THE THEORY OF MOTION GROUPS

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INTRODUCTION

In 1962 in his Ph.D. thesis, David M. Dahm defined the group of motions with compact support of the compact subspace N in the manifold M to be the group of essentially different ways of continuously moving N in M, so that at the end of the motion, N has been returned to its original position. This paper is mostly my exposition of Dahm's unpublished work.

It was the idea of Hurwitz (see [18]), and later of Fox (see [12]), to envisage a braid (see Figure 1) as a continuous 1-parameter family of changing configurations of n distinct points in the xy-plane, where at each time t_0, the configuration is given by the intersection of the braid with the plane at height $z = t_0$. Thus, the motion group $\mathscr{M}(M,N)$ has its origins in the Artin braid group ([1], [2], [3] and [4]).

The sections of this paper are

1. The Braid Groups

2. Motion Groups Defined

3. Properties of Motion Groups

4. The Dahm Homomorphism

5. The Group of Motions of a Collection of n Unknotted, Unlinked Circles in $\mathbf{R}^3$

The background and motivation of the braid groups is discussed in Section 1. In Section 2 I define and give examples of motion groups. Section 3 is mostly a list of short exact sequences containing the group of motions as a term, from which this group may be computed. Section 4 defines a homomorphism

$$D: \mathscr{M}(M,N) \rightarrow \operatorname{Aut}(\pi_1(M-N))$$

from the group of motions of N in M, to the automorphisms of $\pi_1(M-N)$ induced at the end of the motion.

The main result of the paper, in Section 5, is the following calculation: Let $C = C_1 \cup \ldots \cup C_n \subset \mathbf{R}^3$ be a collection of n unknotted, unlinked circles in $\mathbf{R}^3$. Let $F(x_1,\ldots,x_n)$ denote the free group on n generators, x_i, $i = 1,\ldots,n$; note that $\pi_1(\mathbf{R}^3 - C) \simeq F(x_1,\ldots,x_n)$.

THEOREM 5.4. *The group of motions $\mathscr{M}(\mathbf{R}^3,C)$ of the trivial n-component link C in $\mathbf{R}^3$ is generated by the following types of motions:*

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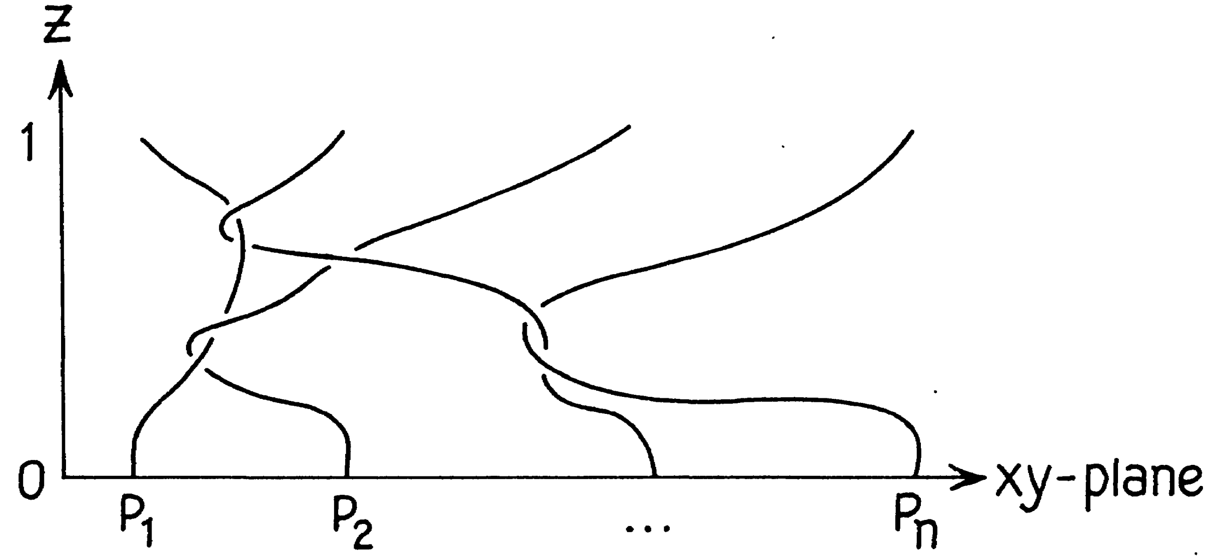

Figure 1

R_i: turn the i^{th} circle over. This motion induces the automorphism

$$\rho_i: x_i \to x_i^{-1}, x_k \to x_k, \qquad k \neq 1.$$

T_i: interchange the i^{th} and $(i+1)^{\text{st}}$ circles. The induced automorphism is $\tau_i: x_i \to x_{i+1}, x_{i+1} \to x_i, x_k \to x_k, k \neq i, i+1$.

A_{ij}: pull the i^{th} circle through the j^{th} circle. This motion induces the automorphism $\alpha_{ij}: x_i \to x_j x_i x_j^{-1}, x_k \to x_k, k \neq 1$.

The homomorphism $D: \mathscr{M}(\mathbf{R}^3, C) \to \mathrm{Aut}(F(x_1, \ldots, x_n))$ is an isomorphism onto the subgroup generated by $\rho_i, \tau_i, \alpha_{ij}, 1 \leq i, j \leq n, i \neq j$.

The maps and spaces in this paper are in the topological category; subspaces $N \subset M^n$ are assumed to be compact, and M^n is assumed to have no boundary. Whenever the P.L. or differentiable categories are invoked, it will be to make use of extra structures on N and M which, in the particular cases being considered, it may be assumed exist. (For example, it is proved by Lashof, Moise, and others (see [9]), that there is little difference between the three categories, if $n = 3$.)

The idea of a motion group was first conceived by R. H. Fox, and later made rigorous by his student, David Dahm. I am publishing Dahm's work because I feel that the motion group is a useful mathematical concept, and to lay the groundwork for a subsequent paper ([17]) in which I compute the group of motions of a certain class of nontrivial links in S^3.

