A GEOMETRIC CONDITION WHICH IMPLIES BMOA

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1. INTRODUCTION

The space BMOA is the collection of analytic functions on the unit disc D which are in the Hardy space H^1 and whose boundary values belong to the space BMO of John and Nirenberg [6].

Recently, Hayman and Pommerenke [5] discovered a geometric characterization of all regions Ω with the property that an analytic function with values in Ω will belong to BMOA. Their characterization uses logarithmic capacities.

At about the same time I independently discovered the sufficiency result along with several applications and generalizations to known results. These applications are given below along with the best norm result which involves a property of logarithmic capacity which may be of independent interest.

2. STATEMENT OF THE RESULTS

The geometric characterization given in [5] is that there exist an $r > 0$ and $\delta > 0$ such that $\operatorname{Cap}(D(w,r)\setminus\Omega) \geq \delta$ for all w in Ω. Here $D(w,r)$ is the closed disc of radius r centered at w and "Cap" denotes the logarithmic capacity of a set.

For a region Ω (Ω is an open, connected subset of $\mathbf{C}$) let

$$\phi(r) = \inf_{w\in\Omega} \frac{\operatorname{Cap}(D(w,r)\setminus\Omega)}{\operatorname{Cap}(D(w,r))}$$

so that $0 \leq \phi \leq 1$. We could replace the denominator with r since the capacity of a disc is its radius. If in the definition of ϕ we replace Cap with a measure, then the condition $\phi(r_0) = \delta > 0$ would not imply a stronger result for large r, i.e., the ratio could remain constant. Surprisingly, the situation with capacities is quite different.

THEOREM 1. *For a region Ω, $\lim_{r\to\infty} \phi(r) = 1$ provided that $\phi(r) \neq 0$ for some $r > 0$. In addition, there exists an $r > 0$ with $2^{-5} \leq \phi(r) \leq 2^{-1/5}$.*

The next is a refinement of that given in [5].

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THEOREM 2. *There are positive functions* $c_1(t)$ *and* $c_2(t)$ *defined on* $(0,1)$ *satisfying the following conditions*:

Let Ω *be a region with* $0 < a \le \phi(r) \le b < 1$ *for some* $r > 0$. *Then*

$$c_1(b) r \le \sup \|f\|_* \le c_2(a) r$$

where the supremum is taken over all analytic functions on D *with values in* Ω *and* $\|\cdot\|_*$ *denotes the* BMO *norm.*

Moreover, $c_2(a)$ *is dominated by a constant multiple of* $\log 2/a$ *and* $c_1(b)$ *is bounded from below by a positive constant on the interval* $(0,1/2]$ *but tends to zero as* b *tends to one.*

We remark that the above result restricted to the case where b is small can be obtained by the methods in [5]. However, for small b the constants c_1, c_2 are not comparable and Theorem 1 is needed to guarantee that $\phi(r)$ can be chosen sufficiently large so that these constants are comparable.

COROLLARY 1. *Let* Ω *be a region with* $\phi(r_0) > 0$ *for some* r_0. *Then there exist an* $r > 0$ *with* $2^{-5} \le \phi(r) \le 2^{-1/5}$ *and the supremum of* $\|f\|_*$ *for* f *with values in* Ω *will be comparable to* r.

We now give some geometric conditions of a more elementary nature which imply BMOA. Let $m_w(t)$ denote the Lebesgue measure of the set of numbers r, $0 \le r \le t$, for which the circle $|z - w| = r$ is contained in Ω.

COROLLARY 2. *The condition* $\sup_{w\in\Omega} \dfrac{m_w(r)}{r} = d < 1$ *implies that* $\phi(r) \ge 1/4(1-d)$.

Proof. The circular projection mapping z into $|z|$ decreases distances and hence decreases capacity, see Pommerenke's book [7, Theorem 11.3, p. 337]. Taking w to be the origin we see that the circular projection of $D(w,r)\setminus\Omega$ is a set whose complement in the interval $[0,r]$ has measure $m_w(r)$. Since the capacity of linear set is at least one quarter of its length the result follows.

