A RESIDUE FORMULA FOR HOLOMORPHIC FOLIATIONS

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INTRODUCTION

In this note we give an explicit formula for the residues of certain of the secondary characteristic classes for holomorphic foliations with trivial normal bundle. These residues exist when the foliation is preserved by a transversely holomorphic vector field. We assume that the vector field has a particularly nice singular set, which allows us to compute some of the residues. The computation of these residues gives a geometric interpretation of the secondary classes, as the form the residues take depends strongly on the local geometry of the vector field and foliation in a neighborhood of the singular set.

Throughout the paper we assume that the reader is very familiar with the construction of characteristic classes using connections and invariant polynomials on Lie groups as given in [9]. In particular note that we observe the Chern convention that if an invariant polynomial does not have enough arguments the last one is repeated until it does; i.e., if ϕ is an invariant polynomial on $gl_q C$ of degree k then

$$\phi(A,B) = \phi(A, \underbrace{B, B, \ldots, B}_{k-1})$$

where $A,B \in gl_q C$.

1. THE RESIDUE THEOREM

Let M be a complex analytic manifold of complex dimension n and

$$T_C M = TM \oplus \bar{T}M$$

the standard splitting of the complexified tangent bundle of M. If ξ is a bundle over M, we denote the space of smooth sections of ξ by $C^\infty(\xi)$. The space of smooth complex valued forms on M is denoted $A(M)$ and the space of smooth complex functions is denoted $C^\infty(M)$.

Let τ be a complex analytic foliation of complex codimension q on M. At each point $z \in M$ there is a coordinate chart $(U, z_1, \ldots, z_n)$ so that $\tau|_U$ is spanned by $\partial/\partial z_{q+1}, \ldots, \partial/\partial z_n$. We call such a chart a flat chart for τ at z. The normal

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bundle of τ is the holomorphic quotient bundle $\nu = TM/\tau$ and we write $\rho: TM \to \nu$ for the natural projection. We assume that ν is a trivial bundle.

Definition 1.1. A basic connection θ on ν is a connection of type (1,0) such that its covariant derivative ∇ satisfies for all $Y \in C^\infty(\tau)$, $Z \in C^\infty(TM)$,

$$\nabla_Y \rho(Z) = \rho([Y,Z])$$

An easy partition of unity argument shows that basic connections always exist.

LEMMA 1.2. *If* (Ω^i_j) *is the local curvature form of a basic connection on* ν *then each* Ω^i_j *annihilates* $\tau \oplus \bar{T}M$, i.e., *if* $Y,Z \in \tau \oplus \bar{T}M$ *then* $\Omega^i_j(Y,Z) = 0$.

COROLLARY 1.3. (Bott Vanishing Theorem [3]). *If* Ω *is the curvature of a basic connection on* ν *and* ϕ *is an invariant polynomial homogeneous of degree* $k > q$ *on* $gl_q C$, *then* $\phi(\Omega) \equiv 0$ *as a form on* M.

For the proofs of these see [3]. Let $C_q[c_1, \ldots, c_q]$ be a polynomial ring on the c_i where degree $c_i = 2i$ and the subscript on C indicates that the ring is truncated above degree $2q$. Let $\Lambda(\hat{c}_1, \ldots, \hat{c}_q)$ be an exterior algebra on the $\hat{c}_i$ where degree $\hat{c}_i = 2i - 1$ and let $W_q = \Lambda(\hat{c}_1, \ldots, \hat{c}_q) \otimes C_q[c_1, \ldots, c_q]$ be the differential graded algebra where $d(\hat{c}_i \otimes 1) = 1 \otimes c_i$, $d(1 \otimes c_i) = 0$. Choose a basic connection θ on ν and a flat connection θ^f on ν, corresponding to a framing σ of ν. These two connections determine a degree preserving map of differential graded algebras $\alpha_{\tau,\sigma}: W_q \to A(M)$ where

$$(1.4)\qquad \begin{aligned} \alpha_{\tau,\sigma}(1 \otimes c_i) &= c_i(\Omega) \\ \alpha_{\tau,\sigma}(\hat{c}_i \otimes 1) &= i \int_0^1 c_i(\theta - \theta^f, \Omega_t)\, dt \end{aligned}$$

Here Ω is the curvature of θ and Ω_t is the curvature of the connection $\theta_t = t\,\theta + (1-t)\theta^f$, $t \in R$. The c_i on the right in equations (1.4) are understood to be the Chern polynomials on $gl_q C$. The map $\alpha_{\tau,\sigma}$ induces the map

$$\alpha^*_{\tau,\sigma} : H^*(W_q) \to H^*(M;R)$$

and this map is independent of the choice of basic connection θ, but depends on the homotopy class of the framing σ. The ambiguity is up to translation by a $\mathbf{Z}$ lattice in $H^*(M;\mathbf{R})$. This $\mathbf{Z}$ lattice may have non-integral periods. This construction, in various forms, is due to Bernstein-Rozenfel'd [2], Bott-Haefliger [4], Kamber-Tondeur [8], and Malgrange (unpublished).

The elements of $H^*(W_q)$ are secondary characteristic classes for complex foliations with trivial normal bundle. A basis for $H^*(W_q)$, due to J. Vey, [cf. 5], is given by elements of the form

$$\hat{c}_{i_1} \cdots \hat{c}_{i_k} c_{j_1} \cdots c_{j_l} \qquad i_1 < \cdots < i_k, j_1 \le \cdots \le j_l,\ i_1 + j_1 + \cdots + j_l > q,\ i_1 \le j_1.$$

Definition 1.5. A Γ vector field for τ is a vector field X on M of type (1,0) such that if $(U, z_1, \ldots, z_n)$ is a flat chart for τ and $X|_U = \sum_{i=1}^{n} f_i \, \partial/\partial z_i$ then $f_1, \ldots, f_q$ are holomorphic functions and

$$\frac{\partial f_i}{\partial z_j} = 0 \qquad 1 \leq i \leq q < j \leq n.$$

We call the set of points $\mathscr{S}$ where X is tangent to τ the singular set of X. $\mathscr{S}$ is a union of leaves of τ and on $M - \mathscr{S}, \tau$ and X span a holomorphic foliation $\hat{\tau}$ of codimension $q - 1$. We assume that $\mathscr{S} = \bigcup_{i=1}^{r} N_i$ is a finite union of closed and separated leaves of τ. For each i we choose an embedded open normal disc bundle D_i of N_i so that its closure $\bar{D}_i$ is an embedded normal disc bundle and so that $\bar{D}_i \cap \bar{D}_j = \emptyset$, $i \neq j$.

