BORDISM OF METACYCLIC GROUP ACTIONS

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In this paper the term *manifold* means a differentiable, compact manifold having a unitary (stably almost complex) structure [9]. A *G-action* on such a manifold is a differentiable action of a finite group G preserving the unitary structure. We write Ω_*^U for the bordism ring of closed unitary manifolds; Milnor [9] proved that Ω_*^U is an integral polynomial ring with one generator in each even dimension. Let $\Omega_*^U(G)$ be the bordism of G-actions, as studied, for example, by Stong [13].

The following question, once under active study, has been dormant for several years. Is $\Omega_*^U(G)$ always a free Ω_*^U-module on even-dimensional generators? Stong [13] proved that this is true for abelian p-groups G, and Ossa [10] showed how to extend Stong's result to all abelian groups. Lazarov [8] showed that the answer is also yes if G is a group of order pq for distinct primes p and q. In this paper, we give an affirmative answer for a well-known class of metacyclic groups.

THEOREM. *Suppose all Sylow subgroups of G are cyclic. Then $\Omega_*^U(G)$ is a free Ω_*^U-module on even-dimensional generators.*

Some readers will recall that Landweber and Lazarov have announced such a theorem [7], although they required an additional hypothesis on the group G. Professor Lazarov was kind enough to send me, several years ago, a copy of a manuscript proving the theorem for groups of order $p^m q^n$. The proof given here is very different; although following the general outline proposed in [8], it uses the methods of [11] to reduce the necessary calculations by at least an order of magnitude.

There are six parts to the proof. The first two list some well-known facts we shall require. Part 3 is an outline of the proof. Part 4 recalls the machinery of [11]. The last two parts contain the computations.

1. GROUP THEORY

Let $\mathscr{M}$ be the class of finite groups G such that every Sylow subgroup of G is cyclic. It is clear that if $G \in \mathscr{M}$ then each subgroup and each factor group of G is also in $\mathscr{M}$. By a well-known theorem, (see [5, pp. 146–148], for example), if $G \in \mathscr{M}$ then G may be written as an extension

$$1 \to H \to G \to K \to 1,$$

so that H and K are cyclic and have relatively prime orders. It follows that if n divides the order of G then G possesses a subgroup of order n. We will need the following refinements of this observation.

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PROPOSITION 1.1. *Suppose $G \in \mathscr{M}$ and L is a normal subgroup of G. Let $[G:L] = p_1^{r_1}, \ldots, p_s^{r_s}$ be the prime factorization of the index of L in G.*

(a) *There is some subscript i, for which G has a normal subgroup K, of index $p_i^{r_i}$* in *G, containing L.*

(b) *For each subscript i, there is a subgroup K_i of G (not always normal), such that K_i contains L and $[K_i:L] = p_i^{r_i}$.*

The proof is an easy exercise. We shall also need a lemma from representation theory. Let θ: $K \times \mathbf{C}^n \to \mathbf{C}^n$ be a complex representation space for some normal subgroup K of G. For each $g \in G$ there is a representation space $g_* \theta$ defined by the rule

$$g_* \theta(k, z) = \theta(g^{-1} kg, z), \ (k, z) \in K \times \mathbf{C}^n.$$

Then θ is *G-invariant* if $g_* \theta \cong \theta$ whenever $g \in G$; more generally the set of $g \in G$ such that $g_* \theta \cong \theta$ forms a subgroup $J(\theta)$ of G, the *isotropy subgroup* of θ.

PROPOSITION 1.2. *Let G be a finite group, and let K be a normal subgroup of G such that G/K is cyclic. Let θ be a G-invariant complex representation of K. Then there is a complex representation σ of G whose restriction σ_K to a K-representation is isomorphic to θ.*

Proof. Our reference is Feit [4]. Let ξ be an irreducible component of θ and let $J = J(\xi)$. Since θ is G-invariant, it contains $\Sigma\, h_* \xi$, where h runs over a set of coset representatives of G/J. We can assume $\theta = \Sigma\, h_* \xi$ without loss of generality. Suppose there is a J-representation ρ such that $\rho_K \cong \xi$; then the induced G-representation ρ^G satisfies $(\rho^G)_K \cong \Sigma\, h_* \xi = \theta$, by [4, (9.10)]. Thus we may also assume that $J = G$. In that case, choose an irreducible component λ of ξ^G, such that λ_K contains ξ. Then $\lambda_K = \xi$, by [4, (9.12) and (9.10)].