For the convenience of the reader we now list the notation used in the remainder of the paper. The first six entries are introduced in Section 2 and the second six in Section 3. Let M be a manifold, and let $N \subset M$ be a compact subspace in the interior of M.

1. $E(M,N)$ is the space of embeddings of N in M, with the compact open topology.

2. $H(M)$ is the space of self-homeomorphisms of M, with the compact open topology.

3. $1_M: M \to M$ is the identity map.

4. $H(M,N)$ is the subspace $\{h \in H(M): h(N) = N\}$.

5. $H_c(M) \subset H(M)$ and $H_c(M,N) \subset H(M,N)$ are the subspaces of homeomorphisms with compact support. These are topological groups.

6. $i_N: N \to M$ is inclusion.

Let $N_1, N_2 \subset M$ be compact, disjoint subspaces.

7. $\mathscr{H}(M)$ is the group $\pi_0(H_c(M); 1_M)$.

8. $\mathscr{H}(M,N)$ is the group $\pi_0(H_c(M,N); 1_M)$.

9. $H_c^+(M,N) = \{h \in H_c(M,N)$ and $h: M \to M$ is orientation preserving$\}$.

10. $\mathscr{H}^+(M,N) = \pi_0(H_c^+(M,N); 1_M)$.

11. $H_c(M,N_1,N_2) = \{h \in H_c(M): h(N_i) = N_i, i = 1,2\}$

12. $\mathscr{H}(M,N_1,N_2) = \pi_0(H_c(M,N_1,N_2); 1_M)$

1. THE BRAID GROUPS

The definition of Artin's braid groups, B_n, is well-known (see [6], [7], [12], [15], [16]). Hurwitz showed (in [18]) that B_n could be viewed as the fundamental group of a certain configuration space. This is, historically, the first motivation for general motion groups. These terms are defined as follows:

(1) A *configuration* of n distinct points in the complex plane is any collection $\{z_1, \ldots, z_n\}$ of n distinct complex numbers.

(2) The *configuration space* of n distinct points in $\mathbf{C}$ is the quotient space $\underbrace{\mathbf{C} \times \ldots \times \mathbf{C}}_{n \text{ times}} \setminus$diagonal modulo the action of the permutation group $S(n)$ on n-tuples $(z_1, \ldots, z_n)$ of complex numbers (disregard the order of the components.)

(3) Let $\vec{c}$ be a particular configuration of n points in $\mathbf{C}$. A *motion* of $\vec{c}$ in $\mathbf{C}$ is a loop in the configuration space based at $\vec{c}$.

(4) The *group of motions* of $\vec{c}$ in $\mathbf{C}$ (which Hurwitz proved in [18] to be isomorphic to B_n) is the fundamental group of the configuration space of n distinct points in $\mathbf{C}$, based at $\vec{c}$.

2. MOTION GROUPS DEFINED

The braid group as a group of motions admits an obvious generalization; that is, replace $\mathbf{C}$ by any manifold, and replace $\vec{c}$ by any compact subspace $N \subset M$ contained in the interior of M. For example, a motion of N = the trefoil knot in $M = S^3$ is illustrated in Figure 2.

Figure 2. Rotate the trefoil knot by $2\pi/3$. Two motions are "multiplied" by following one by the other. We will now make these ideas more precise.

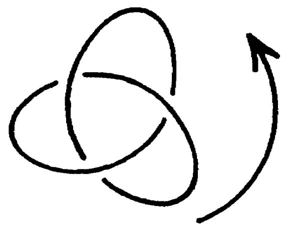

Figure 2

The analogue of the configuration space of n points in the complex plane is the quotient space $E(M,N)/h \sim h'$ if $h(N) = h(N')$. There are several reasons why it would be incorrect to define a motion to be a loop in this space. The topology of $E(M,N)$ is unmanageable. (This is unlike the differentiable category, in which the same quotient space is the base of a fiber bundle with total space $E(M,N)$ and fiber $H(N)$. See [20].) Also, we would like motions of N in M to be extendable to motions of a neighborhood of N in M. This is violated by the space $E(M,N)/\sim$ when $M = S^3$ and N = any knot K; for then there is a path in $E(S^3,K)$ beginning with the knotted embedding i_K and ending with the unknotted circle in S^3, which pulls the knot taut on the string until it "pops." This path cannot be extended to a tabular neighborhood of K.

Now I will present Dahm's definition of a motion group.

Definition 2.1. A *motion* of N in M is a path f_t in $H_c(M)$ such that $f_0 = 1_M$ and $f_1 \in H_c(M,N)$.

Definition 2.2. A *stationary motion* of N in M is a path f_t in $H_c(M,N)$ (thus, the configuration of N in M remains the same for all t. See Figure 3.)

Figure 3. A stationary motion of the trefoil knot: slide K along itself.

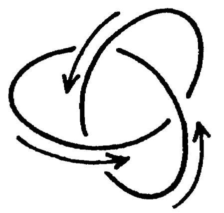

Figure 3

Definition 2.3. The *product* $g \circ f$ *of two motions* f,g of N in M is the path

$$\begin{cases} f_{2t} & 0 \le t \le 1/2 \\ g_{2(t-1/2)} \circ f_1 & 1/2 \le t \le 1. \end{cases}$$

Definition 2.4. The *inverse* g^{-1} *of a motion* g of N in M is the path $g_{(1-t)} \circ g_1^{-1}$.

Definition 2.5. Two motions f,g, of N in M are *equivalent* (denoted by $f \equiv g$) if $g^{-1} \circ f$ is homotopic, keeping endpoints fixed, to a stationary motion. (Thus, stationary motions are equivalent to the trivial motion $f_t = i_N$ for all t.)