Definition 1.6. *A basic X connection* on ν is a basic connection such that on a neighborhood $\mathscr{U}$ of $M - \bigcup_{i=1}^{r} D_i$ its covariant derivative satisfies

$$\nabla_X \rho(Y) = \rho([X, Y])$$

for all $Y \in C^\infty(TM)$. We say the connection has support off $\mathscr{U}$.

A partition of unity argument shows that such connections always exist.

LEMMA 1.7. *Let θ be a basic X connection on ν supported off $\mathscr{U}$. If (Ω^i_j) is any local curvature form of θ defined on an open subset of $\mathscr{U}$ then each Ω^i_j annihilates $\hat{\tau} \oplus \bar{T}M$.*

Definition 1.8. Let $I_q(W_q)$ be the ideal in W_q generated by the elements of the form $c_{j_1} \ldots c_{j_l}$ with $j_1 + \ldots + j_l = q$.

Lemma 1.7 has the following corollary.

COROLLARY 1.9. *If $\phi \in I_q(W_q)$ then $d\phi = 0$ and if a basic X connection with support off $\mathscr{U}$ is used in the construction of the map $\alpha_{\tau,\sigma}: W_q \to A(M)$, then $\alpha_{\tau,\sigma}(\phi)|_{\mathscr{U}} \equiv 0$.*

The proofs of Lemma 1.7 and Corollary 1.9 are the same as the proofs of Lemma 3.8 and Corollary 3.10 of [7]. In analogy with Theorem 3.11 of [7] we have,

THEOREM 1.10. *Let τ, M, X and $\mathscr{S} = \bigcup_{i=1}^{r} N_i$ be as above. Let σ be a framing of the normal bundle ν of τ. Let $\phi \in I_q(W_q)$ be an element homogeneous of degree k. Then ϕ, τ, X and σ determine the cohomology class* $\mathrm{Res}_\phi(\tau, X, N_\iota, \sigma) \in H^{k-2q}(N_i; \mathrm{C})$ *such that*

i) $\mathrm{Res}_\phi(\tau, X, N_i, \phi)$ *depends only on the homotopy class of σ and the behavior of τ and X in a neighborhood of N_i.*

ii) $\sum_i t \, \mathrm{Res}_\phi(\tau, X, N_i, \sigma) = \alpha^*_{\tau,\sigma}([\phi])$.

The map

$$t: H^{k-2q}(N_i;C) \to H_c^k(D_i;C) \to H^k(M;C)$$

is the composition of the Thom isomorphism followed by the natural inclusion. Here we have written $H_c^k(D_i;C)$ for the cohomology of forms with fiber compact support. The symbol $[\phi]$ stands for the cohomology class determined by ϕ.

Compare Theorem 2 of [1]. If $\phi = c_{j_1} \dots c_{j_l}$, $j_1 + \dots + j_l = q$, then the residues we construct are the same as those constructed in [1].

As the proof of Theorem 1.10 is analogous to the proof of Theorem 3.11 of [7] we shall only give the recipe for constructing the residues.

Let θ be a basic X connection on ν supported off $\mathscr{U}$. Let θ^f be the flat connection on ν corresponding to the framing σ. Use these two connections to construct the map $\alpha_{\tau,\sigma}: W_q \to A(M)$. By the Corollary 1.9 we have $\alpha_{\tau,\sigma}(\phi)|_{\mathscr{U}} = 0$. Thus $\alpha_{\tau,\sigma}(\phi)|_{D_i}$ is a form with fiber compact support and it determines the class

$$[\alpha_{\tau,\sigma}(\phi)|_{D_i}] \in H_c^k(D_i;C).$$

Let $\gamma: H_c^k(D_i;C) \to H^{k-2q}(N_i;C)$ be integration over the fiber of the disc bundle D_i. Then

$$\mathrm{Res}_\phi(\tau, X, N_i, \sigma) = \gamma([\alpha_{\tau,\sigma}(\phi)|_{D_i}]).$$

Note that if θ is a basic X connection on ν supported off $\mathscr{U}$ and $Y \in C^\infty(\tau)$, then θ is also a basic $X + Y$ connection supported off $\mathscr{U}$. Thus $\mathrm{Res}_\phi(\tau,X,N_i,\sigma)$ actually depends only on the equivalence class $\rho(X)$ of X in $C^\infty(\nu)$.

2. THE STRUCTURE LEMMA

We now restrict our attention to a single leaf N in the singular set of X. We wish to give a formula for the residue of certain elements of $I_q(W_q)$. To do so we need to know what the structures of τ and X are near N.

PROPOSITION 2.1. (The Structure Lemma). *Let $\alpha \in H_1(N;Z)$. Then there is an open neighborhood B* of $0 \in C^q$ *and a smooth map Φ: $S^1 \times B \to M$ such that*

i) $\Phi(S^1 \times \{0\})$ *represents* α

ii) *for all $t \in S^1$, $\Phi(t \times B)$ is transverse to τ and $\Phi|_{t\times B}$ is a homomorphic map.*

iii) *the induced foliation $\Phi^*\tau$ on $S^1 \times B$ is spanned by a vector field $\partial/\partial t + Y(t,w)$ where $Y(t,w) = \sum_{i=1}^{q} b_i(t,w)\,\partial/\partial w_i$ and each of the b_i's is holomorphic in w. (We write $w = (w_1, \dots, w_q)$ for the usual coordinates on C^q.)*

iv) *there is a vector field* $\hat{X}$ *on* $S^1 \times B$, $\hat{X} = \sum_{i=1}^{q} a_i(w)\, \partial/\partial w_i$ *where the* a_i*'s are holomorphic functions of* w *only, and* $\rho(\Phi_*(\hat{X})) = \rho(X)|_{\Phi(S^1\times B)}$.

v) *there is an is an integer* k *such that the framing of* $\nu|_{\Phi(S^1\times B)}$ *given by* $\Phi_*(t^k\partial/\partial w_1), \Phi_*(\partial/\partial w_2), \ldots, \Phi_*(\partial/\partial w_q)$ *extends to the given framing* σ *of* ν *on* M *(at least up to homotopy). Here* $t^k\partial/\partial w_1$ *is the natural action of* $t \in S^1 \subset C$ *on* $\partial/\partial w_1$.

