I c_J$, $I=(i_1,\ldots,i_r)$, $J=(j_1,\ldots,j_s)$ be an element of the Vey basis for $H^*(WO_{n-k})$. Let $h_I c_J(E)$ be the corresponding DeRham class of E and let $h_I c_J(\nabla)$ be the differential form

$$\lambda(\nabla, \nabla^R)(c_{i_1}) \ldots \lambda(\nabla, \nabla^R)(c_{i_k})\, c_J(\nabla)$$

where ∇^R is any Riemannian connection on $\nu(F)$.

THEOREM 1. *$h_I c_J(\nabla)$ represents $h_I c_J(E)$.*

Remark. The proof of this theorem will come in the next section. It is also true (and the proof is identical) that $h_I c_J(E)$ is represented by the differential form

$$\lambda(D^B, D^R)(c_{i_2}) \ldots \lambda(D^B, D^R)(c_{i_r})\, \lambda(\nabla, \nabla^R)(c_{i_1})\, c_J(\nabla)$$

where D^B is any basic connection for E on $\nu(E)$ and D^R is a Riemannian connection on $\nu(E)$. The chosen form of the theorem is for notational convenience.

LEMMA (1.2). *Each $d\omega_i$ is in the ideal of forms which vanish on E.*

Proof. We must show $d\omega_i(Y,Z)=0$ when Y and Z are tangent to E. By construction $\omega_i(X_j)=\delta^i_j$ and $\omega_i(Y)=0$ if Y is tangent to F. Take $i=1$ for convenience. Extend Y and Z to vector fields. $d\omega_1(Y,Z) = Y\omega_1(Z) - Z\omega_1(Y) - \omega_1([Y,Z])$. If both Y and Z are among $\{X_1,\ldots,X_k\}$ then $\omega_1(Y)$ and $\omega_1(Z)$ are constant, $[Y,Z]=0$ and so $d\omega_1(Y,Z)=0$. If $Y = X_i$ and Z is tangent to F then $\omega_1(Y)$ is constant, $\omega_1(Z)=0$, $[Y,Z]$ is tangent to F so $\omega_1([Y,Z])=0$ and again $d\omega_1(Y,Z)=0$. If Y and Z are tangent to F so is $[Y,Z]$ and thus $\omega_1([Y,Z])=0$. Also $\omega_1(Y)=\omega_1(Z)=0$ and so $d\omega_1(Y,Z)=0$.

Proof of Theorem 2. Let s be a locally flat framing of $\nu(F)$ relative to which all the X_i act linearly. Thus $Ds=0$. We compute the curvature matrix $d\omega + 1/2\,[\omega,\omega]$ of ∇ relative to s. Let $\omega = \nabla s = \sum_{i=1}^{k} \omega_i \otimes L_{X_i}(s)$. Let A_i be the constant matrix of $L_{X_i}(s)$ relative to s. Then the matrix of $L_{[X_i,X_j]}(s)$ relative to s is $\pm[A_i,A_j]$, thus $[A_i,A_j]=0$. Now

$$[\omega,\omega] = \left[\sum \omega_i A_i, \sum \omega_j A_j\right] = \sum_{i,j} \omega_i\omega_j\,[A_i,A_j] = 0.$$

Thus the curvature matrix is $d\omega = \sum_{i=1}^{k} d\omega_i A_i$. Then

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Now, if $h_I c_J$ is a rigid class for E then $i_1 + |J| > n - k + 1$. Consider the differential form $h_I (d\omega_1)^{\alpha_1} \dots (d\omega_s)^{\alpha_s}$. One of the α's, say α_1, is not zero.

$$d(h_I \omega_1 (d\omega_1)^{\alpha_1 - 1} (d\omega_2)^{\alpha_2} \dots (d\omega_s)^{\alpha_s}) = \pm h_I (d\omega_1)^{\alpha_1} \dots (d\omega_s)^{\alpha_s}$$
$$+ \sum \pm h_{i_1} \dots \hat{h}_{i_j} \dots h_{i_r} \omega_1 c_{i_j}(\nabla)(d\omega_1)^{\alpha_1 - 1} (d\omega_2)^{\alpha_2} \dots (d\omega_s)^{\alpha_s}.$$

$$c_J(\nabla) = \sum_{\alpha_1 + \dots + \alpha_s = |J|} b_\alpha (d\omega_1)^{\alpha_1} \wedge \dots \wedge (d\omega_s)^{\alpha_s}.$$

Now, if $h_I c_J$ is a rigid class for E then $i_1 + |J| > n - k + 1$. Consider the differential form $h_I (d\omega_1)^{\alpha_1} \dots (d\omega_s)^{\alpha_s}$. One of the α's, say α_1, is not zero.

