STABILIZATIONS OF PERIODIC MAPS ON MANIFOLDS

David C. Royster

1. INTRODUCTION

In this paper we will study the graded unrestricted unoriented cobordism ring of involutions, $I_*(\mathbf{Z}_2)$, and an endomorphism, Γ, of $I_*(\mathbf{Z}_2)$ of degree $+1$. We use this endomorphism to define ideals in $I_*(\mathbf{Z}_2)$ by

$$\mathscr{A}_n = \{x + \Gamma^n(x) : x \in I_*(\mathbf{Z}_2) \quad \text{and} \quad \varepsilon(\Gamma^j(x) = 0 \quad \text{for} \quad 0 \leq j < n\}.$$

The ideal $\mathscr{A}_1$ plays a more important part in our theory than the remainder of these ideals. We prove that $I_*(\mathbf{Z}_2)/\mathscr{A}_1$ is a polynomial ring over MO_* and over $\mathbf{Z}_2$.

We apply this result about $I_*(\mathbf{Z}_2)/\mathscr{A}_1 \equiv \Lambda(\mathbf{Z}_2)$ to prove Boardman's five-halves theorem. After noting that $\Lambda(\mathbf{Z}_2) \equiv MO_*(BO)$, we use the results of [2] to exhibit explicit polynomial generators for $\Lambda(\mathbf{Z}_2)$ whose underlying manifolds generate MO_* as a polynomial ring over $\mathbf{Z}_2$. We then consider two filtrations on $\Lambda(\mathbf{Z}_2)$. After seeing how these filtrations behave on the polynomial generators for $\Lambda(\mathbf{Z}_2)$ and on polynomials in these generators, the Five-halves theorem and its converse follow. We then look at an application of this theorem and its method of proof to a conjecture about flat manifolds.

We notice that certain elements in the ideals $\mathscr{A}_n$ behave much like the polynomials in $1 + t^n$ in $\mathbf{Z}_2[t]$. Knowing how to factor the cyclotomic polynomials and, hence, $1 + t^n$ in $\mathbf{Z}_2[t]$, we are led to a factorization of the elements $x^k + \Gamma^n(x^k)$ in $\mathscr{A}_n$ in an analogous manner.

2. PRELIMINARY MATERIAL

We will use $I_*(\mathbf{Z}_2)$ to denote the graded unrestricted unoriented cobordism ring of smooth manifolds with involution; $MO_*(\mathbf{Z}_2)$, the graded unoriented cobordism ring of smooth manifolds with fixed point free involutions; MO_*, the graded unoriented Thom cobordism algebra; and $\mathscr{M}_*$, the graded unoriented cobordism ring of principal $O(k)$ bundles with $\mathscr{M}_n = \sum_{j=0}^{n} MO_j(BO(n-j))$ and $MO_n(BO(O)) = MO_n$, by definition.

Conner and Floyd completely determined the additive structure of $I_*(\mathbf{Z}_2)$ in the following theorem.

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THEOREM 2.1. [5;28.1] *The sequence of MO_* modules*

$$0 \to I_*(\mathbf{Z}_2) \xrightarrow{\iota_*} \mathscr{M}_* \xrightarrow{J} MO_{*-1}(\mathbf{Z}_2) \to 0 \tag{2.2}$$

is split exact.

Define the endomorphism $\Gamma: I_n(\mathbf{Z}_2) \to I_{n+1}(\mathbf{Z}_2)$ as follows. Consider the circle as $S^1 = \{z \in \mathbf{C}: ||z|| = 1\}$. Take $\{T, M^n\}_2 \in I_n(\mathbf{Z}_2)$ and consider the manifold $S^1 \times M^n$ with the involutions $T_1(z,m) = (-z, T(m))$ and $T_2(z,m) = (\bar{z}, m)$, for $z \in S^1$ and $m \in M^n$. Note that T_1 is free on $S^1 \times M^n$ and T_1 and T_2 commute, so T_2 induces an involution T' on the quotient manifold $M' = (S^1 \times M^n)/T_1$. Put $\Gamma(\{T, M^n\}_2) = \{T', (S^1 \times M^n)/T_1\}_2 = \{T', M'\}_2 \in I_{n+1}(\mathbf{Z}_2)$. This map is the same on the bordism level as the one defined in [1], cf., [6]. Thus, we have that Γ is well-defined, additive, an MO_*-module map, and $\Gamma(MO_*) = 0$. Further, for any $x, y \in I_*(\mathbf{Z}_2)$

$$\begin{aligned} \Gamma(xy) &= x \cdot \Gamma(y) + \Gamma(x) \cdot \varepsilon(y) \\ &= \Gamma(x) y + \varepsilon(x) \cdot \Gamma(y) \end{aligned} \tag{2.3}$$

where $\varepsilon: I_*(\mathbf{Z}_2) \to MO_*$ is the augmentation map. Also, if F is the fixed point set of T on M^n with normal bundle ν, then the fixed point data of $\Gamma(\{T, M^n\}_2)$ is $\nu \oplus \theta \cup \theta$ over $F \cup M^n$, disjoint, where θ denotes the trivial real line bundle.