PROPOSITION 2.6. *The set of equivalence classes of motions of N in M, with multiplication induced by $\circ$, forms a group.* (*We will denote this group by $\mathcal{M}(M,N)$*).

Proof. $\mathcal{M}(M,N)$ is the relative fundamental group $\pi_1(H_c(M), H_c(M,N), 1_M)$.

Remark 2.7. The relative first homotopy "group" $\pi_1(G,S,e)$ is generally just a *set* of homotopy classes of paths in G which begin at $c \in S$ and end in S. Group multiplication may be defined on the set when G is a topological group, $S \subset G$ is a subgroup, and $e \in G$ is the identity element, by using the multiplication in G. This can be done in one of two ways: paths f,g may be multiplied pointwise, $(g \cdot f)_t = g_t \circ f_t$, or as in Definition 2.3. It is not hard to show that the paths $g \cdot f$ and $g \circ f$ are homotopic, relative to their endpoints.

The following proposition should convince the reader that the definition of a motion agrees with our original notion of a loop in a configuration space:

PROPOSITION 2.8. *Let f,g be motions of N in M. Then $f \equiv g$ if and only if f is homotopic to f', where f' is a motion of N in M such that for all t, $f'_t(N) = g_t(N)$.*

Proof. Since $f \equiv g$, by Definition 2.5 and Remark 2.8 f is homotopic to $g \cdot s$, where s is a stationary motion of N in M. Let $f' = g \cdot s$. Then for all t, $f'_t(N) = g_t(N)$.

Now suppose f is homotopic to f', with $f'_t(N) = g_t(N)$ for all t. Then $s = g^{-1} \cdot f'$ is a stationary motion of N in M, and $g^{-1} \circ f$ is homotopic to s, so $f \equiv g$ by Definition 2.5.

EXAMPLES OF MOTION GROUPS

Example 1. The group of motions $\mathcal{M}(M,p)$ of a point p in a manifold M is the fundamental group $\pi_1(M;p)$ of M based at p.

Example 2. The group of motions of n distinct points $P = \{p_1, \ldots, p_n\} \subset M$ in a connected manifold M is the braid group $B_n(M)$.

We are already aware of Hurwitz' proof that $B_n(\mathbf{C})$ is the fundamental group of the space of configurations of n points in the manifold $\mathbf{C}$. Examples 1 and 2 follow from Lemma 2.9, whose proof is left as an exercise for the reader.

LEMMA 2.9. *The restriction map*

$$(H_c(M), H_c(M,P), 1_M) \to (E(M,P), E(P,P), 1_P)$$

induces isomorphisms on all relative homotopy groups.

Lemma 2.9 is immediate if one first proves that any k-isotopy of n distinct points in a manifold M, (that is, a continuous family of embeddings of n distinct points into M parametrized by the k-simplex) extends to a k-isotopy of all of M.

Example 3. If $n > 2$, then $\mathcal{M}(M,P) \simeq \bigoplus_{i=1}^{n} \pi_1(M;p_i)$.

Example 3 asserts that the braiding phenomenon disappears in manifolds of dimension greater than 2. A proof appears in [10].

The final and most interesting example, the group of motions of n unlinked, unknotted circles in $\mathbf{R}^3$, is computed in Section 5. Generators are the motions which flip a circle, exchange two circles, and move one circle through another.

3. PROPERTIES OF MOTION GROUPS

Some general properties of motion groups which will be used in later sections, are presented here.

Definition 3.1. The homomorphism $\partial: \mathscr{M}(M,N) \to \mathscr{H}(M,N)$ is defined by $\partial([f]) = [f_1]$, where f is a motion of N in M.

PROPOSITION 3.2. *The following sequence is exact:*

$$\pi_1(H_c(M,N);1_M) \to \pi_1(H_c(M);1_M) \to \mathscr{M}(M,N) \xrightarrow{\partial} \mathscr{H}(M,N) \xrightarrow{i} \mathscr{H}(M).$$

Proof. This is just the long exact sequence for relative homotopy groups.

COROLLARY 3.3. *Let* $M = S^n$ or $\mathbf{R}^n$, $n \neq 4$. *Then the following sequence is exact:*

$$\pi_1(H_c(M,N);1_M) \to \pi_1(H_c(M);1_M) \to \mathscr{M}(M,N) \xrightarrow{\partial} \mathscr{H}^+(M,N) \to 1.$$

Proof. $\pi_0(H_c^+(S^n);1_{S^n}) = \pi_0(H_c^+(\mathbf{R}^n)) = 1$ if $n \neq 4$. Clearly,

$$\text{image }(\partial) \subseteq \mathscr{H}^+(M,N).$$

PROPOSITION 3.4. *Suppose there is a subset $K \subset M$ with the following two properties:*

(i) *There is a 1_M-based path h in $H(M)$ with $h_1(N) \subset K$.*

(ii) *Given any motion f of N in M such that $f_1 = 1_M$, there is a homotopic motion $g \simeq f$ such that for all t, $g_t | K = 1$.*

Then the following sequence is exact:

$$1 \to \mathscr{M}(M,N) \xrightarrow{\partial} \mathscr{H}(M,N) \to \mathscr{H}(M).$$

Proof. (Due to Dahm) An element in $\ker(\partial)$ can be represented by a 1_M-based loop in $H(M)$. By (ii) there is then a homotopic motion f, such that $f_t | K = 1$. We will show that $f \equiv$ the trivial motion.