$$d(h_I \omega_1 (d\omega_1)^{\alpha_1 - 1} (d\omega_2)^{\alpha_2} \dots (d\omega_s)^{\alpha_s}) = \pm h_I (d\omega_1)^{\alpha_1} \dots (d\omega_s)^{\alpha_s}$$
$$+ \sum \pm h_{i_1} \dots \hat{h}_{ij} \dots h_{i_r} \omega_1 c_{i_j}(\nabla)(d\omega_1)^{\alpha_1 - 1} (d\omega_2)^{\alpha_2} \dots (d\omega_s)^{\alpha_s}.$$

Let us examine $c_{i_j}(\nabla)(d\omega_1)^{\alpha_1 - 1} (d\omega_2)^{\alpha_2} \dots (d\omega_s)^{\alpha_s}$. Since

$$(\alpha_1 - 1) + \alpha_2 + \dots + \alpha_s = |J| - 1, \qquad i_j + (\alpha_1 - 1) + \alpha_2 + \dots + \alpha_s = i_j + |J| - 1.$$

Since we have a rigid class,

$$i_j + |J| > n - k + 1, \qquad i_j + (\alpha_1 - 1) + \alpha_2 + \dots + \alpha_s > n - k.$$

From the form of $c_{i_j}(\nabla)$ and Lemma (1.2), $c_{i_j}(\nabla)(d\omega_1)^{\alpha_1 - 1}(d\omega_2)^{\alpha_2} \dots (d\omega_s)^{\alpha_s}$ is in the $n - k + 1$ power of the ideal of forms which vanish on E. Thus

$$c_{i_j}(\nabla)(d\omega_1)^{\alpha_1 - 1}(d\omega_2)^{\alpha_2} \dots (d\omega_s)^{\alpha_s} = 0.$$

Hence $h_I c_J(\nabla)$ is exact and so by Theorem 1, $h_I c_J(E) = 0$.

3. PROOF OF THEOREM 1

From the definition of exotic classes, $h_I c_J(E)$ is represented by

$$\lambda(D^B, D^R)(c_{i_1}) \dots \lambda(D^B, D^R)(c_{i_r}) c_J(D^B)$$

where D^B and D^R are basic and Riemannian connections on $\nu(E)$. We wish to replace D^B and D^R by ∇^B by ∇^R (connections on $\nu(F)$).

Let $\bar{D}$ be the connection on the bundle $X_1 + \dots + X_k$ defined by

$$\bar{D}X_i \equiv 0,\ i = 1, \ldots, k.$$

Use the metric to decompose $\nu(F)$ as $\nu(E) + (X_1 + \ldots + X_k)$. Relative to this decomposition we can construct the connections $D^B + \bar{D}$ and $D^R + \bar{D}$. The latter is, of course, Riemannian. Let f be an Ad invariant, homogeneous polynomial of degree l.

Note that $f(\nabla)$ lies in the l^{th} power of the ideal of forms defining E. From the definition of λ, it follows that $\lambda(D^B + \bar{D}, D^R + \bar{D})(f) = \lambda(\hat{D}^B, D^R)(f)$. The *crucial step* in the proof of this theorem will be to show that $\lambda(\nabla, D^B + \bar{D})(f)$ plus an exact form lies in the l^{th} power of the ideal of forms defining E.

Proof of Theorem 2 from This Crucial Step. Let $\nabla^R = D^R + \bar{D}$. Apply Stoke's theorem to the connections ∇, $D^B + \bar{D}$, ∇^R and the polynomial c_i to conclude that $\lambda(\nabla, \nabla^R)(c_i) - \lambda(D^B + \bar{D}, \nabla^R)(c_i) - \lambda(\nabla D^B + \bar{D})(c_i)$ is exact. Multiply by $c_J(\nabla)$ and use Bott vanishing for E to conclude $\lambda(\nabla, \nabla^R)(c_i)\, c_J(\nabla) - \lambda(D^B, D^R)(c_i)\, c_J(\nabla)$ is exact. Then $c_J(\nabla) - c_J(D^B) = d\lambda(\nabla, D^B + \bar{D})(c_J)$ and so (using Bott vanishing) $\lambda(\nabla, \nabla^R)(c_i)\, c_J(\nabla) - \lambda(D^B, D^R)(c_i)\, c_J(D^B)$ is exact. This proves the theorem for a single i. For more than one i replace each $\lambda(D^B, D^R)(c_{i_j})$ by $\lambda(\nabla, \nabla^R)(c_{i_j})$ one at a time by a similar argument. A standard argument shows that ∇^R can be replaced by any Riemannian connection.

Now we prove the crucial step with a sequence of lemmas.

