APPLICATIONS OF CONVOLUTION OPERATORS TO PROBLEMS IN UNIVALENT FUNCTION THEORY

Roger W. Barnard and Charles Kellogg

In this paper we investigate a wide class of problems. We will exploit the strengths and properties of convolution operators. The strength of these methods lies in their ability to unify a number of diverse results. Of the previously known results obtained in this paper, most of the earlier proofs were tedious examinations of the specific properties of the classes of functions involved. In this paper we are able to obtain and generalize many of these results and obtain a number of new results including a verification of Robinson's 1/2 conjecture in the case of spirallike functions. In general, the proofs using convolution operators are clearer and more concise and point out how the unifying linear structure that is common to so many of the problems can be used to solve them via convolution operator techniques.

PRELIMINARY RESULTS

The unit disk in the complex plane will be denoted by U. Let A be the class of analytic functions on U. Let S denote those functions in A that are univalent and normalized by $f(0) = 0$ and $f'(0) = 1$. Let C, S^*, K and S_p be the standard subclasses of S consisting of the convex, starlike, close-to-convex, and spirallike functions, respectively. Let P be the class of functions p in A which have positive real part and are normalized by $p(0) = 1$. Let K_1 be the class of function f in S that have f' in P.

If f and g are in A with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, the convolution of f and g is defined by $(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n$. Given f in A, we define the convolution operator $\Gamma: A \to A$ by $\Gamma(g) = f * g$.

We will use the results and techniques of Ruscheweyh and Sheil-Small developed in [18] in connection with their proof of the Polya-Schöenberg conjecture. Specifically, the following theorem of theirs and the key lemma used in its proof will be used in our work.

THEOREM A. *Let h be in C. If f is in C, S^* or K then $h * f$ is in C, S^* or K respectively.*

In their proof of Theorem A, they proved a most interesting key lemma. We shall need the slightly more general version of their key lemma stated without proof in their paper. For completeness we include a proof of the more general

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version using their result as a basis. The proof given is one suggested by D. Styer and D. Wright since their proof is much more concise than the authors' original.

LEMMA A. *Let* φ *and* g *be analytic in* $|z|<1$ *with* $\varphi(0)=g(0)=0$ *and* $\varphi'(0)g'(0)\neq 0$. *Suppose that for each* $\alpha(|\alpha|=1)$ *and* $\sigma(|\sigma|=1)$ *we have*

$$\left[\varphi * \left(\frac{1+\alpha\sigma z}{1-\sigma z}\right) g\right](z) \neq 0 \qquad \text{on } 0<|z|<r\leq 1 \tag{1}$$

Then for each F *in* A *the image of* $|z|<r$ *under* $(\varphi * Fg)/(\varphi * g)$ *is a subset of the convex hull of* $F(U)$.

Remark. We note that in [18] they showed that if φ is in C and g is in S^* then (1) is satisfied for all z in U.

Proof. Since $\varphi(z) * [(1+\alpha\sigma z)/(1-\sigma z)]g(z)\neq 0$ for $0<|z|<r\leq 1$ is equivalent to $\varphi(rz) * [(1+\alpha\sigma z)/(1-\sigma z)]g(z)\neq 0$ for $0<|z|<1$ we can assume $r=1$. In [18] they proved that if F has positive real part and (1) is satisfied then $\operatorname{Re}\{[(\varphi * gF)/(\varphi * g)](z)\}>0$ for z in U. For arbitrary F in A the convex hull of $F(U)$ is defined to be the total intersection of all half planes containing $F(U)$. If we denote by $\overline{F(U)}$ the closure of $F(U)$ then a line of support ℓ of $F(U)$ is the boundary of a half plane containing $F(U)$ such that $B_\ell = \ell \cap \overline{F(U)} \neq \emptyset$. For a given support line ℓ let b be a point in B_ℓ. Then there exists an α such that the half plane defined by the set $\{e^{-i\alpha}[(1+z)/(1-z)]+b\colon z\in U\}$ contains $F(U)$. For this α and b, if F_1 is defined by $F_1(z)=e^{i\alpha}[F(z)-b]$ for z in U we have that $\operatorname{Re}F_1(z)>0$ for z in U. Thus we can apply the Ruscheweyh-Sheil-Small result to F_1 to obtain

$$\operatorname{Re}\left\{\frac{\varphi * gF_1}{\varphi * g}(z)\right\} = \operatorname{Re}\left\{e^{i\alpha}\frac{\varphi * gF}{\varphi * g}(z) - b\right\} > 0, \qquad z\in U.$$

Therefore, for each z in U we have that $[(\varphi * gF)/(\varphi * g)](z)$ lies in the appropriate half-plane for each support line ℓ of $F(U)$. Hence, it follows that

$$[(\varphi * gF)/(\varphi * g)](U)$$

lies in the convex hull of $F(U)$ as claimed.