First define the following homotopy of f:

$$\begin{aligned} P_{s,t} &= h_{s(3t)} && 0 \leq t \leq 1/3 \\ &= f_{3(t-1/3)} \circ h_s && 1/3 \leq t \leq 2/3 \\ &= h_{s(3[1-t])} && 2/3 \leq t \leq 1. \end{aligned}$$

This is continuous because $f_1 = 1_M$. For all s, $P_{s,0} = P_{s,1} = 1_M$, so that $P_{s,t}$ is indeed a homotopy. $P_0 = f$, and P_1 is a motion which pulls N into the subset K along $h_t(N)$, holds N constant, and then pushes N back along $(h^{-1})_t(N)$. This description of the motion P_1 strongly suggests that it is equivalent to the trivial motion of N in M. To show that this is the case, we define the motion T_0 equivalent to P_1:

$$\begin{aligned} T_{0,t} &= h_{3t} & 0 \le t \le 1/3 \\ &= 1_M \circ h_1 & 1/3 \le t \le 2/3 \\ &= h_{3[1-t]} & 2/3 \le t \le 1. \end{aligned}$$

T_0 is equivalent to P_1 by Proposition 2.8, because for all t, $T_{0,t}(N) = P_{1,t}(N)$. Now $T_0 \simeq$ trivial motion by the homotopy $T_{s,t}$:

$$\begin{aligned} T_{s,t} &= h_{3st} & 0 \le t \le 1/3 \\ &= h_s & 1/3 \le t \le 2/3 \\ &= h_{3s(1-t)} & 2/3 \le t \le 1 \end{aligned}$$

Thus, f trivial motion. This means the image of $\pi_1(H_c(M); 1_M) \to \mathcal{M}(M,N)$ in the exact sequence of Theorem 3.2 is the same as the image of $1 \to \mathcal{M}(M,N)$.

Example 3.5. $\mathbf{R}^n$ is a manifold satisfying the hypothesis of Proposition 3.4, where the subset K may be taken to be a neighborhood of infinity.

COROLLARY 3.6. *The sequence* $1 \to \mathcal{M}(\mathbf{R}^n, N) \to \mathcal{H}^+(\mathbf{R}^n, N) \to 1$ *is exact.*

Proof. This follows from Corollary 3.3 and Proposition 3.4.

PROPOSITION 3.7. *If* $h \in H_c(M)$, *then* $\mathcal{M}(M,N) \simeq \mathcal{M}(M, h(N))$.

Proof. h defines a homeomorphism

$$(H_c(M), H_c(M,N), 1_M) \to (H_c(M), H_c(M, h(N)), 1_M)$$

by $h' \to hh'h^{-1}$ whenever $h' \in H_c(M)$; this induces an isomorphism $\mathcal{M}(M,N) \xrightarrow{\simeq} \mathcal{M}(M, h(N))$.

COROLLARY 3.8. *If* N, N' *are ambient isotopic subspaces of* M, *then* $\mathcal{M}(M,N) \simeq \mathcal{M}(M,N')$.

Now let $N_1, N_2 \subset M$ be a pair of disjoint subspaces.

Definition 3.9. The group of motions of the pair (N_1, N_2) in M (denoted $\mathcal{M}(M, N_1, N_2)$) is the relative fundamental group $\pi_1(H_c(M), H_c(M, N_1, N_2); 1_M)$.

In Proposition 3.10, let e: $\mathcal{M}(M - N_1, N_2) \to \mathcal{M}(M, N_1, N_2)$ be induced by the map $(H_c(M - N_1), H(M - N_1, N_2), 1_{M-N_1}) \to (H_c(M), H_c(M, N_1, N_2), 1_M)$ which sends each $h \in H_c(M - N_1)$ to its extension $e(h) \in H_c(M)$, such that $e(h)|N_1 = 1_{N_1}$.

Let p_1: $\mathcal{M}(M, N_1, N_2) \to \mathcal{M}(M, N_1)$ be induced by the map

$$(H_c(M), H_c(M,N_1,N_2), 1_M) \to (H_c(M), H_c(M,N_1), 1_M)$$

which sends each $h \in H_c(M,N_1,N_2)$ to $h \in H_c(M,N_1)$.

PROPOSITION 3.10. *Suppose N_1, N_2 are disjoint compact subspaces of M. Then the following sequence is exact:*

$$\mathscr{M}(M - N_1, N_2) \xrightarrow{e} \mathscr{M}(M, N_1, N_2) \xrightarrow{\rho_1} \mathscr{M}(M, N_1).$$

Proof. Let f be a motion of (N_1, N_2) in M such that $[f] \in \ker(\rho_1)$. Then $f \simeq f'$ where f' is a stationary motion of N_1 in M. By [19], there is path g in $H_c(M - N_1, N_2)$ such that for all t, $g_t | N_2 = f' | N_2$. By Proposition 2.8, $e(g) \equiv f'$. So, image$(e) = \ker(\rho_1)$.

4. THE DAHM HOMOMORPHISM

The Dahm Homomorphism is a map D: $\mathscr{M}(M,N) \to \mathrm{Aut}(\pi_1(M - N))$ if M is non-compact, and a map D: $\mathscr{M}(M,N) \to \mathrm{Out}(\pi_1(M - N))$ if M is compact (and therefore closed, since $\partial M = \emptyset$). The first map was defined by David Dahm in his Princeton thesis ([10]).

1. In the first case, D is defined as follows:

Since M is non-compact, we may choose a path $p: (0,1) \to M - N$ such that the path $p(x)$ approaches an end of the manifold M as $x \to 0$. Suppose f is any motion of N in M. Since f_t has compact support and the unit interval is compact, there is an $\varepsilon > 0$ such that for all $x \leq \varepsilon$, and for all $t \in [0,1]$, $f_t(p(x)) = p(x)$. Now $f_1: (M - N, p(\varepsilon)) \to (M - N, p(\varepsilon))$ induces an automorphism

$$(f_1)_*: \pi_1(M - N; p(\varepsilon)) \to \pi_1(M - N; p(\varepsilon)).$$

Define $\beta_{p,\varepsilon}(f) = (f_1)_*$.