Proof. Let $\phi: [0,1] \to N$, $\phi(0) = \phi(1)$ be a smooth map representing α. We assume that ϕ is an immersion. Let $(U_j, z_1^j, \ldots, z_n^j)$, $j = 0, \ldots, r$ be open flat coordinate charts on M which cover $\phi([0,1])$, and choose them so that

$$N \cap U_j = \{z \in U_j\colon\ z_1^j = \ldots = z_q^j = 0\}.$$

We further assume that real numbers $0 = \varepsilon_0 < \varepsilon_1 < \ldots < \varepsilon_{r+1} = 1$ are given so that $\phi([\varepsilon_j, \varepsilon_{j+1}]) \subset U_j$. We first construct a map $\psi_r: [0,1] \times B \to M$. To be precise in what follows we might have to cut down our neighborhood B at each step. As there are only a finite number of steps this presents no problem and for convenience sake we ignore this technical detail.

For each $j = 0, \ldots, r$ choose $\gamma_j \in (\varepsilon_j, \varepsilon_{j+1})$. Let $\varepsilon > 0$ be so small that $\varepsilon_j + \varepsilon < \gamma_j$, $\gamma_j + \varepsilon < \varepsilon_{j+1}$ and $\phi(\varepsilon_j, \varepsilon_{j+1} + \varepsilon) \subset U_j$. For $t \in [0, \varepsilon_1 + \varepsilon]$ set

$$\psi_0(t, w_1, \ldots, w_q) = \phi(t) + (w_1, \ldots, w_q, 0, \ldots, 0)$$

$$\text{i.e.}\quad z_i^0(\psi_0(t, w_1, \ldots, w_q)) = \begin{cases} w_i & i = 1, \ldots, q \\ z_i^0(\phi(t)) & i = q+1, \ldots, n. \end{cases}$$

Let $\beta_0, \beta_1, \beta_2$ be smooth functions from $[0,1]$ to R so that

i) $\beta_0(t) = 0$ for $t \le \varepsilon_1$, $\beta_0(t) \equiv 1$ for $t \ge \gamma_1$ and β_0 is strictly increasing on $(\varepsilon_1, \gamma_1)$.

ii) $\beta_1(t) = t$ for $t \le \varepsilon_1 + \varepsilon/2$, $\beta_1(t) = \varepsilon_1 + \varepsilon$ for $t \ge \varepsilon_1 + \varepsilon$ and β_1 is strictly increasing on $[0, \varepsilon_1 + \varepsilon]$.

iii) $\beta_2(t) = \gamma_1$ for $t \le \gamma_1$, $\beta_2(t) = t$ for $t \ge \gamma_1 + \varepsilon$ and β_2 is strictly increasing on $[\gamma_1, 1]$.

For $t \in [0, \varepsilon_2 + \varepsilon]$ set $\psi_1(t, w) = \psi_0(t, w) \qquad 0 \le t \le \varepsilon_1$, and

$$z_i^1(\psi_1(t, w)) = \begin{cases} z_i^1(\psi_0(\varepsilon_1, w)) & i = 1, \ldots, q \\ (1 - \beta_0(t))\, z_i^1\, \psi_0(\beta_1(t), w) & \\ +\, \beta_0(t)\, z_i^1\, \phi(\beta_2(t)) & i = q+1, \ldots, n \end{cases}$$

for $\varepsilon_1 \le t \le \varepsilon_2 + \varepsilon$. See Figure 1. Essentially we are translating a transverse disc along $\phi([0,1])$ using the foliation τ. We are doing this in such a way that each $t \times B$ is mapped to M holomorphically and so that for each $w \in B$,