3. THE IDEALS $\mathscr{A}_n$

Consider the following sets in $I_*(\mathbf{Z}_2)$: $\mathscr{A}_n = \{x + \Gamma^n(x): x \in I_*(\mathbf{Z}_2)$ and $\varepsilon(\Gamma^j(x)) = 0$ for $0 \le j < n\}$. Since Γ and ε are both additive, in order to show that $\mathscr{A}_n$ is an ideal of $I_*(\mathbf{Z}_2)$ we need only show that $y(x + \Gamma^n(x)) \in \mathscr{A}_n$ for any $y \in I_*(\mathbf{Z}_2)$. It is sufficient to see that $\Gamma^n(xy) = \Gamma^n(x) \cdot y$. If $n = 1$, then $\varepsilon(x) = 0$ since $x + \Gamma(x) \in \mathscr{A}_1$. Then $\Gamma(xy) = \Gamma(x) \cdot y + \varepsilon(x) \cdot \Gamma(y) = \Gamma(x) \cdot y$. Assume the result for n and let $x + \Gamma^{n+1}(x) \in \mathscr{A}_{n+1}$. Then

$$\Gamma^{n+1}(xy) = \Gamma(\Gamma^n(xy)) = \Gamma(\Gamma^n(x) \cdot y) = \Gamma^{n+1}(x) \cdot y + \varepsilon(\Gamma^n(x))\Gamma(y) = \Gamma^{n+1}(x) \cdot y,$$

since $\varepsilon(\Gamma^n(x)) = 0$.

Let $\mathscr{P}$ be the principal ideal in $\mathscr{M}_*$ generated by the element $1 + [\theta \to pt]_2$, where θ is the trivial real line bundle.

THEOREM 3.1. *$I_*(\mathbf{Z}_2)/\mathscr{A}_1 \approx \mathscr{M}_*/\mathscr{P}$ as rings.*

Proof. We have the following diagram of exact sequences of MO_* modules.

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & \mathscr{A}_1 & & \mathscr{P} & & \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I_*(\mathbf{Z}_2) & \underset{\rho}{\overset{\iota_*}{\rightleftarrows}} & \mathscr{M}_* & \xrightarrow{J} & MO_*(\mathbf{Z}_2) \longrightarrow 0 \\
 & & \downarrow & \searrow^{\pi\circ\iota_*} & \downarrow \pi & & \\
 & & I_*(\mathbf{Z}_2)/\mathscr{A}_1 & & \mathscr{M}_*/\mathscr{P} & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

$\rho:\mathscr{M}_* \to I_*(\mathbf{Z}_2)$ is the splitting map to ι_* from [5;28.1]. Note that both ι_* and π are ring homomorphisms as well as MO_*-module homomorphisms. Further still, ι_* is a ring monomorphism. The maps ρ and J are only MO_*-module maps.

We first want to show that $\pi\circ\iota_*$ is onto. The only elements in $\mathscr{M}_*$ that are not in the image of ι_* are the elements of the form $M^{n-r}\times D^r$; i.e., the trivial r-plane bundles, θ^r. Now, $M^{n-r}\times D^0 \in \operatorname{im}(\iota_*)(\{id, M^{n-r}\}_2)$. In the quotient ring $\mathscr{M}_*/\mathscr{P}$ we identify θ with 1; that is, identify M^{n-r} with $M^{n-r}\times D^1$. We thus also identify M^{n-r} with $M^{n-r}\times D^r$. Thus, under $\pi\circ\iota_*$, $\{id, M^{n-r}\}_2$ is a preimage for $\pi(M^{n-r}\times D^r)$.

We next need to show that $\ker(\pi\circ\iota_*) = \mathscr{A}_1$. To see that $\mathscr{A}_1 \subseteq \ker(\pi\circ\iota_*)$, let $x + \Gamma(x) \in \mathscr{A}_1$. Let F denote the fixed point set of x and let ν denote its normal bundle. $\Gamma(x)$ has fixed point set $F \cup x$ with normal bundle $\nu \oplus \theta \cup \theta$. Since $\varepsilon(x) = 0$, x contributes nothing to the bordism class of the normal bundle over $\Gamma(x)$. Thus, $\iota_*(x + \Gamma(x)) = [\nu]_2 + [\nu\oplus\theta]_2 = [\nu \to F]_2 \cdot (1 + [\theta \to pt]_2) \in \mathscr{P}$. For the opposite inclusion, let $\alpha \in \ker(\pi\circ\iota_*)$ be nonzero. Since ι_* is monic, $\iota_*(\alpha) \neq 0$; and since $\pi(\iota_*\alpha)) = 0$, $\iota_*(\alpha) \in \mathscr{P}$. Let $\iota_*(\alpha) = [\xi + \xi\oplus\theta]_2$. Now, $J([\xi + \xi\oplus\theta]_2) = 0$ since the sequence (2.2) is exact. Therefore, $J([\xi]_2) + J([\xi\oplus\theta]_2) = 0$. There is no reason to expect that $[\xi]_2$ would be a homogeneous element in $\mathscr{M}_*$. So, let