2. BORDISM THEORY

We adopt the notations and definitions of [12, 13], giving only a brief review. A collection $\mathscr{F}$ of subgroups G is a *family* if $\mathscr{F}$ contains every subgroup and every conjugate of K whenever $K \in \mathscr{F}$. Suppose $\mathscr{F}' \subseteq \mathscr{F}$ and both are families of subgroups of G; then there is a "relative" bordism group $\Omega_*^U(G)(\mathscr{F}, \mathscr{F}')$ whose elements are represented by G-manifolds M such that, for each x in M, the isotropy subgroup $G_x \in \mathscr{F}$, and for each $x \in \partial M$, $G_x \in \mathscr{F}'$. In case $\mathscr{F}' = \emptyset$, then $\partial M = \emptyset$ also, and one writes $\Omega_*^U(G)(\mathscr{F})$ for the bordism group.

If K is a subgroup of G, there is a restriction homomorphism

$$r_K^G \colon \Omega_*^U(G) \to \Omega_*^U(K)$$

which restricts each G-action to a K-action, and an extension homomorphism $e_G^K \colon \Omega_*^U(K) \to \Omega_*^U(G)$ which sends the class of a K-manifold M to the class of $G \times {}_K M$. (See [12] for detailed definitions in terms of the relative groups). A family $\mathscr{F}$ of subgroups of K is *G-invariant* if $g^{-1}Hg \in \mathscr{F}$ for every $H \in \mathscr{F}$

and $g \in G$. Now suppose K is normal. Given a pair $(\mathscr{F}, \mathscr{F}')$ of G-invariant families of subgroups of K, there is an action of G/K on $\Omega_*^U(K)(\mathscr{F}, \mathscr{F}')$: if $\theta: K \times M \to M$ is a K-action and $g \in G$ there is another K-action $g_*\theta$ defined by the rule

$$g_*\theta(k, m) = \theta(g^{-1}kg, m), \, (k, m) \in K \times M;$$

G/K acts on bordism via $(gK)[M, \theta] = [M, g_*\theta]$.

We define abbreviations for certain families of subgroups of G. If K is a subgroup of G, we write AK for the family of all subgroups conjugate in G to subgroups of K, and PK for the family of all subgroups conjugate in G to proper subgroups of K. In particular, $\Omega_*^U(G) = \Omega_*^U(G)(AG)$. If, in addition, L is a normal subgroup of G, we write $AKPL$ for the family of subgroups H of G such that (1) H is conjugate to a subgroup of K, and (2) $H \cap L$ is a proper subgroup of L.

Two families $(\mathscr{F}, \mathscr{F}')$ are *adjacent, differing by L*, if $\mathscr{F}' \subseteq \mathscr{F}$ and $\mathscr{F} - \mathscr{F}'$ consists of the conjugates of L.

PROPOSITION 2.1. *Suppose $(\mathscr{F}, \mathscr{F}')$ are adjacent, differing by L. Then the inclusion $(AL, PL) \subseteq (\mathscr{F}, \mathscr{F}')$ induces an isomorphism*

$$\Omega_*^U(G)(AL, PL) \xrightarrow{\cong} \Omega_*^U(G)(\mathscr{F}, \mathscr{F}').$$

Furthermore, if N is the normalizer of L in G, then the extension e_G^N is an isomorphism,

$$e_G^N: \Omega_*^U(N)(AL, PL) \xrightarrow{\cong} \Omega_*^U(G)(AL, PL).$$

For the proof of this fundamental result, see [12, pp. 14–20].

PROPOSITION 2.2. *There is a canonical isomorphism*

$$\Omega_*^U(G)(AG, PG) \cong \sum_\sigma \Omega_*^U(B\sigma)$$

in which σ runs over the set of complex G-representation spaces, and each $B\sigma$ is a classifying space homotopy equivalent to a Cartesian product of $BU(k)$'s. In particular, $\Omega_*^U(G)(AG, PG)$ is a free Ω_*^U-module on even-dimensional generators.

