# SOME REPRESENTING MEASURES FOR THE BALL ALGEBRA

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In this paper, $M_0$ denotes the class of those (Borel) probability measures $\rho$ on the sphere $S$ (the boundary of the open unit ball $B$ in $\mathbf{C}^n$) that satisfy

$$\int_S f\,d\rho = f(0) \tag{1}$$

for every $f$ in the ball algebra $A(B)$. [Recall that $f \in A(B)$ if and only if $f$ is a continuous complex function on $\bar{B}$ and $f$ is holomorphic in $B$. The members of $M_0$ "represent" the homomorphism $f \to f(0)$ of $A(B)$ onto $\mathbf{C}$.]

When $n = 1$, $M_0$ has exactly one member, namely normalized Lebesgue measure on the unit circle $T$. In general, $M_0$ is convex and weak*-compact, but it turns out to be a very large set when $n > 1$.

The "obvious" members of $M_0$ are the *circular* probability measures $\mu$ on $S$. By definition, these satisfy

$$\int_S v(e^{i\theta}\zeta)\,d\mu(\zeta) = \int_S v\,d\mu \tag{2}$$

for every $v \in C(S)$ and for every real $\theta$. Indeed, if (2) holds and $f \in A(B)$, then $\lambda \to f(\lambda\zeta)$ is in the disc algebra $A(U)$ ($U = B^1$), so that

$$\int_S f\,d\mu = \int_S d\mu(\zeta)\,\frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta}\zeta)\,d\theta = f(0), \tag{3}$$

by Fubini's theorem.

To see some others, take $n = 2$, for simplicity. Let $\tau$ be *any* probability measure on $\bar{U} \subset \mathbf{C}$ that satisfies

$$\int_{\bar{U}} g\,d\tau = g(0) \tag{4}$$

for every $g \in A(U)$. For example, $\tau$ might be concentrated on a simple closed curve $\Gamma$ in $U$ that surrounds the origin, in such a way that $\tau$ solves the Dirichlet problem at 0 relative to the domain bounded by $\Gamma$. The measure $\rho$ that satisfies

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$$\int_S v\,d\rho = \int_U d\tau(z)\,\frac{1}{2\pi}\int_{-\pi}^{\pi} v(z, e^{i\theta}\sqrt{1-|z|^2})\,d\theta \tag{5}$$

for every $v \in C(S)$ belongs then to $M_0$. To see this, simply note that the inner integral on the right side of (5), with $v$ replaced by $f \in A(B)$, equals $f(z,0)$. The support of this $\rho$ is the set of all $(z,w) \in S$ for which $z$ lies in the support of $\tau$.

The set $M_0$ plays a role in the study of Lumer's Hardy spaces $(LH)^p(B)$. We recall, for $0 < p < \infty$, that a holomorphic function $f$ in $B$ belongs to $(LH)^p(B)$ provided that $|f|^p$ has a *pluriharmonic* majorant in $B$, i.e., provided that $|f|^p \leq \operatorname{Re} g$ for some holomorphic $g$ in $B$. (Some of the pathology of these spaces is described in [3].) To see the connection between $M_0$ and $(LH)^p$, associate to every continuous real function $v$ on $S$ the numbers

$$\alpha(v) = \sup\left\{\int_S v\,d\rho : \rho \in M_0\right\} \tag{6}$$

and

$$\beta(v) = \inf\{u(0) : u \in \operatorname{Re} A(B),\ u \geq v \text{ on } S\}. \tag{7}$$

[In (6), the supremum is actually attained, since $M_0$ is weak*-compact.] Since every $\rho \in M_0$ satisfies (1) with $\operatorname{Re} f$ in place of $f$, it is clear that $\alpha(v) \leq \beta(v)$. But more is true, namely

$$\alpha(v) = \beta(v). \tag{8}$$

This is proved on p. 32 of [1] and (as pointed out to me by Stout) implies Lumer's theorem [2; Th.2] which asserts that a holomorphic $f$ in $B$ lies in $(LH)^p(B)$ if and only if

$$\sup_{r,\rho} \int_S |f_r|^p\,d\rho < \infty. \tag{9}$$

Here $0 < r < 1$, $\rho$ ranges over $M_0$, and $f_r(\zeta) = f(r\zeta)$, for $\zeta \in S$.

This characterization of $(LH)^p(B)$ suggests the following two questions:

I. Is the integral in (9) a monotonic function of $r$, for every $\rho \in M_0$ and every holomorphic $f$? [For circular $\rho$, the answer is obviously yes, via the case $n = 1$.]