$$[\xi]_2 = (0, \ldots, 0, [\xi_k]_2, \ldots, [\xi_{k+m}]_2, 0, \ldots).$$

For dimensional reasons

$$J([\xi_{k+m}\oplus\theta]_2) = 0. \tag{3.2}$$

From [3;26.4] we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{M}_{n+1} & \xrightarrow{\ J\ } & MO_n(\mathbf{Z}_2) \\ {\scriptstyle \oplus\,\theta}\uparrow & & \downarrow{\scriptstyle \Delta} \\ \mathcal{M}_n & \xrightarrow[\ J\ ]{} & MO_{n-1}(\mathbf{Z}_2) \end{array}$$

where Δ is the Smith homomorphism. From this and (3.2) we have that $J([\xi_{k+m}]_2) = 0$. But then, $J([\xi_{k+m-1} \oplus \theta]_2) = 0$ and hence $J([\xi_{k+m-1}]_2) = 0$. By continuing this regression argument, we get that $J([\xi_j]_2) = 0$ for all $j = k, \ldots, k + m$, or $J([\xi]_2) = 0$. Again, by the exactness of (2.2), we have that $[\xi]_2 \in \mathrm{im}(\iota_*)$. Let $x \in I_*(\mathbf{Z}_2)$ be a preimage; i.e., $\iota_*(x) = [\xi]_2$. We need to see that $\iota_*(\Gamma(x)) = [\xi \oplus \theta]_2$ and $\varepsilon(x) = 0$. Now, $\iota_*(\Gamma(x)) = [\xi \oplus \theta]_2 + [\theta \to x]_2$. Now, $J(\iota_*(\Gamma(x))) = 0$ and $J([\xi \oplus \theta]_2) = 0$, so $J([\theta \to x]_2) = 0$. Thus, $\varepsilon(x) = 0$ and $\iota_*(\Gamma(x)) = [\xi \oplus \theta]_2$. Therefore, $\alpha = \rho\iota_*(\alpha) = \rho([\xi])_2 + [\xi \oplus \theta]_2) = x + \Gamma(x)$ with $\varepsilon(x) = 0$. Thus, $\alpha \in \mathscr{A}_1$ and $\mathscr{A}_1 = \ker(\pi \circ \iota_*)$. So we have shown that the sequence of rings and ideals

$$0 \longrightarrow \mathscr{A}_1 \longrightarrow I_*(\mathbf{Z}_2) \xrightarrow{\ \pi\circ\iota_*\ } \mathcal{M}_*/\mathscr{P} \longrightarrow 0$$

is exact.

COROLLARY 3.3. *$I_*(\mathbf{Z}_2)/\mathscr{A}_1$ is a polynomial ring over MO_* and over $\mathbf{Z}_2$.*

Proof. Recall that $\mathcal{M}_*$ is a polynomial ring over MO_* in a countable number of variables, one of which is $[\theta \to pt]_2$. Thus, $\mathcal{M}_*/\mathscr{P}$ is a polynomial ring over MO_* in a countable number of variables. Noting that MO_* is a polynomial ring over $\mathbf{Z}_2$ completes the proof.

4. BOARDMAN'S FIVE-HALVES THEOREM REVISITED

Let $\Lambda(\mathbf{Z}_2) = I_*(\mathbf{Z}_2)/\mathscr{A}_1$ and let $MO_*[[t]] = \left\{ \sum_{k=0}^{\infty} [V^k]_2 t^k \colon [V^k]_2 \in MO_k \right\}$ be the subring of "homogeneous" power series of the ring of formal power series over MO_*. Note that we require that the dimension of the coefficient from MO_* be equal to the exponent of t. Define a map $\phi \colon I_*(\mathbf{Z}_2) \to MO_*[[t]]$ by

$$\phi(\{T,M^n\}_2) = \sum_{j=0}^{\infty} \varepsilon(\Gamma^j(\{T,M^n\}_2))t^{n+j}.$$

Clearly, ϕ is additive. From the product formula for Γ, (2.3) and the fact that

ε is multiplicative, we get that $\varepsilon(\Gamma^n(xy)) = \sum_{j=0}^{n} \varepsilon(\Gamma^j(x))\,\varepsilon(\Gamma^{n-j}(y))$. From this, we see that $\phi(xy) = \phi(x)\phi(y)$, or ϕ is multiplicative. Note: $\varepsilon|_{MO_*} =$ identity.