In order to apply these results we shall need the following notation. Let Γ_i, $0\leq i\leq 4$ be the linear operators defined on A by the equations below.

$$\Gamma_0 f(z) = zf'(z) \qquad \Gamma_1 f(z) = [f(z)+zf'(z)]/2$$

$$\Gamma_2 f(z) = \int_0^z \frac{f(\zeta)-f(0)}{\zeta}\,d\zeta \qquad \Gamma_3 f(z) = \frac{2}{z}\int_0^z f(\zeta)\,d\zeta$$

$$\Gamma_4 f(z) = \int_0^z \frac{f(\zeta)-f(x\zeta)}{\zeta - x\zeta}\,d\zeta, \qquad |x|\leq 1, x\neq 1.$$

We note that Γ_4, generalizing Γ_2, was first used by Pommerenke in [14].

We observe that each of these operators can be written as a convolution operator given by $\Gamma_i f = h_i * f$, $0 \le i \le 4$ where

$$h_0(z) = \sum_{n=1}^{\infty} nz^n = \frac{z}{(1-z)^2} \qquad h_1(z) = \sum_{n=1}^{\infty} \frac{n+1}{2} z^n = \frac{z - z^2/2}{(1-z)^2}$$

$$h_2 = \sum_{n=1}^{\infty} \frac{1}{n} z^n = -\log(1-z) \qquad h_3(z) = \sum_{n=1}^{\infty} \frac{2}{n+1} z^n = \frac{-2\,[z + \log(1-z)]}{z}$$

$$h_4 = \sum_{n=1}^{\infty} \frac{1-x^n}{(1-x)n} z^n = \frac{1}{1-x} \log\left[\frac{1-xz}{1-z}\right], \qquad |x| \le 1, x \ne 1.$$

The initial motivation for this paper was the authors' investigation into the properties of the operator Γ_1. The original work and conjecture about this operator were given by R. Robinson in 1947 in [17]. For a given compact subclass X (possibly a singleton) of A let $r_S(X)$ denote the minimum radius of univalence over all functions f in X. We use corresponding notation for the other subclasses of S. For example $r_{S^*}(X)$ denotes the minimum radius of starlikeness over all functions f in X. Robinson observed that for any f in S, the derivative of $\Gamma_1 f(z)$ does not vanish for $|z| < 1/2$. He also noted that for the standard Köebe function $k(z) = z(1-z)^{-2}$, $r_S[\Gamma_1(k)] = 1/2$. He conjectured that $r_X[\Gamma_1(X)] = 1/2$ for $X = S$. Although $1/2$ has been verified to be the correct radius when X is replaced by many of the subclasses of S, Robinson's lower bound of .38 for $r_S[\Gamma(S)]$ has not been improved until just recently. A straightforward argument using convolution techniques and Krzyz's result in [9] determining $r_K(S)$ can be used to show that $r_S[\Gamma_1(S)] > .417$. (The first author has recently proved in [3] that $.49 < r_S[\Gamma_1(S)] \le .50$.)

Many authors have studied the appropriate minimum radii for the different operators Γ_i, $0 \le i \le 4$ on various subclasses of S. We list a few of these results. Each of these and many of their generalizations can be obtained in a fairly straightforward manner by the techniques presented in this paper.

The classical results of Alexander in [1] show that

$$r_{S^*}[\Gamma_0(S)] = r_{S^*}[\Gamma_0(S^*)] = 2 - \sqrt{3}.$$

In [13] A. Livingston proved that

$$r_C[\Gamma_1(C)] = r_{S^*}[\Gamma_1(S^*)] = r_K[\Gamma_1(K)] = 1/2$$

and that

$$r_{K_1}[\Gamma_1(K_1)] = (\sqrt{5} - 1)/2. \tag{2}$$

Generalizations of these results have been given by Libera and Livingston in [12] and by Bernardi in [5]. It has been shown by Causey in [8] and others that $r_K[\Gamma_2(K)] = 1$. The example of Krzyz and Lewandoski in [10] shows that $r_S[\Gamma_2(S)] < 1$. In [11] Libera showed that

$$r_C[\Gamma_3(C)] = r_{S^*}[\Gamma_3(S^*)] = r_K[\Gamma_3(K)] = 1$$

and these results have been generalized by Bernardi in [4]. Pommerenke in [14] has shown that $r_K[\Gamma_4(K)] = 1$.

It is not difficult to find the radius of convexity of each of the functions h_i, $0 \le i \le 4$, previously defined, that is, $r_C(h_0) = 2 - \sqrt{3}$, $r_C(h_1) = 1/2$ and

$$r_C(h_2) = r_C(h_3) = r_C(h_4) = 1.$$

These facts together with Theorem A yield the following theorem and its consequences.