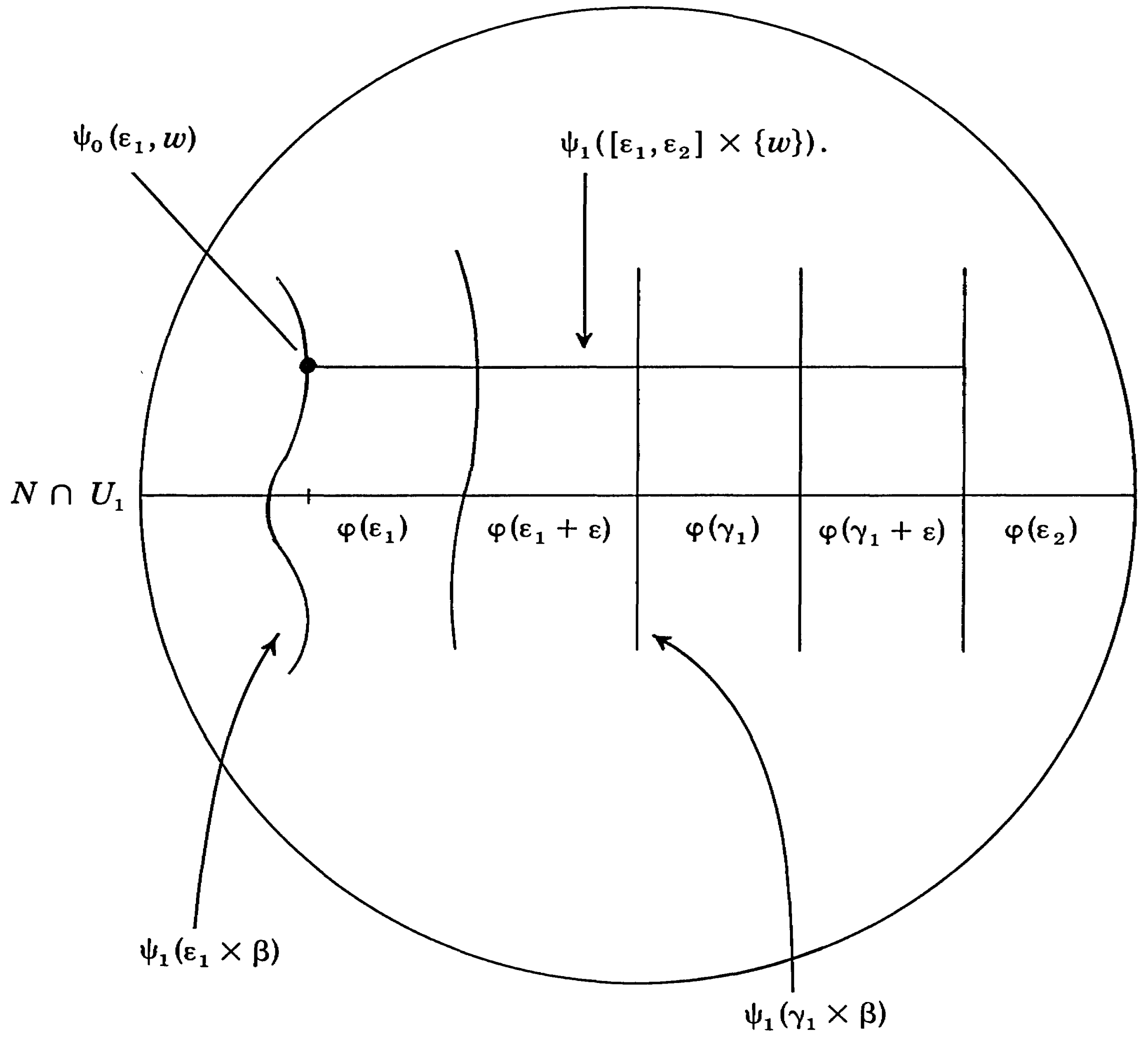

Figure 1.

$$\psi_1([0, \varepsilon_2] \times \{w\})$$

is contained in a leaf of τ.

Continuing in this manner we obtain a smooth map ψ_r: $[0,1] \times B \to M$ satisfying

i) $\psi_r|_{[0,1] \times \{0\}}$ represents α

ii) for all $t \in [0,1]$, $\psi_r|_{t \times B}$ is a holomorphic map

iii) the foliation induced on $[0,1] \times B$ by τ is spanned by $\partial/\partial t$.

iv) $\psi_r(0, w) = \psi_r(1, H(w))$ where $H: B \to B$ is the holonomy around α.

Choose a smooth map $\mu: [0,1] \to C$ which satisfies $\mu(t) = 0$ for t near 0 and $\mu(t) = 1$ for t near 1. Set $H_t = \mu(t)H + (1 - \mu(t))I$, where I is the identity map. Choose μ so that on some neighborhood B of 0, each H_t is a holomorphic diffeomorphism. Define γ: $[0,1] \times B \to [0,1] \times B$ by $\gamma(t, w) = (t, H_t(w))$. Then

$$\psi_r \circ \gamma: [0,1] \times B \to M$$

satisfies $\psi_r \circ \gamma(1, w) = \psi_r \circ \gamma(0, w)$ and so $\psi_r \circ \gamma$ induces a map $\Phi: S^1 \times B \to M$.

It is immediate that Φ satisfies i) and ii) of Proposition 2.1. The induced foliation $\Phi^* \tau$ on $S^1 \times B$ is spanned by the vector field $\gamma_*^{-1}(\partial/\partial t)$ which is equal to

$$\partial/\partial t + \sum_{i=1}^{q} \partial/\partial t(w_i \circ H_t^{-1}(w)) \cdot \partial/\partial w_i.$$

If we set

$$Y(t, w) = \sum_{i=1}^{q} \partial/\partial t(w_i \circ H_t^{-1}(w))\, \partial/\partial w_i$$

then this satisfies iii) of the Proposition since $w_i \circ H_t^{-1}$ is smooth in t and holomorphic in w.

To prove part iv) consider $\psi_r|_{t \times B}$. The action of X on the foliation τ is holomorphic and since $\psi_r(t \times B)$ is transverse to τ and $\psi_r|_{t\times B}$ is holomorphic, X induces a holomorphic vector field X_t on $t \times B$. The union of all these vector fields gives a vector field $\hat{X} = \sum_{i=1}^{q} a_i(t, w)\, \partial/\partial w_i$ on $[0, 1] \times B$ which is smooth in t since ψ_r is smooth in t. Each of the a_i is holomorphic in w as $\psi_r|_{t\times B}$ is holomorphic. Since X preserves τ, $\hat{X}$ preserves the induced foliation on $[0,1] \times B$. But this foliation is apanned by $\partial/\partial t$. Thus $[\hat{X}, \partial/\partial t]$ must be a functional multiple of $\partial/\partial t$. But this is only possible if $[\hat{X}, \partial/\partial t] = 0$, i.e., $\partial/\partial t\ a_i(t, w) = 0$ and each a_i is a function of w only.

Note also that since X preserves τ, the holonomy map H must preserve $\hat{X}$. Thus each H_t preserves $\hat{X}$ and so the vector field $\hat{X}$ on $S^1 \times B$ induced by $\hat{X}$ under γ also has the form $\hat{X} = \sum_{i=1}^{q} a_i(w)\, \partial/\partial w_i$.

Now consider the framing of $\nu|_{\Phi(S^1\times B)}$ given by $\Phi_*(\partial/\partial w_1), \ldots, \Phi_*(\partial/\partial w_q)$. If σ is any other framing of $\nu|_{\Phi(S^1\times B)}$ then by comparing σ with $\Phi_*(\partial/\partial w_i)$ we obtain a map $\sigma^{\#}: S^1 \times B \to Gl_q C$. Two such maps are homotopic if and only if the corresponding framings are homotopic. If $k \in Z$ then the framing of $\nu|_{\Phi(S^1\times B)}$ given by $\Phi_*(t^k \partial/\partial w_1), \Phi_*(\partial/\partial w_2), \ldots, \Phi_*(\partial/\partial w_q)$ represents the element in $\Pi_1(GL_q C) \simeq Z$ corresponding to k.