LEMMA 4.1. *$\phi(\mathscr{A}_1) = 0$, or $\mathscr{A}_1 \subseteq \ker(\phi)$.*

Proof: Let $x + \Gamma(x) \in \mathscr{A}_1$; i.e., $\varepsilon(x) = 0$. Let $\dim(x) = n$.

$$\phi(x + \Gamma(x)) = \phi(x) + \phi(\Gamma(x)) = \sum_{j=0}^{\infty} \varepsilon(\Gamma^j(x))\,t^{n+j} + \sum_{k=0}^{\infty} \varepsilon(\Gamma^k(\Gamma(x)))\,t^{n+1+k}$$
$$= \varepsilon(x)\,t^n + \sum_{j=1}^{\infty} \varepsilon(\Gamma^j(x))\,t^{n+j} + \sum_{j=1}^{\infty} \varepsilon(\Gamma^j(x))\,t^{n+j} = 0.$$

Therefore, ϕ induces a homomorphism $\bar{\phi}: \Lambda(\mathbf{Z}_2) \to MO_*[[t]]$ which is well-defined, additive, and multiplicative. Note that $I_n(\mathbf{Z}_2)$ injects into $\Lambda(\mathbf{Z}_2)$ for each $n \geq 0$. This is true because $I_n(\mathbf{Z}_2)$ injects into $\mathscr{M}_n$ by (2.2). Furthermore, stabilization is monic so $\mathscr{M}_n$ injects in $\mathscr{M}_*/\mathscr{P}$. Thus, $I_n(\mathbf{Z}_2)$ injects into $\mathscr{M}_*/\mathscr{P} \approx \Lambda(\mathbf{Z}_2)$.

As in [2], $MO_*(BO)$ may be interpreted by stabilizing $\mathscr{M}_*$ by ignoring the addition of trivial line bundles. We impose the relation $[\xi]_2 = [\xi \oplus \theta]_2$ for any vector bundle ξ. This is clearly the same as requiring that $1 + [\theta \to pt]_2 = 0$ in $\Lambda(\mathbf{Z}_2)$. Thus, $\Lambda(\mathbf{Z}_2) \approx MO_*(BO)$.

LEMMA 4.2. *There exist elements $\{\tau, M^n\}_2$ in $\Lambda(\mathbf{Z}_2)$ for each $n \geq 0$ such that*

(i) *$\{[M^n]_2\}$ generates MO_* as a polynomial ring over $\mathbf{Z}_2$, for n not of the form $2^S - 1$;*

(ii) *$\{\{\tau, M^n\}_2\}$ generates $\Lambda(\mathbf{Z}_2)$ as a polynomial ring over $\mathbf{Z}_2$.*

Proof: This is [2; Lemma 16].

COROLLARY 4.3. *$\bar{\phi}: \Lambda(\mathbf{Z}_2) \to MO_*[[t]]$ is a monomorphism.*

Proof. By checking $\bar{\phi}$ on the generators from (4.2) this is clear.

Let us introduce two filtrations on $\Lambda(\mathbf{Z}_2)$. The first, fil_{FP}, will be an increasing filtration. We shall say that $\mathrm{fil}_{FP}(\{T, V^n\}_2) = k$ for $\{T, V^n\}_2 \in I_n(\mathbf{Z}_2) \subseteq \Lambda(\mathbf{Z}_2)$ if k is the maximum of the dimensions of the various components of the fixed point set which are nonzero on the bordism level. There is a decreasing filtration on $MO_*[[t]]$ given by $\mathrm{fil}(x) = n$ if the first nonzero coefficient in the power series for x is the coefficient of t^n. Thus, $\bar{\phi}: \Lambda(\mathbf{Z}_2) \to MO_*[[t]]$ induces a decreasing filtration on $\Lambda(\mathbf{Z}_2)$ denoted by $\mathrm{fil}_{\bar{\phi}}$. For an element $\{T, V^n\}_2 \in I_n(\mathbf{Z}_2)$ we have that $\mathrm{fil}_{\bar{\phi}}(\{T, V^n\}_2) = n + j$ if $\varepsilon(\Gamma^j(\{T, V^n\}_2)) \neq 0$ and all of the preceding powers of Γ on $\{T, V^n\}_2$ do augment to 0.

LEMMA 4.4. *For the generators $\{\tau, M^n\}_2$ of $\Lambda(\mathbf{Z}_2)$ from (4.2) we have:*

$$\text{(i)}\ \mathrm{fil}_{FP}(\{\tau, M^n\}_2) = \begin{cases} \dfrac{n}{2}, & \text{if } n \text{ is even} \\[2ex] \dfrac{n-1}{2}, & \text{if } n \text{ is odd;} \end{cases}$$

(ii) $\mathrm{fil}_{\bar\phi}(\{\tau, M^n\}_2) = n$.

Proof. This is clear from the construction of the generators in [2; Lemma 16].

We would like to know what happens to these two filtrations on products and sums. The following are clear from the definitions of the filtrations.

(i) $\mathrm{fil}_{FP}(xy) = \mathrm{fil}_{FP}(x) + \mathrm{fil}_{FP}(y)$, for any $x \neq 0$ and $y \neq 0$.

(ii) $\mathrm{fil}_{\bar\phi}(xy) = \mathrm{fil}_{\bar\phi}(x) + \mathrm{fil}_{\bar\phi}(y)$, for any $x \neq 0$ and $y \neq 0$.