If $\varepsilon' < \varepsilon$, then let $b_t = p(\varepsilon t + \varepsilon'(1 - t))$ be the segment of p between ε and ε'. There is a natural isomorphism ϕ: $\pi_1(M - N; p(\varepsilon)) \to \pi_1(M - N; p(\varepsilon'))$ obtained by mapping a loop q based at $p(\varepsilon)$ to the loop bqb^{-1} based at $p(\varepsilon')$. Now $\phi \circ \beta_{p,\varepsilon}(f) \circ \phi^{-1} = \beta_{p,\varepsilon'}(f)$, since f_1 leaves b fixed. Thus $\beta_{p,\varepsilon}(f)$ and $\beta_{p,\varepsilon'}(f)$ are the same automorphism, modulo the isomorphism ϕ.

Next, let p': $(0,1) \to M - N$ be a different path such that $p'(x)$ approaches the same end of M as the path p, as $x \to 0$. If f is a motion of N in M, then there is an $\varepsilon > 0$, and a path δ: $[0,1] \to M - N$, such that $\delta(0) = p(\varepsilon)$, $\delta(1) = p'(\varepsilon)$, $\forall x f(\delta(x)) = \delta(x)$, and for all $x \leq \varepsilon$, for all $t \in [0,1]$, $f_t(p(x)) = p(x)$ and

$$f_t(p'(x)) = p'(x).$$

Then $\beta_{p,\varepsilon}(f)$ is the same as $\beta_{p',\varepsilon}(f)$, in the same sense that $\beta_{p,\varepsilon}(f)$ and $\beta_{p,\varepsilon'}(f)$ were shown to be the same. We see that $(f_1)_*: \pi_1(M - N) \to \pi_1(M - N)$ is well-defined, up to the choice of an end of M. Fix an end of the manifold M, and define $\beta(f) = (f_1)_*$.

Let g be a motion of N in M such that $f \equiv g$, then $\beta(f) = \beta(g)$ since

$$f_1, g_1 : (M,N) \to (M,N)$$

are isotopic homeomorphisms in $H_c(M,N)$. Then define

$$D : \mathscr{M}(M,N) \to \mathrm{Aut}(\pi_1(M-N))$$

by $D([f]) = \beta(f)$. The map D is a homomorphism because $\beta(g \circ f) = (g_1)_* \circ (f_1)_*$, and because $\beta(f^{-1}) = (f_1)_*^{-1}$.

2. In the second case, D is defined as follows:

Let f be a motion of N in M. Then $f \equiv g$, where g is a motion of N in M satisfying $\forall t\, g_t(e) = e$, for some point $e \in M - N$. Define

$$\beta_e(f) = (g_1)_* : \pi_1(M-N;e) \to \pi_1(M-N;e).$$

Now β_e is well-defined, up to composition with inner automorphisms of $\pi_1(M-N;e)$. So $\beta_e(f) \in \mathrm{Out}(\pi_1(M-N;e))$.

Suppose $e' \in M - N$ is a different basepoint, and let b: $[0,1] \to M - N$ be an arc between e and e'. Then $f \equiv g$, where g is a motion of N in M such that for all $s,t \in [0,1]$, $g_t(b(s)) = b(s)$. Now if ϕ: $\pi_1(M-N;e) \to \pi_1(M-N;e')$ is defined as in 1, then $\beta_e(f)$ and $\beta_{e'}(f)$ are really the same outer automorphism of $\pi_1(M-N)$, in the sense that $\phi \circ \beta_e(f) \circ \phi^{-1} = \beta_{e'}(f)$.

Define D: $\mathscr{M}(M,N) \to \mathrm{Out}(\pi_1(M-N))$ by $D([f]) = \beta_e(f)$. The proof that D is a homomorphism is essentially the same as in 1.

5. THE GROUP OF MOTIONS OF A TRIVIAL LINK OF UNKNOTTED CIRCLES IN $\mathbf{R}^3$

Let the trivial link $C = C_1 U \ldots UC_n \subset \mathbf{R}^3$ be a collection of n unknotted, unlinked circles, C_i. The exact position of C is irrelevant to the calculation of $\mathscr{M}(\mathbf{R}^3,C)$, by Corollary 3.8; so we will assume that C is contained in the xy-plane, in the unit cube $I^3 \subset \mathbf{R}^3$. Let $x_i \subset I^3$, $i = 1,\ldots,n-1$, be the portion of the plane $y = c_i$ contained in I^3, where the plane $y = c_i$ separates C_i from C_{i+1}. Denote by R_i the region between X_{i-1} and X_i in I^3, containing the component C_t. Let $l_1,\ldots,l_n$ be vertical line segments between the planes $z = 0$, $z = 1$, such that l_i touches C_i (thus, $l_i \subset R_i$, $i = 1,\ldots,n$). Let $D_1,\ldots,D_n$ be disks in the xy-plane such that $\partial D_i = C_i$.

Choose a basepoint $e \in \mathbf{R}^3 - I^3$. Let $b_i \subset Cl(\mathbf{R}^3 - I^3)$ be a line segment which joins e to l_i, and let k_i be a small circle winding once about C_i, which intersects l_i. Then $\pi_1(\mathbf{R}^3 - C;e)$ is the free group $F(x_1,\ldots,x_n)$ on the generators x_i, $i = 1,\ldots,n$, where x_i may be represented by a loop which runs around k_i once, and then runs back to e along l_i and b_i.

If the link C has only one component (as in Proposition 5.1), we will drop all subscripts.

PROPOSITION 5.1. *Let $C \subset \mathbf{R}^3$ be an unknotted circle. Then the Dahm homomorphism*

$$\mathscr{M}(\mathbf{R}^3, C) \xrightarrow{D} \operatorname{Aut}(F(x)) = \mathbf{Z}_2$$

is an isomorphism.

Proof. D is surjective, because the motion which flips the circle C induces the automorphism $x \to x^{-1}$.