3. THE RESIDUE FORMULA

In this section we give an explicit formula for the residues of certain elements in $I_q(W_q)$. To wit

THEOREM 3.1. *Let τ, X, N, M and σ be as above and let $\hat{c}_1 c_{j_1} \ldots c_{j_\ell} = \hat{c}_1 c_J$ be in $I_q(W_q)$. Let $\alpha \in H_1(N; Z)$ and choose Φ, Y, $\hat{X}$ and k as in the Structure lemma. Form the matrices of partial derivatives $L = \|\partial a_j/\partial w_i\|$, $L_Y = \|\partial b_j/\partial w_i\|$. Then*

$$\mathrm{Res}_{\hat{c}_1 c_J}(\tau, X, N, \sigma)(\alpha) = (-1)^{q+1}(2\pi i)^q \int_{S^1} \mathrm{Res}(t)dt.$$

Res(t) is defined to be the value at 0 of the Grothendieck residue symbol

$$\mathrm{Res}_0 \begin{bmatrix} (c_1(L_Y) + k(2\pi i t)^{-1} c_J(L)\, dw_1 \dots dw_q \\ a_1 \dots a_q \end{bmatrix}$$

for fixed t.

In [1] an algorithm for computing this residue symbol, due to R. Hartshorne, is given. We quote with appropriate changes.

Since the origin is an isolated zero of the a_i there exist positive integers $\alpha_1, \dots, \alpha_q$ with $w_i^{\alpha_i}$ in the ideal generated by $a_1, \dots, a_q$. Hence there exist holomorphic functions b_{ij} defined near 0 with $w_i^{\alpha_i} = \sum_{j=1}^{q} b_{ij} a_j$. The Grothendieck residue is evaluated by expanding $(c_1(L_Y) + k(2\pi i t)^{-1})c_J(L) \det \|b_{ij}\|$ in a power series in the w_i. The coefficient of $dw_1 \dots dw_q/(w_1 \dots w_q)$ is the resulting Laurent series for

$$(c_1(L_Y) + k(2\pi i t)^{-1})c_J(L) \det \|b_{ij}\|\, dw_1 \dots dw_q/(w_1^{\alpha_1} \dots w_q^{\alpha_q})$$

is the desired answer.

Proof. Let $\rho(X_1), \dots, \rho(X_q)$ be a framing of ν on a neighborhood U_0 of $\Phi(S^1 \times B)$ in M which extends the framing given by

$$\rho(\Phi_*(\partial/\partial w_1)), \dots, \rho(\Phi_*(\partial/\partial w_q)).$$

Let $\bar{\theta}$ be a basic connection on ν whose local form $\bar{\theta}_{U_0}$ on U_0 with respect to the framing $\rho(X_1), \dots, \rho(X_q)$ satisfies $\bar{\theta}_{U_0}(Z) = 0$ for all $Z \in \Phi_*(TB) \subset TM$. This uniquely determines $\bar{\theta}$ on $\Phi(S^1 \times B)$ as at each point $z \in \Phi(S^1 \times B)$, the image of the holomorphic tangent bundle of B, $\Phi_*(TB)$ is complementary to τ in TM_z.

Let D be an embedded open normal disc bundle of N in M so that

$$D|_{\Phi(S^1 \times \{0\})} = \Phi(S^1 \times B)$$

as bundles over $\Phi(S^1 \times \{0\})$. Let ω be a (1,0) form on M so that ω annihilates τ and on a neighborhood $\mathscr{U}$ of $M - D$ in M, $\omega(X) = 1$. (We have assumed for simplicity that the singular set of X is precisely N). On the neighborhood U_0 of $\Phi_*(S^1 \times B)$ define the matrix $A = \|A_j^i\|$ by the equation $\rho([X, X_i]) = \sum_{i=1}^{q} A_j^i \rho(X_j)$. Then let θ be a basic X connection on ν supported off $\mathscr{U}$ so that on U_0 θ has the local form $\theta_{U_0} = \bar{\theta}_{U_0} + \omega \cdot A$ with respect to the framing $\rho(X_1), \dots, \rho(X_q)$.

The normal bundle of $\Phi^*(\tau)$ on $S^1 \times B$ is naturally isomorphic to the holomorphic tangent bundle TB of B in $T(S^1 \times B)$, so we will identify them. We now compute the local form of the connection $\Phi^*(\theta)$ on TB with respect to the framing

$$\partial/\partial w_1, \ldots, \partial/\partial w_q.$$

We denote by ∇ the covariant derivative of θ and by $\hat{\nabla}$ the convariant derivative of $\Phi^*\theta$.

For each $t \in S^1$, $\Phi|_{t\times B}$ is holomorphic. Since θ is a connection of type $(1,0)$ and $\Phi_*(\partial/\partial w_i)$ are holomorphic for fixed t, we have $\nabla_{\Phi_*(\partial/\partial \bar{w}_l)}\rho(\Phi_*(\partial/\partial w_i)) = 0$, and so $\hat{\nabla}_{\partial/\partial\bar{w}_l}\partial/\partial w_i = 0$. Since $\rho(\Phi_*(\hat{X})) = \rho(X)$, θ is a basic $\Phi_*(\hat{X})$ connection and so we may assume that in fact $X = \Phi_*(\hat{X})$ on $\Phi(S^1 \times B)$. Then

$$\begin{aligned}
\nabla_{\Phi_*(\partial/\partial w_l)}\rho(\Phi_*(\partial/\partial w_i) &= \nabla_{\Phi_*(\partial/\partial w_l)}\rho(X_i)\\
&= \omega(\Phi_*(\partial/\partial w_l))\sum_j A^i_j\rho(X_j)\\
&= \omega(\Phi_*(\partial/\partial w_l))\,\rho([X, X_i])\\
&= \omega(\Phi_*(\partial/\partial w_l))\,\rho([\Phi_*(\hat{X}), \Phi_*(\partial/\partial w_i)])\\
&= \omega(\Phi_*(\partial/\partial w_l))\,\rho\left(\Phi_*\left(\sum_j dw_j([\hat{X}, \partial/\partial w_i])\,\partial/\partial w_j\right)\right)\\
&= \Phi^*\omega(\partial/\partial w_l)\sum_j(-L^i_j)\,\rho\Phi_*(\partial/\partial w_j).
\end{aligned}$$