(iii) $\mathrm{fil}_{FP}(x + y) = \max\{\mathrm{fil}_{FP}(x), \mathrm{fil}_{FP}(y)\}$ if x and y have no monomials in common.

(iv) $\mathrm{fil}_{\bar\phi}(x + y) = \min\{\mathrm{fil}_{\bar\phi}(x), \mathrm{fil}_{\bar\phi}(y)\}$ if x and y have no monomials in common.

It then follows that the fixed point filtration of a polynomial in the generators is the maximum of the fixed point filtration of its terms. For the $\bar\phi$-filtration of a polynomial in the generators, we get the minimum of the $\bar\phi$-filtrations of its terms.

From (4.4) for any generator $\{\tau, M^n\}_2$, n not of the form $2^s - 1$, we have that $\mathrm{fil}_{\bar\phi}(\{\tau, M^n\}_2) \leq \frac{5}{2}\,\mathrm{fil}_{FP}(\{\tau, M^n\}_2)$. We cannot improve on this inequality because the fixed point dimension of the generator $\{\tau, M^5\}_2$ is 2.

THEOREM 4.5. (Boardman's Five-Halves Theorem):

Let T be a smooth involution on a closed manifold V^n of dimension n and let k be the fixed point dimension; i.e., *the maximum of the dimensions of the various components of the fixed point set which are nonzero on the bordism level. If V^n does not bound,* $[V^n]_2 \neq 0$ in MO_n, *then $n \leq 5k/2$.*

Proof. Note that $\{T, V^n\}_2 \in I_n(\mathbf{Z}_2) \subseteq \Lambda(\mathbf{Z}_2)$. Since $[V^n]_2 \neq 0$ in MO_n, $\mathrm{fil}_{\bar\phi}(\{T, V^n\}_2) = n$. Write $\{T, V^n\}_2$ as a polynomial, $p_0(V^n)$, in the generators $\{\tau, M^n\}_2$. Then,

$$\begin{aligned} n = \mathrm{fil}_{\bar\phi}(\{T, V^n\}_2) &\leq \mathrm{fil}_{\bar\phi}(p_0(V^n)) \leq \frac{5}{2}\,\mathrm{fil}_{FP}(p_0(V^n)) \\ &\leq \frac{5}{2}\,\mathrm{fil}_{FP}(\{T, V^n\}_2) = \frac{5}{2}\,k. \end{aligned}$$

Since n is an integer, we can improve this to $n \leq [[5k/2]]$, the greatest integer in $5k/2$.

COROLLARY 4.6. *With the same assumptions as* (4.5),

$$n \leq \begin{cases} \dfrac{5}{2}\,k \text{ if } k \text{ is even}, \\[2ex] \dfrac{5k-1}{2} \text{ if } k \text{ is odd}. \end{cases}$$

COROLLARY 4.7. *Let T be a smooth involution on a smooth closed manifold V^n and let* $\mathrm{fil}_{FP}(\{T, V^n\}_2) = k$. *If* $[V^n]_2$ *is indecomposable in* MO_* *then*

$$n \leq \begin{cases} 2k+1, & \text{if } n \text{ is odd;} \\ 2k, & \text{if } n \text{ is even.} \end{cases}$$

Proof. Since $[V^n]_2$ is indecomposable in MO_*, we have that

$$\{T, V^n\}_2 = \{\tau, M^n\}_2 + \text{decomposables.}$$

Thus

$$k = \mathrm{fil}_{FP}(\{T, V^n\}_2) \geq \mathrm{fil}_{FP}(\{\tau, M^n\}_2) = \begin{cases} \dfrac{n-1}{2}, & \text{if } n \text{ is odd;} \\ \dfrac{n}{2}, & \text{if } n \text{ is even.} \end{cases}$$

Theorem (4.5) tells us that for at least one m with

$$0 \leq m \leq 5k/2,\ \varepsilon(\Gamma^m(\{T, V^n\})) \neq 0$$

in MO_{n+m}, where $k = \mathrm{fil}_{FP}(\{T, V^n\}_2)$. Since $k < n$, we actually have that $0 \leq m \leq 3k/2$.

COROLLARY 4.8. *Let T be a smooth involution on a smooth closed manifold V^n and let* $\mathrm{fil}_{FP}(\{T, V^n\}_2) = k$. If $n > 5k/2$, *then* $\{T, V^n\}_2$ *bounds in* $I_*(\mathbf{Z}_2)$.