COROLLARY 3. *If the image* $f(D)$ *of an analytic function* f *does not contain circles centered in* $f(D)$ *of radius larger than* r *then* f *is in* BMOA *and* $\|f\|_* \le cr$ *for some constant* c *independent of* f *and* r.

COROLLARY 4. *If the vertical cross-sectional measures of* $f(D)$ *are bounded by* d *then* f *is in* BMOA *and* $\|f\|_* \le cd$.

Proof. Take $\Omega = f(D)$ and $r = d$ then $D(w,r)\setminus\Omega$ contains a linear set of measure at least equal to d.

We remark that Corollary 3 is a generalization of Pommerenke's result [8] that a univalent function f is in BMOA if and only if $f(D)$ contains no discs of arbitrary large radii. Obviously, if a circle is contained in a simply connected region then the entire disc is also. Also, Corollary 4 is a generalization of Baernstein's result [2] that a nonvanishing univalent function f satisfies $\log f \in$ BMOA. In this case $\log f(D)$ has vertical cross-sectional measures bounded by 2π. See also [9].

Finally, let $\Omega_w(r)$ be the component of $D(w,r) \cap \Omega$ containing w.

COROLLARY 5. *If Ω is a region and* $\sup_{w\in\Omega} \dfrac{area(\Omega_w(r))}{\pi r^2} = d^2 < 1$ *for some* $r > 0$ *then* $f(D) \subset \Omega$ *implies that f is in BMOA and* $\|f\|_* \le cr \log \dfrac{2}{1-d}$.

Proof. A calculation shows that $\pi m_w(r)^2 \le area(\Omega_w(r))$ and hence Corollary 2 and Theorem 2 imply the norm estimate.

A special case of Corollary 5 is the case that area $(f(D))$ is finite and the resulting norm inequality is that $\|f\|_* \le c\ (area\, f(D))^{1/2}$. This problem was first considered in [1, Theorem 1] where it is shown that finite area implies H^2. Later, this result was improved in [3] to all H^p for $p < \infty$. Since BMOA is contained in H^p for all $p < \infty$, see [6], the above corollary generalizes these results.

3. PROOFS OF THE THEOREMS

The proofs of Theorem 1 and Theorem 2 are based on the following lemmas.

LEMMA 1. *There is a positive constant t_0 satisfying*

$$(*) \qquad \phi(2tr) \ge \exp\left(-\frac{1}{t-2}\left[\log\frac{t}{\phi(r)} + 4\right]\right)$$

whenever $\phi(r) > 0$ and $t \ge t_0$.

Proof. Assume that $\phi(r/2) = \delta > 0$ then Cap $(D(w,r/2)\setminus\Omega) \ge \delta r/2$ for all w in Ω. It follows that Cap $(D(w,r)\setminus\Omega) \ge \delta r/2$ for all w in $\mathbf{C}$. Fix an odd integer $n = 2k+1$ and $R \ge (1 + n/3)r$. Put

$$A_m = \left\{z : R - 2r \le |z| \le R;\ \frac{2\pi}{n}\left(m - \frac{1}{2}\right) \le \arg z \le \frac{2\pi}{n}\left(m + \frac{1}{2}\right)\right\}$$

for $m = 0, 1, \ldots, n-1$. If n is sufficiently large a computation shows that A_m contains a disc of radius r and hence there is a subset E_m of $A_m\setminus\Omega$ with Cap $(E_m) \ge \delta r/2$. Then there exist a positive measure μ_m on E_m with unit mass and such that the potential

$$U^{\mu_m}(z) = \int \log\frac{1}{|z-\zeta|}\, d\mu_m(\zeta)$$

is bounded by log $[2/\delta r]$ [4, p. 235].