By Corollary 3.6 and Section 4, D factors as follows:

$$\mathscr{M}(\mathbf{R}^3, C) \underset{\simeq}{\overset{\partial}{\rightarrow}} \mathscr{H}^+(\mathbf{R}^3, C) \rightarrow \operatorname{Aut}(F(x)),$$

where ∂ is an isomorphism. Therefore, D is shown to be injective if we can prove that for each homeomorphism $f \in H_c^+(\mathbf{R}^3, C)$ such that $f_*: F(x) \to F(x)$ is the identity, f is isotopic in $H_c^+(\mathbf{R}^3, C)$ to $1_{\mathbf{R}^3}$.

The proof will proceed in three steps.

Step 1. $f \simeq f'$ in $H_c^+(\mathbf{R}^3, C)$ such that $f|(\mathbf{R}^3 - I^3) = id$, and $f(l) \cap \operatorname{int} D = \emptyset$.

Proof. We may assume that $f(\mathbf{R} - I^3) =$ identity (since f has compact support), and by [9], we may assume that f is a P.L. homeomorphism. Consider a projection of C and $f(l)$ onto the plane $z = -1$. We may arrange this so that all points of intersection in the projection are double points. We may put a parameter on l so that $f(l(0))$ is the point at which l touches C and $f(l(1))$ is the endpoint on the plane $z = 1$. We may arrange the projection so that $f(l(1))$ lies outside the projections of the circle and so that $f(l(\varepsilon))$ lies outside the projection of the circle for ε close to 0.

There are four steps involved in the process of unwinding $f(l)$.

(i) There exists an isotopy moving $f(l)$ in such a way that the new projection is identical to the old projection except that if $f(l(t_1))$ is a crossing point and $f(l(t_2))$ is the other point at the crossing then the new $f(l(t_1))$ lies over $f(l(t_2))$ if and only if $t_1 < t_2$. To see this suppose that $f(l(t_2))$ lies over $f(l(t_1))$. We move $f(l(t_2))$ and a neighborhood of it back along $f(l)$ by isotopy until we reach a small neighborhood of the point where $f(l)$ touches C. Then we move this piece of arc over $f(D)$ and back along the other side of $f(D)$. Then we move the arc back along $f(l)$ to its original position in the projection. Step (i) does not change the projection of $f(l)$.

(ii) We now look at the overpasses and underpasses of $f(l)$ with C in the order in which they occur. We will write o for an overpass and u for an underpass. We may then write a sequence of o's and u's to represent the crossings as they appear in order. For example, *ouuo* means an overpass followed by two underpasses followed by an overpass. If o is never followed by o and u is never followed by u then proceed to step (iii). Otherwise we eliminate the pairs of similar crossings. Since between the two crossings $f(l)$ lies under the preceeding part of $f(l)$ and over the following part we may move $f(l)$ so that the two crossings disappear. At the conclusion of step (ii) $f(l)$ crosses C in fewer places than before. Return to step (i). This process must terminate since there are only a finite number of crossings. Hence, we will eventually proceed to step (iii).

(iii) We either have no crossings of $f(l)$ with C, in which case we proceed to step (iv), or we have *ouou . . . ou* or *uouo . . . uo* as our pattern of projections. By rotating the points in the neighborhood of C around C (a precise definition of this operation is given below) we may produce a diagram in which we have *uo . . . uoou . . . ou* or *ou . . . ouuo . . . uo* as our pattern of projection.

We now return to step (i). When step (iii) is next reached there will be no more crossings of $f(l)$ and C and so step (iv) will be reached.

By rotation we now mean that we pick a tubular neighborhood $N \simeq C \times D^2$ of C. (Here, D^2 is the unit disk in $\mathbf{R}^2$.) Then on each meridial disk $c \times D^2$ of N perpendicular to C, at $c \in C$, we move the points within radius $1/2$ around the center by a complete rotation of 2π. For points at distance s, $1/2 \leq s \leq 1$, we move the points a rotation of $4\pi(1-s)$.

(iv). $f'(l)$ now crosses C at no points at all. Hence $f'(l) \cap \operatorname{int} D = \emptyset$.

Step 2. $f' \simeq f''$ in $H_c^+(\mathbf{R}^2, C)$ such that $f'' \mid \mathbf{R}^3 - I^3 = id$ and $f'' \mid D = id$.

Proof. We may assume that $f' \mid C = id$, since $\mathscr{H}^+(C) = 1$, and every isotopy of C may be extended to an isotopy of $\mathbf{R}^3$, without destroying the properties of f' (namely, f' is P.L., and $f' \mid (\mathbf{R}^3 - I^3) = id$). Now if $f'(D)$ and D have disjoint interiors, then $f'(D) \cup D$ is a P.L. 2-sphere in I^3, which bounds a 3-cell B (by the Schoenflies theorem, [8]). There is then an isotopy of f' to f'' in $H_c^+(\mathbf{R}^3, C)$ which has its support in a neighborhood of B, and which moves $f'(D)$ onto $D = f''(D)$. It is an easy matter to adjust f'' so that $f'' \mid D = id$. Therefore, it remains to eliminate circles of intersection of $f'(D)$ with $\operatorname{int}(D)$.

Let c be an innermost such circle in D. Let $d \subset D$ be the disk such that $\partial d = c$, and let d' be the disk intercepted on $f(D)$ by c. Since $(\operatorname{int} d) \cap (\operatorname{int} d') = \emptyset$, the above argument implies the existence of a 3-cell $b \subset I^3$ bounded by the 2-sphere $d \cup d'$. Now b does not contain C, because $d \cap d'$ is disjoint from $f'(l)$ (this follows from the property $f'(l) \cap \operatorname{int} D = \emptyset$). There is an isotopy, with support in a neighborhood of b, which moves d' through b onto d, and then past d to the other side of D, thereby removing at least one circle of intersection of $f'(D)$ with $\operatorname{int}(D)$ (namely, c) without creating any new ones. In this way, all of these circles may be removed.