Proof. Recall that $\bar{\phi}(\{T, V^n\}_2) = \sum_{j=0}^{\infty} \varepsilon(\Gamma^j(\{T, V^n\}_2))t^{n+j}$. Since $n > 5k/2$, by (4.5) $[V^n]_2 = \varepsilon(\{T, V^n\}_2) = 0$. If there is a $j > 0$ such that $\varepsilon(\Gamma^j(T, V^n\}_2)) \neq 0$, then $j > 3k/2$. Otherwise, (4.5) implies that $n \leq 5k/2$. The fixed point set of $\Gamma^j(\{T, V^n\}_2)$ is $F(T) \cup \bigcup_{i=0}^{j-\iota} \Gamma^i(V^n)$, where $F(T)$ is the fixed point set of T on V^n. Let j be the first integer for which $\varepsilon(\Gamma^j(\{T, V^n\}_2)) \neq 0$. Then,

$$\mathrm{fil}_{\bar{\phi}}(\Gamma^j(\{T, V^n\}_2)) = n + j$$

and $\mathrm{fil}_{FP}(\Gamma^j(\{T, V^n\}_2)) = k$ since $\varepsilon(\Gamma\{T, V^n\}_2)) = 0$ for all $i < j$. Thus

$$\mathrm{fil}_{\bar{\phi}}(\Gamma^j(\{T, V^n\}_2)) > \frac{5}{2}\,\mathrm{fil}_{FP}(\Gamma^j(\{T, V^n\}_2)).$$

By (4.5), $\varepsilon(\Gamma^j(\{T, V^n\}_2)) = 0$. Thus, we have that $\bar{\phi}(T, V_2^n) = 0$. Since $\bar{\phi}$ is monic, $\{T, V^n\}_2 = 0$ in $\Lambda(\mathbf{Z}_2)$. Recalling that $I_n(\mathbf{Z}_2)$ injects into $\Lambda(\mathbf{Z}_2)$ gives the desired result.

5. AN APPLICATION TO FLAT MANIFOLDS

An abstract group B is a *Bieberbach* group if B has a normal free abelian subgroup of finite index. M^n is a *flat* manifold if $\pi_1(M^n)$ is a Bieberbach group and the rank of the free abelian subgroup is n.

If M^n is a flat manifold, then so is $\Gamma(M^n)$. Consider the fibration $M^n \to \Gamma(M^n) \to RP(1)$. We have the sequence $0 \to \pi_1(M^n) \to \pi_1(\Gamma(M^n)) \to \mathbf{Z} \to 1$ is split exact. So, $\pi_1(\Gamma(M^n)) \approx \pi_1(M^n) \cdot \mathbf{Z}$, the semi-direct product. The normal abelian subgroup of $\pi_1(\Gamma(M^n))$ will be the direct sum of that subgroup of $\pi_1(M^n)$ and the subgroup of index 2 in $\mathbf{Z}$. The rank of this subgroup is $n + 1$.

There is a standing conjecture that all flat manifolds bound mod 2; see [3], [4], and [7]. Let T be a smooth involution on a flat manifold M^n with $\mathrm{fil}_{FP}(\{T,M^n\}_2) \geq 0$, i.e., the fixed point set is nonempty. Since M^n is flat, $\Gamma^j(M^n)$ is flat for all j. If this conjecture is true, we have

$$\bar{\phi}(\{T,M^n\}_2) = \sum_{j=0}^{\infty} \varepsilon(\Gamma^j(\{T,M^n\}_2))\, t^{n+j}$$
$$= \sum_{j=0}^{\infty} [\Gamma^j(M^n)]_2\, t^{n+j} = 0.$$

Since $\bar{\phi}$ is a monomorphism, $\{T,M^n\}_2 = 0$ in $I_n(\mathbf{Z}_2)$. Of course, if T is fixed point free, then $\{T,M^n\}_2 = 0$. Therefore, if the conjecture is true, then any smooth involution on a flat manifold bounds. If the conjecture is false, then $\varepsilon(\Gamma^j(\{T,M^n\}_2)) \neq 0$ for some j. Thus, there is a smooth involution on a flat manifold which does not bound. This may be a method of constructing a counterexample.

6. FACTORIZATIONS IN $I_*(\mathbf{Z}_2)$

In this section we shall return to the elements of the form $x + \Gamma^n(x) \in \mathscr{A}_n$ in $I_*(\mathbf{Z}_2)$. In $\mathscr{M}_*$ we have that $1 + \theta^n = (1 + \theta)\left(\sum_{j=0}^{n-1} \theta^j\right)$. We should expect a similar occurence in $I_*(\mathbf{Z}_2)$ due to (2.2).

PROPOSITION 6.1. $(x + \Gamma(x))\left(\sum_{j=0}^{n-1} \Gamma^j(x)\right) = x^2 + \Gamma^n(x^2)$ *if* $\varepsilon(\Gamma^j(x)) = 0$ *for* $0 \leq j < n$.