II. If the supremum in (9) is finite when $\rho$ ranges just over the *circular* measures in $M_0$, does it follow that $f \in (LH)^p(B)$? [Of course, it does follow, trivially, that $f$ lies in the ordinary Hardy space $H^p(B)$.]

When $n = 2$ (hence also when $n > 2$), both questions have negative answers:

THEOREM I. *Put $f(z,w) = (1 - z)w$. Then there exists $\rho \in M_0$ such that*

$\int_S |f_r|^p d\rho$ *is not a monotonic function of* $r$ *in* $[0,1]$, *for any* $p \in (0,\infty)$.

THEOREM II. *There is a holomorphic* $f$ *in* $B$ *which extends continuously to* $\bar{B}$, *except for one boundary point, such that*

$$\sup_r \int_S |f_r|^2 d\rho \leq 1 \tag{10}$$

*for every circular* $\rho \in M_0$, *but*

$$\int_S |f_r|^2 d\rho^* \to \infty \text{ as } r \to 1, \tag{11}$$

*for a certain* $\rho^* \in M_0$.

Both theorems will be proved by means of the representing measures described by (5), with one of the following measures $\tau_x$ in place of $\tau$.

For $0 \leq x < 1$, let $\psi_x(\lambda) = (x + \lambda)/(1 + x\lambda)$, let $I_x$ be the interval $[\psi_x(0), \psi_x(1/2)]$, put $\Omega_x = U \setminus I_x$, and (for $0 < x < 1$) let $\tau_x$ be the probability measure on $\partial\Omega_x = T \cup I_x$ that satisfies

$$\int_{\partial\Omega_x} h\, d\tau_x = h(0) \tag{12}$$

for every $h \in C(\bar{\Omega}_x)$ which is harmonic in $\Omega_x$.

LEMMA 1. *There is a constant* $\gamma > 0$ *such that*

$$\tau_x(I_x) \geq \gamma(1 - x) \qquad (0 < x < 1). \tag{13}$$

*Proof.* fix $u \in C(\bar{\Omega}_0)$, harmonic in $\Omega_0$, such that $u = 0$ on $T$, $u = 1$ on $I_0$. There is a $\gamma > 0$ such that $u(-x) \geq \gamma(1 - x)$ for $0 < x < 1$. The composition $u \circ \psi_x^{-1}$ is continuous on $\bar{\Omega}_x$, harmonic in $\Omega_x$, 0 on $T$, 1 on $I_x$. Hence, by (12),

$$\tau_x(I_x) = u(\psi_x^{-1}(0)) = u(-x) \geq \gamma(1 - x).$$

*Proof of Theorem* I. For $(z,w) \in S$, $0 < r < 1$,

$$f_r(z,w) = (1 - rz)rw, \tag{14}$$

so that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f_r(z, e^{i\theta}\sqrt{1 - |z|^2})|^p d\theta = |r - r^2 z|^p (1 - |z|^2)^{p/2}.$$

If $\rho$ is now defined by (5), with $\tau_x$ (as in Lemma 1) in place of $\tau$ and $x = 3/4$, it follows that

$$\int_S |f_r|^p d\rho = \int_{I_x} (r - r^2 t)^p (1 - t^2)^{p/2} d\tau_x(t). \tag{15}$$

This integral is 0 when $r = 0$. It is positive for all $r \in (0,1)$. It decreases as $r$ increases from $2/3$ to 1, since the integrand on the right of (15) is then a decreasing function of $r$, for every $t \in I_x$.

LEMMA 2. *For $m = 1, 2, 3, \ldots,$ define*

$$g_m(z,w) = (1 - z)^{-m-1} w^{2m+1}. \tag{16}$$

*Then*

$$|g_m(z,w)| < (2/\delta)^{1/2} (2 - \delta)^m \tag{17}$$

*if $(z,w) \in \bar{B}$, $0 < \delta < 1$, $|1 - z| \geq \delta$, and*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |g_m(re^{i\theta}\zeta)|^2 d\theta \leq \binom{2m}{m} \tag{18}$$

*if $\zeta = (z,w) \in S$, $0 < r < 1$.*

*Proof.* Since $|w|^2 \leq 1 - |z|^2$ in $\bar{B}$, $|g_m(z,w)|$ is at most

$$|1 - z|^{-m-1} (1 - r^2)^{m+1/2}, \tag{19}$$

where $r = |z|$. Let $E_\delta$ be the set of all $z \in \bar{U}$ with $|1 - z| \geq \delta$. Fix $r$, and let $z$ range over the part of this circle that lies in $E_\delta$. When $0 \leq r \leq 1 - \delta$, then (19) attains its maximum at $z = r$; when $1 - \delta < r \leq 1$, the maximum occurs when $|1 - z| = \delta$. Hence (19) attains its maximum in $E_\delta$ *at the point* $z = 1 - \delta$. At that point, (19) equals

$$\delta^{-1/2} (2 - \delta)^{m+1/2}. \tag{20}$$

Hence (17) holds.