The essentials of the proof are present in Conner and Floyd [2, section 38]. Recall that the construction assigns, to a G-manifold M, the disjoint union of tubular neighborhoods of the various components F of the fixed set of G in M. Each such neighborhood is regarded as the total space of a G-vector bundle $\nu \to F$ whose fiber exhibits a G-representation σ. Corresponding to a decomposition of σ into irreducible components, $\sigma = \Sigma\xi_i$, there exist bundles ν_i so that $\nu = \Sigma\, \nu_i \otimes \xi_i$. Classifying the ν_i gives a mapping $F \to \Pi_i\, BU(k_i)$, $k_i = \dim \nu_i$, so the latter space serves as $B\sigma$. Finally the last statement holds because $H_*(BU(k_i))$ is free abelian and vanishes in odd dimensions [3, Proposition (3.3)].

Now let L be a normal subgroup of G. We shall use (2.2) to describe the action of G/L on $\Omega_*^U(L)(AL, PL)$. Suppose $[M, \theta] \in \Omega_*^U(L)(AL,PL)$. If F is a component of the fixed set of L in (M, θ), then it is also a component of the fixed set of L in $(M, g_*\theta)$ for each $g \in G$, but the associated representation σ is changed to $g_*\sigma$ in the latter case. Thus the effect of g_* is to permute the summands of $\Sigma\, \Omega_*^U(B\sigma)$ in the obvious fashion. This gives the following result.

PROPOSITION 2.3. *Suppose L is a normal subgroup of G, and C is a set of representatives of G-conjugacy classes of representations of L. Then there is an isomorphism of (G/L)-modules,*

$$\Omega_*^U(L)(AL, PL) \cong \sum_{\sigma \in C} \Omega_*^U(B\sigma) \otimes Z(G/J(\sigma)).$$

3. OUTLINE OF THE PROOF

If M_* is a graded Ω_*^U-module, we write $M_+ = \sum_i M_{2i}$ and $M_- = \sum_i M_{2i+1}$.

PROPOSITION 3.1. *Suppose $G \in \mathscr{M}$ and suppose $(\mathscr{F}, \mathscr{F}')$ is an adjacent pair of families of subgroups of G. Then $\Omega_+^U(G)(\mathscr{F}, \mathscr{F}')$ is a free Ω_*^U-module, and $\Omega_-^U(G)(\mathscr{F}, \mathscr{F}')$ has projective dimension one over Ω_*^U.*

Proof. See section 5 below.

PROPOSITION 3.2. *Given the same hypothesis, the homomorphism*

$$\Omega_-^U(G)(\mathscr{F}, \mathscr{F}') \to \Omega_-^U(G)(AG, \mathscr{F}')$$

is the zero homomorphism.

Proof. See section 6 below.

The theorem appears as the case $\mathscr{F}' = \emptyset$ of the following proposition.

PROPOSITION 3.3. *Suppose $G \in \mathscr{M}$ and $\mathscr{F}'$ is a family of subgroups of G. Then $\Omega_*^U(G)(AG, \mathscr{F}')$ is a free Ω_*^U-module on even-dimensional generators.*

Proof. Select a chain of families $\mathscr{F}' \subset \mathscr{F} \subset \ldots \subset PG \subset AG$ such that each pair of successive entries is adjacent. By (2.2) the Proposition holds for $\Omega_*^U(G)(AG, PG)$. Applying an obvious inductive argument, assume the Proposition holds for $\Omega_*^U(G)(AG, \mathscr{F})$. In particular, $\Omega_-^U(G)(AG, \mathscr{F}) = 0$.