Proof. For $n = 1$ this is true since $\mathscr{A}_1$ is an ideal.

$$(x + \Gamma(x))(x + \Gamma(x)) = x^2 + (\Gamma(x))^2.$$

Now, $\Gamma^2(x^2) = \Gamma(x(\Gamma(x))) = \Gamma(x)\Gamma(x) = (\Gamma(x))^2$, since $\varepsilon(x) = 0$. So, $(x + \Gamma(x))(x + \Gamma(x)) = x^2 + \Gamma^2(x^2)$. By the inductive hypothesis, we assume that

$$(x + \Gamma(x))\left(\sum_{j=0}^{n-1} \Gamma^j(x)\right) = x^2 + \Gamma^n(x^2)$$

if $\varepsilon(\Gamma^j(x)) = 0$ for $0 \le j < n$. Assuming that $\varepsilon(\Gamma^j(x)) = 0$ for $0 \le j < n+1$, we have

$$\begin{aligned}(x+\Gamma(x))\left(\sum_{j=0}^{n}\Gamma^j(x)\right) &= (x+\Gamma(x))\left(\sum_{j=0}^{n-1}\Gamma^j(x)+\Gamma^n(x)\right)\\ &= x^2+\Gamma^n(x^2)+x\Gamma^n(x)+\Gamma(x)\Gamma^n(x).\end{aligned}$$

By repeated use of the product formula for Γ, (2.3), it is easily seen that $x\Gamma^n(x) = \Gamma^n(x^2)$ and $\Gamma(x)\Gamma^n(x) = \Gamma^{n+1}(x^2)$. Making these substitutions gives us the result.

Consider the polynomial $t^n - 1$ over $\mathbf{Q}$ and let ζ_n denote the primitive n^{th} root of unity. The minimal polynomial for ζ_n over $\mathbf{Q}$ is the n^{th} cyclotomic polynomial, denoted by $\Phi_n(t)$. From the definition we have that $t^n - 1 = \prod_{d|n}\Phi_d(t)$, where $d|n$ means that d divides n. Let ϕ denote the Euler ϕ function.

THEOREM 6.2. [8;7-2-4] *If n is odd, then 2 factors in $\mathbf{Q}(\zeta_n)$ into the product of r distinct prime ideals of degree f, where $rf = \phi(n)$ and f is the smallest positive integer such that $2^f \equiv 1 \pmod{n}$.*

THEOREM 6.3. [8;7-4-3] *If n is even write $n = 2^s n'$,* with *n' odd. Then 2 factors in $\mathbf{Q}(\zeta_n)$ in the form*

$$2\,\mathcal{O}_{\mathbf{Q}(\zeta_n)} = (\mathcal{B}_1 \ldots \mathcal{B}_r)^{\phi(2^s)}$$

where $\mathcal{B}_1, \ldots, \mathcal{B}_r$ are distinct prime ideals of $\mathcal{O}_{\mathbf{Q}(\zeta_n)}$ of degree f with $rf = \phi(n')$ and f being the smallest positive integer such that $2^f \equiv 1 \pmod{n'}$.

THEOREM 6.4. [8;7-5-4] $\mathcal{O}_{\mathbf{Q}(\zeta_n)} = \mathbf{Z}[\zeta_n]$.

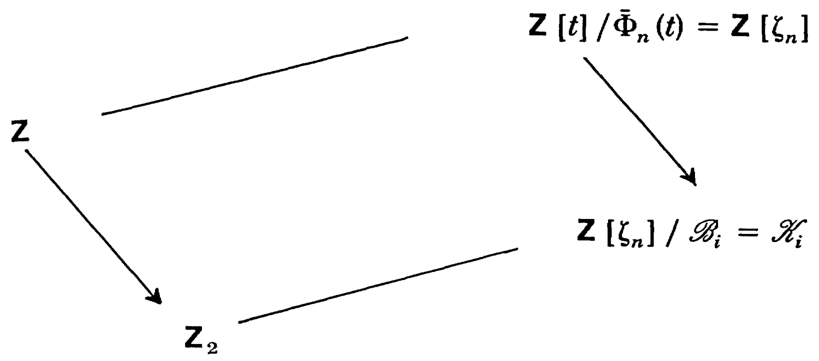
Let $n = 2^s n'$, with $s \ge 0$ and n' odd.

PROPOSITION 6.5. *If 2 factors in $\mathbf{Z}[\zeta_n]$ as $2\,\mathbf{Z}[\zeta_n] = \left(\prod_{i=1}^{r}\mathcal{B}_i\right)^{\phi(2^s)}$ with the above restrictions on r, f, and the ideals $\mathcal{B}_i$, then*

$$\Phi_n(t) \equiv \left(\prod_{i=1}^{r} p_i(t)\right)^{\phi(2^s)} \mod 2$$

with $p_i(t) \in \mathbf{Z}_2[t]$ being distinct and irreducible and $\deg(p_i(t)) = f$ *for $i = 1, \ldots, r$.*

Proof: We have the following diagram of rings



By (6.4) $\mathbf{Z}[\zeta_n]$ is a Dedekind domain. Thus $\mathscr{B}_i$ is a maximal ideal. So, $\mathbf{Z}[\zeta_n]/\mathscr{B}_i = \mathscr{K}_i$ is a field; in fact, an extension field of $\mathbf{Z}_2$. By assumption $[\mathscr{K}_i:\mathbf{Z}_2] = f$. Take $\bar{\zeta}_n$ in $\mathscr{K}_i$ and let $p_i(s)$ be its minimal polynomial over $\mathbf{Z}_2$. Then, $p_i(t)^{\phi(2^s)}$ divides $\Phi_n(t)$ modulo 2. Doing this for each i gives us the result.