If $\mu = \dfrac{1}{n}\sum_m \mu_m$ then the inequality $\sup_{z\in E} U^\mu(z) \le c$ where $E = \bigcup_m E_m$ implies the lower bound Cap $E \ge e^{-c}$. Since $E \subset D(0,R)\setminus\Omega$ this will result in a lower bound estimate for $\phi(R)$.

Let $z \in E_{m'}$, then for $U_m = U^{\mu_m}$ we have

$$\begin{aligned} U^{\mu}(z) &= \frac{1}{n}\left[U_{m'-1}(z) + U_{m'}(z) + U_{m'+1}(z) + \sum_{\text{rest}} U_m(z)\right] \\ &\le \frac{1}{n}\left[3\log\frac{2}{\delta r} + \sum \sup_{\zeta\in E_m}\log\frac{1}{|z-\zeta|}\right] \\ &\le \frac{1}{n}\left[3\log\frac{2}{\delta r} + 2\sum_{m=2}^{k}\log\frac{1}{(R-2r)|e^{2\pi i(m-1)/n}-1|}\right] \\ &\le \frac{1}{n}\left[3\log\frac{2}{\delta r} + 2n\int_0^{(k-1)/n}\log\frac{1}{(R-2r)|e^{2\pi i t}-1|}\,dt\right] \end{aligned}$$

Since $\dfrac{k-1}{n} = \dfrac{1}{2} - \dfrac{3}{2n}$ and $\displaystyle\int_0^{1/2}\log|e^{2\pi i t}-1|\,dt = 0$ we get after some simplification that

$$\begin{aligned} \sup_{z\in E} U^{\mu}(z) \le \frac{3}{n}\left[\log\frac{2}{\delta} + \log\left(\frac{R}{r}-2\right) + \log 2\right] \\ + \log\frac{R}{R-2r} - \log R. \end{aligned}$$

We now set $R = tr$ and assume that $(1+n/3)r \le R \le \left(1+\dfrac{n+2}{3}\right)r$. If $t \ge t_0$ where t_0 is sufficiently large then the value of n will be large enough to apply the above argument. Since $n/3 \ge t-2$ we deduce that

$$\begin{aligned} \log\phi(tr) &\ge -\frac{3}{n}\left[\log\frac{t}{\phi(r/2)} + \log\left(1-\frac{2}{t}\right) + \log 4\right] + \log\left(1-\frac{2}{t}\right) \\ &> \frac{1}{t-2}\left[\log\frac{t}{\phi(r/2)} + \log 4\right] - \frac{2}{t-2}. \end{aligned}$$

Replacing r with $2r$ in the above yields (*).

LEMMA 2. *If $\phi(r) \ne 0$ for some $r > 0$ then there exists an $R > 0$ with $2^{-5} \le \phi(R) \le 2^{-1/5}$.*

Proof. Since $r\phi(r)$ is a nondecreasing function, $\phi(r)$ has left and right limits everywhere. In fact, by the outer regularity of capacity we see that ϕ is continuous from the right. By (*), the set $\{r : \phi(r) \ge 2^{-5}\}$ is nonempty. Let R be the infimum of this set so that $R > 0$, $\phi(R-0) \le 2^{-5}$, and $\phi(R) = \phi(R+0) \ge 2^{-5}$.

Let $\varepsilon > 0$. Since the capacity of a semicircle of radius R is $R/\sqrt{2}$, an open neighborhood will have capacity bounded by $(1+\varepsilon)R/\sqrt{2}$. Let $w' \in \Omega$ and $R' < R$. If R' is sufficiently close to R then there exists $w \in \Omega$ with

$$\operatorname{Cap}(D(w,R)\setminus D(w',R')) < (1+\varepsilon)R/\sqrt{2}.$$

Put $E = D(w,R)\setminus\Omega$ and $E' = D(w',R')\setminus\Omega$. Then E can be split into two sets E_1, E_2 where $E_1 \subset E'$ and $\operatorname{Cap}(E_2) \le (1+\varepsilon)R/\sqrt{2}$. Now the subadditivity of capacity gives

$$\begin{aligned} 1/\log\frac{2R}{\operatorname{Cap} E} &\le 1/\log\frac{2R}{\operatorname{Cap} E_1} + 1/\log\frac{2R}{\operatorname{Cap} E_2} \\ &\le 1/\log\frac{2R}{\operatorname{Cap} E'} + 1/\log\frac{2\sqrt{2}}{1+\varepsilon} \end{aligned}$$