The exact sequence [13, Prop. 2.2] for the triple $(AG, \mathscr{F}, \mathscr{F}')$ then has the following form:

$$0 \to \Omega_+^U(G)(\mathscr{F}, \mathscr{F}') \to \Omega_+^U(G)(AG, \mathscr{F}')$$

$$\overset{a_*}{\to} \Omega_+^U(G)(AG, \mathscr{F}) \to \Omega_-^U(G)(\mathscr{F}, \mathscr{F}')$$

$$\overset{b_*}{\to} \Omega_-^U(G)(AG, \mathscr{F}') \to 0.$$

By (3.2), we have $b_* = 0$; thus $\Omega_-^U(G)(AG, \mathscr{F}') = 0$. By (3.1), the image of a_* is a free Ω_*^U-module. Thus the exact sequence

$$0 \to \Omega_+^U(G)(\mathscr{F}, \mathscr{F}') \to \Omega_+^U(G)(AG, \mathscr{F}') \to \operatorname{Im} a_* \to 0$$

must split. By (3.1) again, this implies that $\Omega_+^U(G)(AG, \mathscr{F}')$ is free, as required.

Notice that the plan of attack is the same as that of Lazarov [8] for groups of order pq; the problem is to find tolerable proofs of (3.1) and (3.2). One should also notice that (3.1) often fails without the requirement that $G \in \mathscr{M}$. If $\mathscr{F} = \{\{1\}\}$ and $\mathscr{F}' = \emptyset$, then $\Omega_*^U(G)(\mathscr{F}, \mathscr{F}')$ becomes the bordism $\bar{\Omega}_*^U(G)$ of free G-actions; by [2, (19.1)] the latter is isomorphic to the bordism $\Omega_*^U(BG)$ of a classifying space for principal G-bundles. Landweber [6, Theorem 3] has shown that $\dim \Omega_*^U(BG) \leq 1$ if and only if G has periodic cohomology.

4. THE EQUIVARIANT BORDISM SPECTRAL SEQUENCE

We record here some results from [11].

PROPOSITION 4.1. *Suppose K is a normal subgroup of G, $(\mathscr{F}, \mathscr{F}')$ is a pair of G-invariant families of subgroups of G, and $\Omega_*^U(K)(\mathscr{F}, \mathscr{F}')$ is a (G/K)-module as described in section 2. Then there is a spectral sequence $\{E^r_{a,b}: r \geq 2;\ a, b \geq 0\}$ such that*

(i) $E^2_{a,b} = H_a(G/K; \Omega_b^U(K)(\mathscr{F}, \mathscr{F}'))$;

(ii) $E^\infty_{a,b}$ *is associated with a filtration of* $\Omega^U_{a+b}(G)(\mathscr{F}, \mathscr{F}')$; *and*

(iii) *the edge homomorphism* $\Omega_b^U(K)(\mathscr{F}, \mathscr{F}') \cong E^\infty_{0,b} \to E^2_{0,b}\, \Omega_b^U(G)(\mathscr{F}, \mathscr{F}')$ is the extension, e_G^K.

For the proof, see [11, (2.1)]. Our only applications of (4.1) will be in the case $(\mathscr{F}, \mathscr{F}') = (AL, PL)$ for some normal subgroup L of G which is contained in K. Thus (2.3) will compute the E^2 term of the spectral sequence.

If K is a subgroup of G, $P(G{:}K)$ is the collection of primes not dividing the index of K in G. If P is a collection of primes, Z_P denotes the P-local integers (integers not in P have inverses).

PROPOSITION 4.2. *Suppose L is a subgroup of G, and $P = P(G{:}L)$. Then $\Omega_*^U(G)(AL, PL) \otimes Z_P$ is a free module over $\Omega_*^U \otimes Z_P$.*

Proof. see [11, (3.2)]. In particular, note that $\Omega_-^U(G)(AL, PL)$ contains only torsion, of orders involving primes that divide the index of L in its normalizer.

PROPOSITION 4.3. *Let $(\mathscr{F}, \mathscr{F}', \mathscr{F}'')$ be a triple of families of subgroups of G, and let $P = \cap\ \{P(G{:}K): K \in \mathscr{F}' - \mathscr{F}''\}$. Then the forgetful homomorphism*

$$i_*: \Omega_*^U(G)(\mathscr{F}', \mathscr{F}'') \otimes Z_P \to \Omega_*^U(G)(\mathscr{F}, \mathscr{F}'') \otimes Z_P$$

is a split monomorphism.

Proof. See [11,(3.3)]. We use the following corollary: suppose $x \in \Omega_*^U(G)(\mathscr{F}, \mathscr{F}'')$ is a q-torsion class for some prime q, and $x = i_*(y)$. Then we may assume y is also a q-torsion class.

