A TRULY ISOLATED UNIVALENT FUNCTION

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Let R denote the space of functions

$$f(z) = \sum_{n=1}^{\infty} a_n z^n \tag{1}$$

that are holomorphic in the unit disc D and let the space R be metrized by the metric

$$\|f\| = \sup_n |a_n|^{1/n}. \tag{2}$$

Let S denote the subset of R which consists of the functions $\Sigma a_n z^n$ that are univalent (but not necessarily normalized) in D. Hornich [2] studied the structure of the subset S. In [3] he exhibited a function f in S with the following property: for some positive number r none of the functions $f(z) + cz$ $(0 < |c| < r$, c not positive) belongs to S. Hornich's example suggested that S may have isolated points. Piranian [5] claimed there is a function $f(z) = \Sigma a_n z^n$ which belongs to S and lies at a distance one from $S - \{f\}$. His proof relied heavily on certain geometric constructions for which it is difficult to obtain detailed proofs. The following proof is much simpler and the claim is much stronger. Instead of a suggestive geometric argument it utilizes known geometric properties of normal analytic functions together with an explicit analytic construction.

THEOREM. *Let LS denote the set of functions in R which are locally univalent in D. Then there exists a univalent function f(z) in R which lies at a distance one from LS* $- \{f\}$.

Thus not only is this univalent function isolated from other univalent functions, it is even isolated from all other locally univalent functions. The proof also provides a univalent function such that no function k in R with $\|k - f\| < 1$ is univalent (or even locally univalent) in *any sector of the unit disc,* a result which Piranian observed would be at best tedious to derive from his geometrical approach [5, p. 238].

We first prove a lemma which provides a simple explicit construction of an analytic function without finite radial limits.

LEMMA 1. *Let* $h(z) = \sum_{n=1}^{\infty} z^{g(n)}$ *where* $g(n) = 2^{2^n}$. *Then* $h(z)$ *has no finite radial limits and furthermore* $(1 - |z|)\,|h'(z)| \leq 2$.

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Proof. The high index theorem of Hardy [1, p. 215, Corollary 1] shows that $h(z)$ has no finite radial limits. Clearly

$$\begin{aligned}(1-|z|)^{-1}|zh'(z)| &\le \left(\sum_{n=0}^{\infty}|z|^n\right)\left(\sum_{n=1}^{\infty}g(n)\,|z|^{g(n)}\right)\\ &\le \sum_{k=1}^{\infty}\left(\sum_{g(n)\le k}g(n)\right)|z|^k\\ &\le 2\sum_{k=1}^{\infty}k|z|^k = 2|z|\,(1-|z|)^{-2}\end{aligned}$$

which implies $(1-|z|)\,|h'(z)| \le 2$.

Proof of the theorem. Let $f(z)$ be the solution of $f''(z)/f'(z) = h'(z)/3$, $f'(0) = 1$, $f(0) = 0$, where $h(z)$ is the function of lemma 1. Since

$$(1-|z|)\,|f''(z)/f'(z)| \le 2/3 < 1,$$

the function $f(z)$ is univalent and even has a quasiconformal extension to $\mathbf{C}$ [6, p. 172, 294]. Since $\log f'(z) = h(z)/3$, lemma 1 guarantees that $\log f'(z)$ is a normal analytic function with no finite radial limits. Thus $f'(z)$ can have only 0 and ∞ as radial limits. We claim that $f(z)$ is the desired isolated function.

Let $g(z)$ be any function in R with $\|f-g\| < 1$ and write $g(z)$ as $f(z)+k(z)$ *where* $k(z)$ *is analytic on a region which contains* $|z| \le 1$ since $\|k\| < 1$. Since $k'(z)$ is analytic on $|z| \le 1$ there are only a finite number of points on $|z| = 1$ at which $k'(z)$ vanishes. Choose any $e^{i\theta}$ such that $k'(e^{i\theta}) \ne 0$ and any neighborhood N of $e^{i\theta}$ such that $|k'(z)| > |k'(e^{i\theta})/2|$ for all z in N.

We now prove that $g'(z) = f'(z) + k'(z)$ has a zero in $N \cap D$. Suppose to the contrary that $f'(z) + k'(z) \ne 0$ in $N \cap D$. We first note that $k'(z)$ is bounded in $|z| < 1$, since it is analytic on $|z| \le 1$, while $f'(z)$ is normal, since it is the derivative of a univalent function [6, p. 262]. The sum of a *bounded* analytic function and a normal function is normal. The reciprocal of a normal function is normal. Therefore, the assumption that $f'(z) + k'(z) \ne 0$ in $N \cap D$ would force $(f'(z) + k'(z))^{-1}$ to be normal *analytic* in $N \cap D$. Since normal analytic functions are in MacLane's class $\mathscr{A} = \mathscr{B} = \mathscr{L}$ [4] we can find a neighborhood N^* of $e^{i\theta}$, $N^* \subset N$, with $N^* \cap \partial D$ an arc, and $|(f'(z) + k(z))^{-1}| \le M < \infty$ on $\partial N^* \cap D$.

We now prove that $(f'(z) + k'(z))^{-1}$ must be unbounded in $N^* \cap D$. Suppose $(f'(z) + k'(z))^{-1}$ were bounded in $N^* \cap D$. Since $(f'(z) + k'(z))^{-1} \not\equiv (k'(z))^{-1}$ in $N^* \cap D$ and since both would be bounded analytic functions in $N^* \cap D$, there would be a point (in fact a set of positive measure) on $N^* \cap \partial D$ on which

$$(f'(z) + k'(z))^{-1}$$

and $(k'(z))^{-1}$ would have different nonzero radial values. But this would violate the fact that $f'(z)$ has only 0 and ∞ as radial values.

Thus $(f'(z) + k'(z))^{-1}$ is unbounded in $N^* \cap D$ and we can choose a point z_0 in $N^* \cap D$ such that

$$|(f'(z_0) + k'(z_0))^{-1}| > \max \{4|(k'(e^{i\theta}))^{-1}|, M\}.$$

Consider the infinite ray $t(f'(z_0) + k'(z_0))^{-1}$, $t \geq 1$. We lift $(f'(z_0) + k'(z_0))^{-1}$, the beginning point of this ray, back to a point in N^*. We continue to lift along this ray as far as we can. The lift is a path γ in N^* which can not go to $\partial N^* \cap D$ since $|(f'(z) + k'(z))^{-1}| \leq M$ on $\partial N^* \cap D$. Since $(f'(z) + k'(z))^{-1}$ is a normal function the path can not form a Koebe arc [6, p. 267]. Therefore $\gamma(t)$ must end at a point, say $e^{i\theta_1}$, of $N^* \cap \partial D$. Thus $(f'(z) + k'(z))$ has a *finite* asymptotic value along $\gamma(t)$ ending at $e^{i\theta_1}$. Since $f'(z) + k'(z)$ is normal the asymptotic value is a radial value [6, p. 268] with modulus less than $|k'(e^{i\theta})|/4$. On the other hand, $k'(z)$ has a radial value at $e^{i\theta_1}$ of modulus greater than $|k'(e^{i\theta})|/2$. Therefore $f'(z)$ will have a finite radial value of modulus greater than $|k'(e^{i\theta})|/4$. This violates $f'(z)$ having radial limits of 0 and ∞ only.

Thus $g'(z) = f'(z) + k'(z)$ must have a zero in every neighborhood of $e^{i\theta}$ which proves $g(z)$ is not locally univalent in any sector of the unit disc. This concludes the proof of the theorem.

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