Thus

$$\hat{\nabla}_{\partial/\partial w_l}\partial/\partial w_i = -\sum_j \Phi^*\omega(\partial/\partial w_l)\,L^i_j\cdot\partial/\partial w_j.$$

Recall that $\partial/\partial t + Y(t,w)$ spans $\Phi^*(\tau)$ where $Y(t,w) \in C^\infty(TB)$ and is holomorphic in w. Now

$$\begin{aligned}
\nabla_{\Phi_*(\partial/\partial t+Y(t,w))}\rho\Phi_*(\partial/\partial w_i) &= \rho([\Phi_*(\partial/\partial t + Y(t,w)), \Phi_*(\partial/\partial w_i)])\\
&= \rho\left(\Phi_*\left(\sum_j dw_j([Y(t,w), \partial/\partial w_i])\,\partial/\partial w_j)\right)\right)\\
&= \sum(-L^i_{Yj})\rho\Phi_*(\partial/\partial w_j).
\end{aligned}$$

Thus

$$\hat{\nabla}_{\partial/\partial t}\partial/\partial w_i = (\hat{\nabla}_{\partial/\partial t+Y} - \hat{\nabla}_Y)(\partial/\partial w_i) = \sum(-L^i_{Yj} + \Phi^*\omega(Y)L^i_j)\partial/\partial w_j.$$

By the above we see that the local form of $\Phi^*\theta$ with respect to $\partial/\partial w_1, \ldots, \partial/\partial w_q$ is $(-\Phi^*\omega|_B)\cdot L + (-L_Y + (\Phi^*\omega)(Y)L)dt$. The form $\Phi^*\omega|_B$ is a one form on $S^1 \times B$ whose restriction to $t \times B$ is of type $(1,0)$ and which is a projector for X off some neighborhood of 0. In fact any one form $\hat{\omega}$ of type $(1,0)$ on B which has $\hat{\omega}(X) = 1$ off a neighborhood of 0 may be obtained in this way. We need merely require that the one form ω on M satisfy $\omega(\Phi_*(\partial/\partial w_i)) = \hat{\omega}(\partial/\partial w_i)\cdot\Phi^{-1}$

on $\Phi(S^1 \times B)$. Thus we may assume that the local from $\hat{\theta}$ of $\Phi^*(\theta)$ is given by $-\hat{\omega}L + (-L_Y + \hat{\omega}(Y)L)dt$ where $\hat{\omega} = \sum_j c_j(w, \bar{w})dw_j$ is any one form on B which satisfies $\hat{\omega}(X) = 1$ off some neighborhood of $0 \in B$.

Let θ^f be the flat connection on ν corresponding to the framing σ. Since

$$\mathrm{Res}_{\hat{c}_1 c_J}(\tau, X, N, \sigma)$$

depends only on the homotopy class of σ, we may assume that the induced connection $\hat{\theta}^f = \Phi^*(\theta^f)$ on TB is the flat connection corresponding to the framing $t^k \partial/\partial w_1, \partial/\partial w_2, \ldots, \partial/\partial w_q$. Then with respect to the framing $\partial/\partial w_1, \partial/\partial w_2, \ldots, \partial/\partial w_q, \hat{\theta}^f$ has the local form $\|\hat{\theta}_j^{fi}\|$, where $\hat{\theta}_1^{fi} = -kt^{-1}dt$ and all other entries are zero.

In order to compute $\mathrm{Res}_{\hat{c}_1 c_J}(\tau, X, N, \sigma)(\alpha)$ we must compute $c_1(\hat{\theta} - \hat{\theta}^f)c_J(\hat{\Omega})$ where $\hat{\Omega}$ is the curvature of $\hat{\theta}$. Now $\hat{\Omega} = d\hat{\theta} - \hat{\theta} \wedge \hat{\theta}$ and

$$d\hat{\theta} = -d\hat{\omega} \cdot L - \hat{\omega} \wedge dL - dL_Y \wedge dt + d(\hat{\omega}(Y)) \cdot L \wedge dt + \hat{\omega}(Y)dL \wedge dt$$

while $\hat{\theta} \wedge \hat{\theta} = f\,\hat{\omega} \wedge dt$ for some function f. Since each entry of $\hat{\theta}$ and θ^f involves either an $\hat{\omega}$ or a dt we have $c_1(\hat{\theta} - \hat{\theta}^f)c_J(\hat{\Omega}) = c_1(\hat{\theta} - \hat{\theta}^f)c_J(d\,\hat{\theta})$. Now observe that $d\hat{\omega}L$ is a $(2,0) + (1,1) + (1,0) \wedge dt$ form $\hat{\omega} \wedge dL$ is a $(2,0) + \hat{\omega} \wedge dt$ form $dL_Y \wedge dt$ is a $(1,0) \wedge dt$ form, and $\hat{\omega}(Y)dL \wedge dt$ is a $(1,0) \wedge dt$ form. The degree of c_J is q, so for dimensional and type reasons we have

$$\begin{aligned} c_1(\hat{\theta} - \hat{\theta}^f)c_J(d\hat{\theta}) &= c_1(-L_Y)c_J(L)(-d\hat{\omega})^q \wedge dt \\ &+ c_1(L)c_J(L)(\hat{\omega}(Y))(-d\hat{\omega})^q \wedge dt \\ &+ c_1(L)c_J(L)q\,(-\hat{\omega}) \wedge (-d\hat{\omega})^{q-1} \wedge (-d(\hat{\omega}(Y)) \wedge dt) \\ &- k(2\pi i t)^{-1}c_J(L)(-d\hat{\omega})^q \wedge dt \\ &= (c_1(-L_Y) - k(2\pi i t)^{-1})c_J(L)(-d\hat{\omega})^q \wedge dt \\ &+ c_1(L)c_J(L)\{\hat{\omega}(Y)(-d\hat{\omega})^q \\ &+ q(-\hat{\omega}) \wedge i(Y)(-d\hat{\omega}) \wedge (-d\hat{\omega})^{q-1} \\ &+ q\hat{\omega} \wedge \mathscr{L}_Y(-\hat{\omega}) \wedge (-d\hat{\omega})^{q-1}\} \wedge dt. \end{aligned}$$