By letting R' tend to R so that $\operatorname{Cap} E'$ tends to $R\phi(R-0)$ and by letting ε tend to zero we deduce that

$$1/\log\frac{2}{\phi(R)} \le 1/\log\frac{2}{\phi(R-0)} + 1/\log 2\sqrt{2}$$

Since $\phi(R-0) \le 2^{-5}$ this implies that $\phi(R) \le 2^{-1/5}$. Thus, $2^{-5} \le \phi(R) \le 2^{-1/5}$. See [7, Chapter 11.1] for the regularity and subadditivity results used in the above.

The author is indebted to the referee for suggesting the above lemma.

Proof of Theorem 1. Clearly (*) implies $\lim_{r\to\infty}\phi(r) = 1$ and the remaining statement is Lemma 2.

Proof of Theorem 2. In [5] it is shown that:

(1) $\phi(r) \ge \delta > 0$ implies $\|f\|_* \le c(\delta)r$ whenever f takes values in Ω.

(2) There exist $0 < \delta_0 < 1$ such that $\phi(r) \le \delta_0$ implies there exists a function f with values in Ω and $\|f\|_* \ge cr$.

Actually, the upper estimate given in [5] is of the form $c(\delta, r)$ but an easy dilation argument places it in the above form.

Since $\lim_{r\to\infty}\phi(r) = 1$ we can define $r_0 = \inf\{r : \phi(r) \ge \delta_0\}$. Thus, there exists $r_0 \le r_1 \le 2r_0$ with $\phi(r_1) \ge \delta_0$ and hence (1) implies $\sup\|f\|_* \le c(\delta_0)2r_0$. In addition, $\phi(r_0/2) < \delta_0$ so (2) implies $\sup\|f\|_* \ge cr_0/2$. Thus, the best norm estimate is given by r_0.

By (1) $c_2(t)$ can be taken to be constant on $[\delta_0, 1)$. By (2) $c_1(t)$ can be constant on $(0, \delta_0)$.

Let $\delta < \delta_0$ and $\phi(r) \ge \delta$. Assuming as we may that δ_0 is small we use $t = \log 2/\delta \ge t_0$ in (*) to get $\phi(2(\log 2/\delta)r) \ge \delta_1$ where δ_1 is independent of $\delta < \delta_0$. Hence by (1) $\|f\|_* \le c(\delta_1)2(\log 2/\delta)r \le c(\log 2/\delta)r$ whenever f takes values in Ω. It follows that $c_2(\delta)$ can be chosen to be a constant multiple of $\log 2/\delta$ on $(0, \delta_0)$ and hence also on $(0, 1)$.

Finally, we must determine $c_1(\delta)$ for $\delta_0 < \delta < 1$. Let $\delta_0 < \delta < 1$ and $\phi(r) \leq \delta$. Now $r = 2tr_1$ for some t. From (*) and the fact that $\phi(r_1) \geq \delta_0$ we obtain an upper bound for t in terms of δ, say $8t \leq \psi(\delta)$. Since $\|f\|_* \geq cr_0/2$ for some f with values in Ω and $2r_0 \geq r_1$ we obtain $\|f\|_* \geq c\psi(\delta)^{-1}r$. Thus, taking $c_1(\delta) = c\psi(\delta)^{-1}$ for $\delta_0 < \delta < 1$ we are done.

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