PROPOSITION 4.4. *Suppose $L \leq K \leq G$, and L is normal in G. Let q be a prime which does not divide $[G:K]$. Then the q-torsion submodule of $\Omega_*^U(G)(AL, PL)$ is contained in the image of e_G^K.*

Proof. By (2.1) we may suppose K is normal in G. The result then follows from (4.1), since $E^2_{a,b} = H_a(G/K; \Omega_b^U(K)(AL, PL))$ has no q-torsion for $a > 0$.

5. THE PROOF OF 3.1

For the rest of the paper, we assume $G \in \mathscr{M}$. However, this restriction is not actually needed in the proofs of (5.1), (5.2), or (6.1).

Suppose $(\mathscr{F}, \mathscr{F}')$ are adjacent, differing by L. By (2.1) and induction on the order of G, it suffices to assume that L is normal in G, and that

$$(\mathscr{F}, \mathscr{F}') = (AL, PL).$$

PROPOSITION 5.1. *If L is normal in G, and G/L is cyclic, then the extension*

$$e_G^L\colon \Omega_+^U(L)(AL, PL) \to \Omega_+^U(G)(AL, PL)$$

is a split epimorphism.

Proof. Let t be a generator of G/L, $t_*\colon \Omega_*^U(L)(AL, PL) \to \Omega_*^U(L)(AL, PL)$, $D_* = 1 - t_*$, and $N_* = \Sigma\{t_*^i : 0 \leq i \leq [G:L] - 1\}$. Of course,

$$H_a(G/L; \Omega_b^U(L)(AL, PL))$$

is Ker N_*/Im D_* for even $a > 0$, and Ker D_*/Im N_* for odd a.

Let σ be a complex representation of L. Since G/L is cyclic, the isotropy subgroup $J(\sigma)$ is normal in G. Note that $H_a(G/L; Z(G/J(\sigma))) = 0$ for even $a > 0$. By (2.3),

$$H_a(G/L; \Omega_b^U(L)(AL, PL)) = 0$$

for $a > 0$ even, or for b odd. By (4.1), $e_G^L\colon \Omega_+^U(L)(AL, PL) \to \Omega_+^U(G)(AL, PL)$ is surjective.

$\mathscr{F} = \{\{1\}\}$ and $\mathscr{F}' = \emptyset$

Moreover, (2.3) implies that Im D_* (= Ker N_*) is an Ω_*^U-module summand of $\Omega_+^U(L)(AL, PL)$. By [2, pp. 52–54] it is easy to see that

$$\text{Im}\, D_* \subseteq \text{Ker}\, e_G^L \subseteq \text{Ker}\, N_*;$$

thus Ker e_G^L is a summand and e_G^L is a split epimorphism.

Note. This argument also shows that the spectral sequence collapses. Since $H_0(G/L;\ \Omega_b^U(L)(AL,PL)) = \Omega_b^U(L)(AL,PL)/\text{Im}D_*$, and Ker $e_G^L = \text{Im}D_*$, the homomorphism $E^2_{0,b} \to \Omega^U_b(G)(AL, PL)$ is injective. It follows that there can be no nonzero differentials in the spectral sequence.

PROPOSITION 5.2. *If L is normal in G, and G/L is cyclic, then* $\Omega^U_-(G)(AL, PL)$ *has projective dimension* 1 *over* Ω^U_*.

Proof. First observe that $H_{2n+1}(G/L; Z(G/J(\sigma)))$ is cyclic; by (2.3) it follows that $H_{2n+1} = H_{2n+1}(G/L; \Omega^U_*(L)(AL, PL))$ has projective dimension one. By (4.1) there is a filtration

$$0 = V^{-1} \subseteq V^1 \subseteq V^3 \subseteq \dots \subseteq V^{2n+1} \subseteq \dots \subseteq \Omega^U_-(G)(AL, PL)$$

such that $V^{2n+1}/V^{2n-1} \cong H_{2n+1}$. In particular, $V^1 \cong H_1$ and thus dim $V^1 = 1$. Using the exact sequences $0 \to V^{2n-1} \to V^{2n+1} \to H_{2n+1} \to 0$ it follows that dim $V^{2n+1} = 1$ for each n.