LEMMA (2.1). *There is a basic connection D on $\nu(F)$ for which $DX_i = 0$, $i = 1, \ldots, k$.*

Proof. Let D' be any connection on $\nu(F)$ for which $D'X_i = 0$. Let $Y = Y_1 + Y_2$ be orthogonal decomposition of Y according to $T(M) = T(F) + \nu(F)$. $D_Y s = [Y_1, s] + D'_{Y_2} s$ yields the desired connection.

Now define ∇ as in (1.1) except take the D to be as in (2.1).

LEMMA (2.2). *$\lambda(\nabla, D^B + \bar{D})(f)$ is in the l^{th} power of the ideal defining E.*

Proof. First, we need only consider polynomials $f(A) = \text{Trace}\,(A^i)$ since these generate all polynomials.

Next, for any two connections ∇ and ∇^0, let θ and θ^0 be local connection matrices and let $\rho = \theta - \theta^0$, $\Theta = d\rho + [\rho, \theta^0]$, $\Omega_0 = d\theta^0 + 1/2\,[\theta^0, \theta^0]$. ρ, Θ, and Ω_0 are tensorial [5, p. 75]. A direct computation shows

$$\text{(2.3)} \qquad \lambda(\nabla, \nabla^0)(f) = \sum_{i+j+k=l-1} b_{(i,j,k)}\, \text{Trace}\,(\rho^{2i+1} \wedge \Theta^j \wedge \Omega_0^k)$$

where the b's are constant. For our connections ∇ and $D^B + \bar{D}$ it will be sufficient to show $\text{Trace}\,(\rho^{2i+1} \wedge \Theta^j \wedge \Omega_0^k)$ is in the l^{th} power of the ideal of E.

From [7] we can choose local coordinates $\{u_1, \ldots, u_p, x_1, \ldots, x_k, y_1, \ldots, y_{n-k}\}$ such that

(a) $y_1 = \ldots = y_{n-k} =$ constant define the leaves of E.

(b) $y_1 = \ldots = y_{n-k} = x_1 = \ldots = x_k =$ constant define the leaves of F.

(c) $X_i - \partial/\partial x_i$ is tangent to F.

Let $v_i = \partial/\partial y_i - \sum_j < \partial/\partial y_i, X_j > X_j$. Then the orthogonal projection of $\{v_1,\ldots,v_{n-k},X_1,\ldots,X_k\}$ onto $\nu(F)$ is a local framing which respects the decomposition $\nu(F) = \nu(E) + (X_1 + \ldots + X_k)$.

Let θ and θ^0 be local connection matrices for ∇ and $D^B + \bar{D}$ relative to this framing. Direct computation shows that for Y tangent to E, $\nabla_Y v_i$ is a linear combination of $X_1, \ldots, X_k$ and $\nabla X_i = 0$ for all i. Thus,

$$\theta = \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix} \tag{2.4}$$

where A is an $(n-k)$ square matrix of forms all of which are in the ideal of E. θ^0 is a direct sum $\begin{pmatrix} \theta^1 & 0 \\ 0 & \theta^2 \end{pmatrix}$. $\theta^2 = 0$ from the definition of $\bar{D}$ and direct computation shows that θ^1 is in the ideal of E. Thus θ^0 is of the form (2.4) (with $B = 0$). Thus, ρ, θ^0 and also Θ and Ω_0 are of the form (2.4). So $\rho^{2i+1} \wedge \Theta^j \wedge \Omega_0^k$ is of the form $\begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix}$ with A in the l^{th} power of the ideal of E. Thus applying trace we have the desired result.

The crucial step then follows from the next lemma.

LEMMA (2.5). *Let ∇ be a connection as in* (1.1) *except the D can be any basic connection for F. Then $\lambda(\nabla, D^B, + \bar{D})(f)$ + exact is in the l^{th} power of the ideal defining E.*

Proof. Let ∇^0 be the ∇ of the previous lemma ($DX_i = 0$). By Stoke's theorem, it is sufficient to show $\lambda(\nabla,\nabla^0)(f)$ is in the l^{th} power of this ideal. Use the local coordinates of the previous lemma. Take the local framing of $\nu(F)$ given by $\{\partial/\partial y_1,\ldots,\partial/\partial y_{n-k},X_1,\ldots,X_k\}$ mod $T(F)$. Direct computation shows $\nabla_Y\, \partial/\partial y_i = \nabla_Y X_i = 0$ for Y tangent to E. Of course the same is true of ∇^0. Thus in the expansion (2.3) of $\lambda(\nabla,\nabla^0)(f)$ the forms θ, θ^0, ρ and hence Θ and Ω_0 are matrices all of whose entries are in the ideal of E. Now the lemma follows as in (2.2).

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