Step 3. $f'' \simeq 1_{\mathbf{R}^3}$ in $H_c^+(\mathbf{R}, C)$.

Proof. The map f'' is isotopic in $H_c^+(\mathbf{R}^3, C)$ to f''', which is the identity on a 3-cell neighborhood $N(D)$ of D. The space between I^3 and $N(D)$ is $S^2 \times I$, by the annulus theorem. Step 3 now follows from a lemma of Gluck's [13].

THEOREM 5.2. *The Dahm homomorphism* D: $\widehat{(\mathbf{R}^3, C)} \to \operatorname{Aut}(F(x_1, \ldots, x_n))$ *is injective.*

Proof. As in Proposition 5.1, we need only to show that if $f \in H_c^+(\mathbf{R}^3, C)$ is a homeomorphism inducing the identity automorphism

$$f_*: F(x_1, \ldots, x_n) \to F(x_1, \ldots, x_n),$$

then f is isotopic in $H_c^+(\mathbf{R}^3, C)$ to $1_{\mathbf{R}^3}$. The proof of this proceeds in three steps.

Step 1. $f \simeq f'$ in $H_c^+(\mathbf{R}^3, C)$ where $f|(\mathbf{R}^3 - I^3) = id$, and $f(l_i) \subset R_i$, $i = 1, \ldots, n$.

Proof. As in Step 1, Proposition 5.1, we may assume that $f|(\mathbf{R}^3 - I^3) = id$, and that f is a P.L. homeomorphism.

Let λ_i denote the arc $f(l_i)$. Take a projection of I^3 to the plane $z = -1$ in which there are no points of higher order than double points in the projection of $\left(\bigcup_{i=1}^{n} \lambda_i\right) \cup C$. Parametrize λ_i, $i = 1, \ldots, n$, so that $\lambda_i(0)$ lies on C_i. We may assume that for small values of the parameter, the projection of λ_i lies outside the projection of C_i, $i = 1, \ldots, n$. (If this is not so, we can arrange for it to be so by a small isotopy of λ_i.)

We will show that for each λ_k, $1 \leq k \leq n$, there is an isotopy which is supported in I^3, moving λ_k into R_k, without moving $\left(\bigcup_{i \neq k} \lambda_i\right) \cup C$. Once this has been shown, the proof of Step 1 will be complete.

Consider the overpasses, o_i, and the underpasses u_i, of the projection of λ_k with the projection of C_i as they occur in order on λ_k. We show that we can move λ_k so that o_i is not followed by o_i and u_i is not followed by u_i. If this condition is met, we will have each o_i followed or preceded by u_i and each u_i followed or preceded by o_i. Now the projection of λ_k cannot cross the projection of C_i an odd number of times in succession. So there exists a natural pairing of the crossings of the same index so that we may write x_i for o_i followed by u_i and x_i^{-1} for u_i followed by o_i. Hence the patterns of crossings can be expressed by writing $W = x_{i_1}^{\varepsilon_1} \ldots x_{i_m}^{\varepsilon_m}$. Notice that W is a reduced word in $F(x_1, \ldots, x_n)$ since o_i is not followed by o_i and u_i is not followed by u_i. However, if we are to have that the induced automorphism $f_*: F(x_1, \ldots, x_n) \to F(x_1, \ldots, x_n)$ is the identity, then we must have $W = x_k^s$ for some s. Hence if we move λ_k in such a fashion that no o_i follows o_i and no u_i follows u_i then λ_k crosses no circles except C_k.

We shall only show that the isotopy exists which removes pairs of overpasses. The same argument removes underpasses.

Move that part of λ_k between the crossings so that λ_k lies entirely over all λ_i, $i \neq k$. This is done by moving a small piece of λ_k along λ_ι until we reach C_ι. Then we move this piece along one side of the disk $f(D_i)$, over the boundary of this disk, and back along the other side of $f(D_i)$. Then move the piece back to its original position in the projection. The projection has not been disturbed except that λ_k now lies above λ_i.

Now move λ_k so that the part of λ_k between the two crossings lies entirely above the rest of λ_k. This is done without disturbing the projection in the same manner as in the preceding argument, using the disk $f(D_k)$ instead of $f(D_i)$. At the conclusion of the operation we have the part of λ_k between the crossings lying entirely over the rest of the projection. Hence we may move λ_k so as to remove the two overpasses. This reduces the total number of overpasses and underpasses by two. Hence this operation will terminate.

This shows that we can move λ_k so that its projection intersects only C_k. Move λ_k without disturbing the projection so that λ_k lies above λ_i, $i \neq k$. This is done as above. Then, since the projection of λ_k lies above the rest of the projection, with the exception of C_k, we may move λ_k into the region R_k. In none of these isotopies has the region $\mathbf{R}^3 - I^3$, or the graph $\left(\bigcup_{i \neq k} \lambda_i\right) \cup C$, been moved.

Step 2. $f' \simeq f''$ in $H_c^+(\mathbf{R}^3, C)$ satisfying $f''|(\mathbf{R}^3 - I^3) = id$ and $f''\Big|\bigcup_{i=1}^{n} X_i = id.$

Proof. The proof is by induction. Suppose f' is a P.L. homeomorphism satisfying $f'|(\mathbf{R}^3 - I^3) = id$, $f'(\lambda_i) \subset R_i$, $i = 1, \ldots, n$, and $f'|X_i = id$ if $i < k$. We will show that $f' \simeq f''$ in $H_c^+(\mathbf{R}^3, C)$ such that f'' is a P.L. homeomorphism satisfying $f''|(\mathbf{R}^3 - I^3) = id$, $f''(\lambda_i) \subset R_i$, $i = 1, \ldots, n$, and $f''|X_i = id$, $i \leq k$.