The form $\hat{\omega} \wedge d\hat{\omega}^q \wedge dt = 0$ for dimensional reasons so

$$-i(Y)(\hat{\omega} \wedge d\hat{\omega}^q \wedge dt) = -\omega(Y)d\hat{\omega}^q \wedge dt + q\hat{\omega} \wedge i(Y)\,d\hat{\omega} \wedge d\hat{\omega}^{q-1} \wedge dt = 0.$$

For fixed t, Y is holomorphic and $\hat{\omega}$ is of type $(1,0)$. Thus the Lie derivative $\mathscr{L}_Y\hat{\omega}$ is of type $(1,0)$ and the form $\hat{\omega} \wedge \mathscr{L}_Y\hat{\omega} \wedge d\hat{\omega}^{q-1} \wedge dt$ is at least a $(q+1, q-1)$ form $\wedge\, dt$ and so is zero. Thus

$$\hat{c}_1(\hat{\theta} - \hat{\theta}^f)c_J(\hat{\Omega}) = (c_1(-L_Y) - k(2\pi i t)^{-1})c_J(L)(-d\hat{\omega})^q \wedge dt,$$

and

$$\begin{aligned}\operatorname{Res}_{\hat{c}_1 c_J}(\tau, X, N, \sigma)(\alpha) &= \int_{S^1}\left(\int_{B} c_1(\hat{\theta} - \hat{\theta}^f)c_J(\hat{\Omega})\right)\\ &= \int_{S^1}\left(\int_{B} (-1)^{q+1}(c_1(L_Y) + k(2\pi i t)^{-1})c_J(L)d\hat{\omega}^q\right) dt.\end{aligned}$$

We finish the proof by showing that the integral over B is the Grothendieck residue. We follow [1, pp. 321-324].

Recall that $\hat{X} = \sum a_i(w)\partial/\partial w_i$ where the a_i have an isolated zero at 0, as N is the singular set of X. Thus there are positive integers $\alpha_1, \ldots, \alpha_q$ such that $w_i^{\alpha_i}$ is in the ideal generated by the $a_1, \ldots, a_q$ and there exist holomorphic functions b_{ij} defined near 0 with $w_i^{\alpha_i} = \sum_{j=1}^{q} b_{ij}a_j$.

Let $B_\alpha \subset B$ be the set

$$B_\alpha = \left\{(w_1, \ldots, w_q) \in B : \sum_{i=1}^{q} (w_i\bar{w}_i)^{\alpha_i} < 1\right\}.$$

We may assume that the b_{ij} are defined on an open set $U \subset B$ such that $B_\alpha \subset U$. See [1]. On U define the one form ω by

$$\omega = \sum_{1 \le i,j \le q} (\bar{w}_i)^{\alpha_i} b_{ij}\, dw_j$$

and note that on ∂B_α, the boundary of B_α, $\omega(\hat{X}) = 1$. Let $\hat{\omega}$ be a form of type (1,0) on U so that on ∂B_α, $\hat{\omega} = \omega$ and $\hat{\omega}(\hat{X}) = 1$ off B_α. Let $\hat{\theta}$ be the basic X connection corresponding to $\hat{\omega}$, and write $\hat{\Omega}$ for the curvature of $\hat{\theta}$. Then for fixed $t \in S^1$

$$\begin{aligned}\int_B c_1(\hat{\theta} - \hat{\theta}^f)c_J(\hat{\Omega}) &= \int_{B_\alpha} c_1(\hat{\theta} - \hat{\theta}^f)c_J(\hat{\Omega})\\ &= \int_{B_\alpha} (-1)^{q+1}(c_1(L_Y) + k(2\pi i t)^{-1})c_J(L)d\hat{\omega}^q\\ &= \int_{B_\alpha} (-1)^{q+1} d((c_1(L_Y) + k(2\pi i t)^{-1})c_J(L)\,\hat{\omega} \wedge d\hat{\omega}^{q-1})\\ &= \int_{\partial B_\alpha} (-1)^{q+1}((c_1(L_Y) + k(2\pi i t)^{-1})c_J(L)\hat{\omega} \wedge d\hat{\omega}^{q-1})\\ &= \int_{\partial B_\alpha} (-1)^{q+1}((c_1(L_Y) + k(2\pi i t)^{-1})c_J(L)\,\omega \wedge d\omega^{q-1})\end{aligned}$$

$$= \int_{B_\alpha} (-1)^{q+1} d((c_1(L_Y) + k(2\pi it)^{-1})c_J(L)\omega \wedge d\omega^{q-1})$$

$$= \int_{B_\alpha} (-1)^{q+1} (c_1(L_Y) + k(2\pi it)^{-1})c_J(L) d\omega^q.$$

The first equality follows from the fact that $c_J(\hat{\Omega}) = 0$ on $B - B_\alpha$ as $\hat{\theta}$ is supported off $B - B_\alpha$. The third is because both $c_1(L_Y)$ and $c_J(L)$ are holomorphic for fixed t and $\hat{\omega} \wedge d\hat{\omega}^{q-1}$ is a $(q, q-1)$ form. The fourth and sixth are Stokes Theorem, and the last is the same as the third.