To prove (18), insert the binomial expansion

$$(1 - z)^{-m-1} = \sum_{k=0}^{\infty} \binom{k+m}{k} z^k \tag{21}$$

into (16). By Parseval's theorem, the integral in (18) is then

$$J(r,\zeta) = |w|^{4m+2} \sum_{k=0}^{\infty} \binom{k+m}{k}^2 |z|^{2k} r^{2k+4m+2}. \tag{22}$$

It is easily verified that

(23)
$$\binom{k+m}{k}^2 \leq \binom{2m}{m}\binom{k+2m}{k}.$$

Since $|w|^2 = 1 - |z|^2$ on $S$, another application of the binomial theorem gives therefore

$$J(r,\zeta) < \binom{2m}{m} |w|^{4m+2} \sum_{k=0}^{\infty} \binom{k+2m}{k} |z|^{2k}$$
$$= \binom{2m}{m} |w|^{4m+2} (1 - |z|^2)^{-2m-1} = \binom{2m}{m}.$$

*Proof of Theorem* II. Let $C_m\,(m = 1,2,3,\ldots)$ be non-negative numbers that satisfy

(24)
$$\sum_{m=1}^{\infty} C_m = 1 \quad \text{but} \quad \sum_{m=1}^{\infty} C_m^2 m^{1/2} = \infty.$$

For example, put $C_m = (p(p+1))^{-1}$ if $m = p^{10}\,(p = 1,2,3,\ldots)$, put $C_m = 0$ otherwise. Define

(25)
$$f(z,w) = \sum_{m=1}^{\infty} C_m \binom{2m}{m}^{-1/2} (1-z)^{-m-1} w^{2m+1}.$$

By Stirling's formula,

(26)
$$\binom{2m}{m} \sim 4^m / \sqrt{\pi m}.$$

The estimate (17) implies therefore that the series (25) converges at every point of $\bar{B}$, and that the convergence is uniform outside any neighborhood of the point (1,0).

Since $f$ is a convex combination of the functions $\binom{2m}{m}^{-1/2} g_m$, it follows from (18) that

(27)
$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta}\zeta)|^2 d\theta \leq 1$$

for all $\zeta \in S$, $r \in (0,1)$. If now $\rho \in M_0$ is circular, we can apply Fubini's theorem as in (3), and conclude from (27) that (10) holds.

We turn to the proof of (11). The definition of $f$ shows, for $0 < t < 1$, that

(28)
$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t, e^{i\theta}\sqrt{1-t^2})|^2 d\theta = (1-t)^{-1}\psi(t)$$

where

$$\psi(t) = \sum_{m=1}^{\infty} C_m^2 \binom{2m}{m}^{-1} (1+t)^{2m+1}. \tag{29}$$

By (24) and (26), it follows that

$$\psi(t) \to \infty \text{ as } t \to 1. \tag{30}$$

Now pick $x \in (0,1)$, and define $\rho_x \in M_0$ as in (5), with $\tau_x$ in place of $\tau$. (See Lemma 1.) By (28),

$$\int_S |f|^2 \, d\rho_x = \int_{I_x} (1-t)^{-1} \psi(t) \, d\tau_x(t) \tag{31}$$

and hence Lemma 1 implies that

$$\int_S |f|^2 \, d\rho_x \geq \gamma \psi(x), \tag{32}$$

since $\psi(t) \geq \psi(x)$ on $I_x$.

By (30), we can choose $x_i \to 1$ so that $\psi(x_i) > 4^i$. Define

$$\rho^* = \sum_{i=1}^{\infty} 2^{-i} \rho_{x_i}. \tag{33}$$

Since $M_0$ is convex and weak*-compact, $\rho^* \in M_0$. Also, for each $i$, (32) gives

$$\int_S |f|^2 \, d\rho^* \leq 2^{-i} \cdot \gamma \psi(x_i) > \gamma \cdot 2^i. \tag{34}$$

Thus $\int_S |f|^2 d\rho^* = \infty$, and (11) follows from Fatou's lemma.

## REFERENCES

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