Example 6.6. $n = 31$, $s = 0$, $\phi(n) = 30$, $r = 6$, and $f = 5$.

$$\begin{aligned} t^{31} + 1 &\equiv (t+1)\Phi_{31}(t) \equiv (t+1)\sum_{i=0}^{30} t^i \\ &\equiv (t+1)(t^5+t^2+1)(t^5+t^3+1) \\ &\cdot\ (t^5+t^4+t^3+t^2+1)(t^5+t^4+t^3+t+1) \\ &\cdot\ (t^5+t^4+t^2+t+1)(t^5+t^3+t^2+t+1) \pmod 2. \end{aligned}$$

From (6.5) and the definition of cyclotomic polynomials we have

$$t^n + 1 \equiv \prod_{d\mid n}\left[\prod_{i_d=1}^{r_d} p_{i_d}(t))^{\phi(2s_d)}\right] \bmod 2$$

where $\deg(p_{i_d}(t)) = f_d$, $i_d = 1, \ldots, r_d$; $d = 2^{s_d}d'$; and $r_d f_d = \phi(d')$ with f_d being the smallest positive integer with $2^{f_d} \equiv 1 \pmod{d'}$ for each d which divides n.

We need some computational tools for Γ. Assume that $n + m < N$ and $\varepsilon(\Gamma^j(x)) = 0$ for all $0 \le j < N$ for the next three lemmas.

LEMMA 6.7. $\Gamma^n(x)\Gamma^m(x) = \Gamma^{n+m}(x^2)$.

Proof: The proof is only a simple induction argument on n and m, using only the product formula for Γ (2.3) and the assumption about the augmentations.

LEMMA 6.8. $\Gamma(x)\Gamma^n(x^r) = \Gamma^{n+1}(x^{r+1})$.

Proof: This is again an induction argument as in (6.7).

LEMMA 6.9. $\Gamma^n(x^r)\Gamma^m(x^s) = \Gamma^{n+m}(x^{r+s})$.

Let $p(t)$ be a polynomial over $\mathbf{Z}_2$ defined by $p(t) = \sum_{i=0}^{n} a_i t^i$, $a_i \in \mathbf{Z}_2$. Define the polynomial operator $p(\Gamma)$ by $p(\Gamma)(x) = \left(\sum_{i=0}^{n} a_i\Gamma^i\right)(x) = \sum_{i=0}^{n} a_i\Gamma^i(x)$. If $q(\Gamma)$ is another polynomial operator, $q(\Gamma)(x) = \sum_{j=0}^{m} b_j\Gamma^j(x)$, $b_j \in \mathbf{Z}_2$; define the product of these two polynomial operators to be

$$p(\Gamma)\cdot q(\Gamma) = pq(\Gamma) = \sum_{k=0}^{n+m} c_k\Gamma^k,$$

$c_k = \sum_{i=0}^{k} a_i b_{k-i} \in \mathbf{Z}_2$. (6.7), (6.8), and (6.9) give us that

$$[p(\Gamma)(x)] \cdot [q(\Gamma)(x)] = pq(\Gamma)(x^2).$$

Combining this, the above lemmas, and (6.5) we have the following theorem.

THEOREM 6.10: *If* $t^n + 1 = \prod_{d|n} \left(\prod_{j_d=1}^{r_d} p_{j_d}(t) \right)^{\phi(2^{s_d})}$ *with the* $p_{j_d}(t)$ *not necessarily distinct for different values of* d *and with the above restrictions of* r_d, f_d, *and* s_d, *then*

$$\prod_{d|n} \left(\prod_{j_d=1}^{r_d} [p_{j_d}(\Gamma)(x)] \right)^{\phi(2^{s_d})} = x^k + \Gamma^n(x^k)$$

where $k = \sum_{d|n} r_d \phi(2^{s_d})$.

Example 6.11: We saw how to factor $t^{31} + 1$ (mod 2). The above gives us that

$$\begin{aligned} x^7 + \Gamma^{31}(x^7) = {} & (x + \Gamma(x))(x + \Gamma^2(x) + \Gamma^5(x))(x + \Gamma^3(x) + \Gamma^5(x))(x + \Gamma^2(x) \\ & + \Gamma^3(x) + \Gamma^4(x) + \Gamma^5(x))(x + \Gamma(x) + \Gamma^3(x) + \Gamma^4(x) \\ & + \Gamma^5(x))(x + \Gamma(x) + \Gamma^2(x) \\ & + \Gamma^4(x) + \Gamma^5(x))(x + \Gamma(x) + \Gamma^2(x) + \Gamma^3(x) + \Gamma^5(x)). \end{aligned}$$

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Mathematics Department, RLM 8-100
University of Texas at Austin
Austin, Texas 78712