In fact, if $0 \to G^{2n+1} \to F^{2n+1} \to H^{2n+1} \to 0$ is a projective resolution (= free resolution, by [3], Proposition (3.2)), then one may construct free resolutions

$$\begin{array}{ccccccccc} 0 \to & \sum_{i=1}^{n} G^{2i+1} & \to & \sum_{i=1}^{n} F^{2i+1} & \to & V^{2n+1} & \to 0 \\ & \downarrow\alpha & & \downarrow\beta & & \downarrow\gamma & \\ 0 \to & \sum_{i=1}^{n+1} G^{2i+1} & \to & \sum_{i=1}^{n+1} F^{2i+1} & \to & V^{2n+3} & \to 0 \end{array}$$

so that α and β are the obvious injections $x \mapsto (x, 0)$ and γ is the inclusion. This is an easy exercise, or the construction may be found in [1, p.79]. Thus, by taking direct limits we obtain a free resolution

$$0 \to \sum_{i=1}^{\infty} G^{2i+1} \to \sum_{i=1}^{\infty} F^{2i+1} \to \Omega^U_-(G)(AL, PL) \to 0,$$

as required.

We now complete the proof of (3.1). If L has prime-power index in G, then G/L is cyclic and (3.1) would follow from (5.1) and (5.2). Otherwise, use (1.1a) to select a normal subgroup K of G containing L, such that $[G{:}K] = q^s$ for some prime q which does not divide $[K{:}L]$. By induction on the order of G, assume $\Omega^U_+(K)(AL, PL)$ is free, $e^L_K\colon \Omega^U_+(L)(AL, PL) \to \Omega^U_+(K)(AL, PL)$ is a split epimorphism, and $\Omega^U_-(K)(AL, PL)$ has projective dimension one.

Consider the action of the cyclic group G/K on $\Omega^U_+(K)(AL, PL)$. Let $t \in G - K$ be an element of order n whose coset generates G/K. Then t induces a Z/n action on $\Omega^U_+(L)(AL, PL)$, and with this action e^L_K becomes a Z/n-morphism. In fact, there is a K-equivariant diffeomorphism $t_*(K \times_L M) \to K \times_L (t_*M)$ defined by $[k, m] \mapsto [t\,k\,t^{-1}, m]$. Therefore $\Omega^U_+(K)(AL, PL)$ is isomorphic as a G/K-module to a summand of $\Omega^U_+(L)(AL, PL)$.

As in the proof of (5.1), this implies that $H_{2a}(G/K; \Omega^U_+(K)(AL, PL)) = 0$ for $a > 0$. On the other hand, $H_-(G/K; \Omega^U_-(K)(AL, PL)) = 0$ because the coefficients contain only torison of orders prime to q (by (4.2)). Thus

$$e^K_G: \Omega^U_+(K)(AL, PL) \to \Omega^U_+(G)(AL, PL)$$

is surjective, by (4.1). As in (5.1), we see that e^K_G is in fact a split epimorphism, so $\Omega^U_+(G)(AL, PL)$ is free.

In the odd dimensions, let us write $\Omega^U_-(G)(AL, PL) = Q_- + T_-$, where Q_- is the q-torsion submodule. By (4.4), Im $e^K_G = T_-$. By [2, p. 54],

$$e^K_G r^G_K e^K_G = [G:K]\ e^K_G = q^s e^K_G;$$

hence $e^K_G r^G_K | T_-$ is an isomorphism. Therefore T_- is isomorphic to a summand of $\Omega^U_-(K)(AL, PL)$ and must have projective dimension one. Since

$$H_-(G/K; \Omega^U_+(K)(AL, PL))$$

has projective dimension one, while $H_{2a}(G/K; \Omega^U_-(K)(AL, PL)) = 0$ by (4.2), it follows as in the proof of (5.2) that Q_- also has projective dimension one. The proof of (3.1) is thus complete.

6. THE PROOF OF 3.2

We continue to suppose that $(\mathscr{F}, \mathscr{F}')$ are adjacent, differing by L. As before, by (2.1) and induction on the order of G it will suffice to assume that L is normal, and that $(\mathscr{F}, \mathscr{F}') = (AL, PL)$. By (2.2) we may assume that L is a proper subgroup. Fix a prime q which divides $[G:L]$; by (4.2) it suffices to show that the q-torsion of $\Omega^U_-(G)(AL, PL)$ is sent to zero in $\Omega^U_-(G)(AG, PL)$.