Now $d\omega = \partial\omega + \bar{\partial}\omega$ where $\partial\omega$ is a form of type $(2,0)$ and $\bar{\partial}\omega$ is a form of type $(1,1)$. It is easy to see that for type reasons $d\omega^q = \bar{\partial}\omega^q$. The form

$$\bar{\partial}\omega = \sum_{1 \le i,j \le q} \alpha_i (\bar{w}_i)^{\alpha_i - 1} b_{ij} d\bar{w}_i \wedge dw_j$$

and an easy computation shows

$$(\bar{\partial}\omega)^q = q!\, \alpha_1 \ldots \alpha_q (\bar{w}_q)^{\alpha_1 - 1} \ldots (\bar{w}_q)^{\alpha_q - 1} \det \|b_{ij}\|\, W$$

where $W = d\bar{w}_1 \wedge dw_1 \wedge \ldots \wedge d\bar{w}_q \wedge dw_q$. The integral

$$\int_{B_\alpha} (w_1\bar{w}_1)^{\alpha_1 - 1} \ldots (w_q\bar{w}_q)^{\alpha_q - 1}\, W = (q!\,\alpha_1 \ldots \alpha_q)^{-1} (2\pi i)^q$$

while

$$\int_{B_\alpha} \bar{w}_1^{\alpha_1 - 1} w_1^{\beta_1} \ldots \bar{w}_q^{\alpha_q - 1} w^{\beta_q}\, W = 0$$

if $\beta_1, \ldots, \beta_q$ is a q tuple of nonnegative integers and

$$(\beta_1, \ldots, \beta_q) \neq (\alpha_1 - 1, \ldots, \alpha_q - 1).$$

Expand $(c_1(L_Y) + k(2\pi it)^{-1})c_J(L) \det \|b_{ij}\|$ in a power series in $w_1, \ldots, w_q$. Denote by $\mathrm{Res}(t)$ the coefficient of $w_1^{\alpha_1 - 1} \ldots w_q^{\alpha_q - 1}$. Then by the above

$$(-1)^{q+1}(2\pi i)^q \,\mathrm{Res}(t) = \int_B c_1(\hat{\theta} - \hat{\theta}^f) c_J(\hat{\Omega}).$$

If we compare $\mathrm{Res}(t)$ with

$$\mathrm{Res}_0 \left[\frac{(c_1(L_Y) + k(2\pi it)^{-1})\, c_J(L)\, dw_1 \ldots dw_q}{a_1 \ldots a_q} \right]$$

we see that they are the same.

4. EXAMPLES

Example 1. Let $M = T^2 \times C^q$ where $T^2 = C/(Z \times Z)$ and has coordinate z. We write $w_1, \ldots, w_q$ for the usual coordinates on C^q. Let $\lambda_1, \ldots, \lambda_q, \delta_1, \ldots, \delta_q$ be nonzero complex numbers. Set $X = \sum_{i=1}^{q} \lambda_i w_i \, \partial/\partial w_i$ and let τ be the foliation on M spanned by the vector field

$$Y = \partial/\partial z + \sum_{i=1}^{q} \delta_i w_i \, \partial/\partial w_i.$$

The vector fields X and Y commute so X is a Γ vector field for τ and its singular set is $N = T^2 \times \{0\}$. The normal bundle ν of τ is isomorphic to the pullback of TC^q and is a trivial bundle. Let $\sigma = \partial/\partial w_1, \ldots, \partial/\partial w_q$ be the usual framing of ν and suppose $\hat{c}_1 c_J \in I_q(W_q)$. Then

$$\operatorname{Res}_{\hat{c}_1 c_J}(\tau, X, N, \sigma) = \frac{c_1(\delta_1, \ldots, \delta_q) c_J(\lambda_1, \ldots, \lambda_q)}{\lambda_1 \ldots \lambda_q} [dt_1 + dt_2]$$

where $dz = dt_1 + i dt_2$ and $[dt_1 + dt_2]$ is the class in $H^1(T^2; C)$ determined by $dt_1 + dt_2$. We have written $c_1(\delta_1, \ldots, \delta_q)$ for c_1 applied to the diagonal matrix $\operatorname{diag}(\delta_1, \ldots, \delta_q)$ and $c_J(\lambda_1, \ldots, \lambda_q)$ for c_J applied to the diagonal matrix $\operatorname{diag}(\lambda_1, \ldots \lambda_q)$. For more on this example in the special case that $Y = \partial/\partial z + X$, see [6].

Example 2. Consider the situation above but change the framing σ as follows. Think of T_2 as $S^1 \times S^1 \subset C \times C$ with coordinates t_1 and t_2. Let σ be the framing of ν given by $\sigma = t_1^{k_1} t_2^{k_2} \partial/\partial w_1, \partial/\partial w_2, \ldots, \partial/\partial w_q$. Then

$$\operatorname{Res}_{\hat{c}_1 c_J}(\tau, X, N, \sigma) = \frac{c_J(\lambda_1, \ldots, \lambda_q)}{\lambda_1 \ldots \lambda_q} \{(c_1(\delta_1, \ldots, \delta_q) - k_1) dt_1 + (c_1(\delta_1, \ldots, \delta_q) - k_2) dt_2\}.$$

REFERENCES

1. P. Baum and R. Bott, *Singularities of holomorphic foliations.* J. Differential Geometry 7 (1972), 279–342.
2. I. N. Bernstein and B. I. Rozenfel'd, *Characteristic classes of foliations.* Funkcional Anal. i Priložen. 6 (1972), no. 1, 68–69.
3. R. Bott, *On a topological obstruction to integrability.* Global Analysis (Proc. Sympos. Pure Math., Vol. XVI, Berkeley, Calif., 1968) pp. 127–131. Amer. Math. Soc., Providence, R.I., 1970.
4. R. Bott and A. Haefliger, *On characteristic classes of* Γ*-foliations.* Bull. Amer. Math. Soc. 78 (1972), 1039–1044.
5. C. Godbillon, *Cohomologie d'algebres de Lie de Champs de Vecteurs Formels.* Sem. Bourbaki 421.01–421.19.

6. J. Heitsch, *A remark on the residue theorem of Bott.* Indiana Univ. Math. J. 25 (1976), no. 12, 1139–1147.

7. ———, *Independent variation of secondary classes. preprint.* Ann. of Math. 108 (1978), 421–460.

8. F. Kamber and Ph. Tondeur, *Characteristic invariants of foliated bundles.* Manuscripta Math. 11 (1974), 51–89.

9. S. Kobayashi and K. Nomizu, *Foundations of differential geometry.* Vol. II. Interscience Tracts in Pure and Applied Mathematics, No. 15. Vol. II. Interscience Publishers (John Wiley and Sons Inc.) New York-London-Sydney, 1969.

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