Suppose $[\operatorname{int} X_k] \cap f'(X_k) = \emptyset$. Then, by the Shoenflies theorem [8], there is a 3-cell $B \subset I^3$ whose boundary is the P.L. 2-sphere $X_k \cup f'(X_k)$. Now B is disjoint from $\bigcup_{i=1}^{n} \lambda_i$ (where $\lambda_i = f'(l_i)$, as in Step 1), because $\lambda_i \subset R_i$, $i = 1, \ldots, n$, and therefore $\lambda_i \cap X_k = \emptyset$, $i = 1, \ldots, n$. (Clearly $\lambda_b \cap f'(X_k) = \emptyset$ since $l_i \cap X_k = \emptyset$, $i = 1, \ldots, n$.) Thus B is also disjoint from C, as well as from $\bigcup_{i<k} X_i$ (because $f'\Big|\bigcup_{i<k} X_i = id$). There is an isotopy of f' to f'' in $H_c^+(\mathbf{R}^3, C)$, supported in a neighborhood of B, which moves $f'(X_k)$ through B, onto $f''(X_k) = X_k$. Clearly, this isotopy moves neither $\mathbf{R}^3 - I^3$, C, nor $\bigcup_{i=1}^{n} \lambda_i$; hence $f''|(\mathbf{R}^3 - I^3) = id$, $f''(\lambda_i) \subset R_i$, $i = 1, \ldots, n$, and $f''(X_k) = X_k$. From here, it is easy to alter f'' in a neighborhood of X_k, so that $f''|X_k = id$.

If $[\operatorname{int}(X_k)] \cap [f'(X_k)] \neq \emptyset$, then choose an innermost circle of the intersection in X_k, and proceed as in Step 2 of Proposition 5.1. This completes the induction step.

Step 3. $f'' \simeq 1_{\mathbf{R}^3}$ in $H_c^+(\mathbf{R}^3, C)$.

Proof. Apply Proposition 5.1 to the restrictions $f''|R_i$, $i = 1, \ldots, n$.

Now let the motions $R_i 2, T_i, A_{ij}$, and their induced automorphism

$$\rho_i, \tau_i, \alpha_{ij} \in \operatorname{Aut}(F(x_1, \ldots, x_n)),$$

respectively, be those defined in the Introduction.

THEOREM 5.3. *Let* $T(F)$ *be the subgroup of* $\operatorname{Aut}(F(x_1, \ldots, x_n))$ *consisting of all automorphisms of the form* $\alpha: x_i \to q_i x_{j(i)} q_i^{-1}$, $i = 1, \ldots, n$, *where* $j(i)$ *is some permutation of the numbers* $1, \ldots, n$. *Then* $T(F)$ *is generated by the automorphisms* ρ_i, τ_i, α_{ij}, $1 \leq i, j \leq n$, $i \neq j$.

Proof. (This is an adaptation of an argument due to Artin. See [3] or [7], Theorem 1.9, p. 30.) It suffices to show that each automorphism $\alpha: x_i \to q_i x_i q_i^{-1}$ is a product of the α_{ij}, $i \neq j$.

We assume that $q_i x_i q_i^{-1}$ is a reduced word in $F(x_1, \ldots, x_n)$ and hence q_i is a reduced word. We will show that α is a product of α_{ij} by induction on the sum of the lengths of the q_i, where by length we mean the number of letters in q_i. Let $l(q)$ be the length of q. Our induction is on $\sum_{i=1}^{n} l(q_i)$.

Since α is an automorphism, there exist ξ_i, $i = 1, \ldots, n$, such that $\alpha(\xi_i) = x_i$. Let $\xi_i = \prod_{j=1}^{\nu_i} x_{\lambda(i,j)}^{\mu(i,j)}$ where $\lambda(i,j) \neq (i,j+1)$. Then

$$x_i = \alpha(\xi_i) = \prod_{j=1}^{\nu_i} q_{\lambda(i,j)} x_{\lambda(i,j)}^{\mu(i,j)} q_{\lambda(i,j)}^{-1}.$$

Since $q_i x_i q_i^{-1}$ is a reduced word and since $\lambda(i,j) \neq \lambda(i,j+1)$ we must have cancellations operating only at the places where the q's meet.

If we write the above expressions as $x_k = \ldots q_i x_i^{\mu_i} q_i^{-1} q_j x_j^{\mu_j} q_j^{-1} \ldots$ we must have one of four cases:

1) $q_i^{-1} = a x_j q_j^{-1}$, or 2) $q_i^{-1} = a x_j^{-1} q_j^{-1}$, or 3) $q_j = q_i x_i a$, or 4) $q_j = q_i x_i^{-1} a$,

where a is a reduced word. By changing i to j and taking reciprocals of equations (3) and (4) we reduce the number of cases to be considered to two.

In both cases we have $l(q_i) = l(a^{-1}) + 1 + l(q_j)$, and so $l(q_i) > l(q_j a^{-1})$. If case (1) holds, then

$$\begin{aligned} \alpha\alpha_{ij}: x_k &\to q_k x_k q_k^{-1}, \qquad k \neq i \\ x_i &\to q_j x_j q_j^{-1} q_i x_i q_i^{-1} q_j x_j q_j^{-1} = q_j a^{-1} x_i a q_j^{-1} \end{aligned}$$

and the lengths of the q's for $\alpha\alpha_{ij}$ is smaller than those for α.

If case (2) holds, then

$$\begin{aligned} \alpha\alpha_{ij}^{-1}: x_k &\to q_k x_k q_k^{-1}, \qquad k \neq i \\ x_i &\to q_j x_j^{-1} q_j^{-1} q_i x_i q_i^{-1} q_j x_j q_j^{-1} = q_j a^{-1} x_i a q_j^{-1} \end{aligned}$$

and the lengths of the q's for $\alpha\alpha_{ij}^{-1}$ is smaller than those for α.

Proof of Theorem 5.4. The proof is an immediate consequence of Theorems 5.2 and 5.3.

This provides us with a complete classification of the motions of unknotted, unlinked circles in 3-space.

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