PROPOSITION 6.1. *Suppose G/L is cyclic of order q^s. Then*

$$\Omega^U_-(G)(AL, PL) \to \Omega^U_+(G)(AG, AGPL)$$

is the zero homomorphism.

Proof. We continue the computations of (5.1) and (5.2), which cover this case. Suppose $x \in V^{2j+1} \cap \Omega^U_{2k+1}(G)(AL, PL)$. Then x determines a certain coset

$$[x] \in E^2_{2j+1,2(k-j)} = H_{2i+1}(G/L; \Omega^U_{2(k-j)}(L)(AL, PL)).$$

Let $y \in \text{Ker } D_* \subseteq \Omega^U_{2(k-j)}(L)(AL, PL)$ determine this same coset. We show that there exists some $[M] \in \Omega^U_{2(k-j)}(G)(AG, AGPL)$ such that $r^G_L[M] = y$.

By (2.3), it suffices to assume that

$$y \in \sum_{i=0}^{[G:J]-1} \Omega^U_*(Bt^i_* \sigma)$$

for some L-representation σ and $J = J(\sigma)$. Thus, since $t_* y = y$, we may write $y = \sum_{i=0}^{[G:J]-1} t_*^i(z)$ for some $z \in \Omega_*(B\sigma)$. By (1.2) there exists a J-action M' such that $[M'] \in \Omega_*^U(J)(AJ, AJPL)$ and $r_L^J[M'] = z$. It follows that $r_L^G e_G^J[M'] = y$, so we choose $[M] = e_G^J[M'] \in \Omega_*^U(G)(AG, AGPL)$.

Next, let S^{2j+1} have the standard free action of G/L, in which t acts as multiplication by a primitive $[G:L]$-th root of one in each coordinate of complex $(j+1)$-space. We regard this as an action of G via the projection $G \to G/L$. The product $S^{2j+1} \times M$ is thus a G-space, admitting a G-equivariant map $S^{2j+1} \times M \to S^{2j+1} \to E(G/L)$ to a classifying space for free G/L- actions. By the proof of [11, (2.1)] it follows that $[S^{2j+1} \times M] \in \Omega_*(G)(AL, PL)$ represents the coset $[x]$. Now $S^{2j+1} \times \partial M = \partial(D^{2j+2} \times \partial M)$, and in $D^{2j+2} \times \partial M$ all isotropy groups lie in $AGPL$. Therefore $[S^{2j+1} \times M] = 0 \in \Omega_*^U(G)(AG, AGPL)$. It follows, by a straightforward induction on j, that V^{2j+1} has zero image in $\Omega_*^U(G)(AG, AGPL)$.

By (1.2b), there is a subgroup K of G containing L, so that $[K:L] = q^s$ and $[G:K]$ is prime to q. By (4.4), $e_G^K: \Omega_-^U(K)(AL, PL) \to \Omega_-^U(G)(AL, PL)$ maps onto the q-torsion. It follows by induction on the order of G that $\Omega_-^U(G)(AL, PL) \to \Omega_-^U(G)(AG, AKPL)$ kills q-torsion, for each prime q dividing $[G:L]$, and must therefore be the zero homomorphism.

PROPOSITION 6.2. *Suppose $G \in \mathscr{M}$, G/L is cyclic of order q^s, and $\mathscr{F}_0$ is a family of subgroups of G such that $PL \subseteq \mathscr{F}_0 \subseteq AGPL$. Then*

$$\Omega_*^U(G)(AGPL, \mathscr{F}_0) \to \Omega_*^U(G)(AG, \mathscr{F}_0)$$

is zero on q-torsion.

Proof. There is nothing to prove if $\mathscr{F}_0 = AGPL$. Suppose $\mathscr{F}_0 \subset \mathscr{F}_1 \subseteq AGPL$, $(\mathscr{F}_1, \mathscr{F}_0)$ are adjacent differing by H, and $\Omega_*^U(G)(AGPL, \mathscr{F}_1) \to \Omega_*^U(G)(AG, \mathscr{F}_1)$ is zero on q-torsion. Consider the commutative diagram

$$\begin{array}{ccccccccc}
\dots \to & \Omega_*^U(G)(\mathscr{F}_1, \mathscr{F}_0) & \xrightarrow{j_*} & \Omega_*^U(G)(AGPL, \mathscr{F}_0) & \to & \Omega_*^U(G)(AGPL, \mathscr{F}_1) & \to \dots \\
& \downarrow id & & \downarrow k_* & & \downarrow & \\
\dots \to & \Omega_*^U(G)(\mathscr{F}_1, \mathscr{F}_0) & \to & \Omega_*^U(G)(AG, \mathscr{F}_0) & \to & \Omega_*^U(G)(AG, \mathscr{F}_1) & \to \dots
\end{array}$$

If $y \in \Omega_*^U(G)(AGPL, \mathscr{F}_0)$ is a q-torsion class, then $y = j_*(z) + w$ for certain z and w such that $k_*(w) = 0$. We may as well assume $y = j_*(z)$; by (4.3) we can assume z is a q-torsion class also.

Now $\Omega_*^U(G)(\mathscr{F}_1, \mathscr{F}_0)$ is without q-torsion unless q divides $[G:H]$, by (2.1) and (4.2). Suppose $[G:H] = q^t$; then either $H \le L$ or $L \le H$, according as $t \ge s$ or $t \le s$, respectively. Since $H \notin \mathscr{F}_0$ and $\mathscr{F}_0 \supseteq PL$, we must have $L \le H$. This is not possible, since $H \in \mathscr{F}_1$, and $\mathscr{F}_1 \subseteq AGPL$. Thus some other prime p divides $[G:H]$.

We wish to prove that $\Omega_*^U(G)(\mathscr{F}_1, \mathscr{F}_0) \to \Omega_*^U(G)(AG, \mathscr{F}_0)$ kills q-torsion. By (2.1) we may assume H is normal in G. Use (1.1b) to select a subgroup Q so that $H < Q < G$, $[Q:H] = q^t$, and $[G:Q]$ is prime to q. Then there is a commutative diagram

$$\begin{array}{ccccc}
 & & \Omega_*^U(Q)(AH, PH) & \xrightarrow{i'_*} & \Omega_*^U(Q)(AQ, PH) \\
 & & \downarrow e_G^Q & & \downarrow e_G^Q \\
\Omega_*^U(G)(\mathscr{F}_1, \mathscr{F}_0) & \cong & \Omega_*^U(G)(AH, PH) & \xrightarrow{i_*} & \Omega_*^U(G)(AG, PH)
\end{array}$$

By (4.4), e_G^Q maps onto the q-torsion. By induction on the order of G, i'_* is zero. Thus i_* is zero on q-torsion, which completes the proof.

We can now finish the proof of (3.2). Consider the commutative diagram:

$$\begin{array}{ccccc}
 & & \Omega_{-}^U(G)(AL, PL) & \xrightarrow{id} & \Omega_{-}^U(G)(AL, PL) \\
 & & \downarrow l_* & & \downarrow j_* \\
\Omega_{-}^U(G)(AKPL, PL) & \xrightarrow{i_*} & \Omega_{-}^U(G)(AG, PL) & \to & \Omega_{-}^U(G)(AG, AKPL) \\
\uparrow e_L^K & & \uparrow e_L^K & & \\
\Omega_{-}^U(K)(AKPL, PL) & \to & \Omega_{-}^U(K)(AK, PL) & &
\end{array}$$

If $y \in \Omega_{-}^U(G)(AL, PL)$ is a q-torsion class, then by (6.1) we know $j_*(y) = 0$. Let $l_*(y) = i_*(x)$ for suitable $x \in \Omega_{-}^U(G)(AKPL, PL)$. By (4.3) we may assume x is a q-torsion class. If $[G:L] = q^s$ then $K = G$ and we are done, by (6.2). Otherwise $K < G$ and we apply induction on the order of G and the knowledge that e_L^K maps onto the q-torsion.

This finishes the proof of (3.2), and thus the proof of the theorem.

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