THEOREM 1. *If f is in C, S^*, or K then $\Gamma_i f = h_i * f$ is convex, starlike or close to convex, respectively, up to $r_C(h_i)$ for each i, $0 \le i \le 4$.*

It is clear that Theorem 1 encompasses most of the major results previously mentioned concerning the various radii with the sharpness following by using the standard extremal functions for the specific subclasses. The generalizations of these results can be obtained by an appropriate modification of the function defining the operator. We shall later show how this can be done with Bernardi's work in [5] as an example.

APPLICATIONS

Livingston's results in [13] follow from Theorem 1. That is, Robinson's 1/2 conjecture is valid when X is replaced by C, S^* or K simply because $r_C(h_1) = 1/2$. We shall now prove that X can also be replaced by S_p the class of spirallike functions. However, the result does not follow directly from the convexity of h_1 up to 1/2 because, unlike C, S^*, and K, S_p is not preserved under convolution with convex functions as is shown by the example given in [10]. We shall, however, still be able to obtain the result using convolution techniques by going directly to Lemma A. We shall need the following lemma which we can prove for the more general class S.

LEMMA 1. *Let f be in S and $F(z) = 1 + a_1 z + \ldots$ be regular in U. Then, the image of $|z| < 1/2$ under $(h_1 * fF)/(h_1 * f)$ is a subset of the convex hull of $F(U)$.*

Proof. This result follows from Lemma A upon showing that for all α and σ, $(|\alpha| = |\sigma| = 1)$,

$$h_1(z) * \{[f(z)][(1 + \alpha\sigma z)/(1 - \sigma z)]\} = H(z) \neq 0 \qquad \text{for } 0 < |z| < 1/2.$$

From the definition of h_1 we see that

$$2H(z) = \left(\frac{1+\alpha\sigma z}{1-\sigma z}\right) f(z)\left[1 + \frac{zf'(z)}{f(z)} + \frac{(1+\alpha)\sigma z}{(1-\sigma z)(1+\alpha\sigma z)}\right].$$

Since f is in S, if we put $\zeta = \sigma z$, it is sufficient to show that for all $\alpha(|\alpha| = 1)$

$$1 + \frac{zf'(z)}{f(z)} + \frac{(1+\alpha)\zeta}{(1-\zeta)(1+\alpha\zeta)} \neq 0$$

for $|z|$, $|\zeta| < 1/2$. We note that since f is in S we have that

$$|\log\{[zf'(z)]/f(z)\}| \leq \log[(1+r)/(1-r)] \text{ for } |z| \leq r.$$

Let $r = 1/2$. We then have that

$$|\log\{[zf'(z)]/f(z)\}| \leq \log 3 \qquad \text{for } |z| \leq 1/2.$$

Now, we claim that $|1 + e^{\omega}| \geq 4/3$ if $|\omega| \leq \log 3$. To verify this, it clearly suffices to consider $\omega = -\rho e^{i\theta}$ for θ in $(-\pi, \pi]$ and $\rho > 0$, and note that

$$|1 + e^{\omega}|^2 = 1 + e^{-2\rho\cos\theta} + 2[\cos(\rho\sin\theta)]\, e^{-\rho\cos\theta}.$$

It is clear that the replacement of θ by $-\theta$ does not change the above expression and thus it suffices to consider $0 \leq \theta \leq \pi$. Letting $h(\rho, \theta) = |1 + e^{\omega}|^2$, we see that

$$\frac{\partial h}{\partial\theta} = 2\rho e^{-2\rho\cos\theta}[\sin\theta + \sin(\theta - \rho\sin\theta)\, e^{\rho\cos\theta}]$$

It now follows that $\partial h/\partial\theta \geq 0$ and therefore

$$|1 + e^{\omega}|^2 \geq h(\rho, 0) = |1 + e^{-\rho}|^2 \geq \left(\frac{4}{3}\right)^2$$

and the claim follows. We now have that $|1 + zf'(z)/f(z)| > 4/3$ for $|z| < 1/2$.

Lemma 1 will follow upon showing that for all α, $(|\alpha| = 1)$,

(3) $$\frac{(1+\alpha)\zeta}{(1-\zeta)(1+\alpha\zeta)} < \frac{4}{3}$$

provided $|\zeta| < 1/2$. Inequality (3) will follow from our next lemma which we will use a number of times in this paper.

LEMMA 2. *For* $|\alpha| = 1$, *let* $f_\alpha(z) = [(1+\alpha)z]/[(1-z)(1+\alpha z)]$. *If* $|z| \leq r < 1$, *then* $|f_\alpha(z)| \leq 2r/(1-r^2)$.

Proof. Write $\alpha = e^{2i\varphi}$, $-\pi/2 < \varphi \leq \pi/2$ and $z = re^{i\theta}$. If we put $t = \theta + \varphi$ we see that

$$f_\alpha(z) = \frac{r(e^{-i\varphi} + e^{i\varphi})e^{it}}{1 + re^{it}(e^{i\varphi} - e^{-i\varphi}) - r^2e^{2it}}$$
$$= \frac{2r\cos\varphi}{(1-r^2)\cos t + i[2r\sin\varphi - (1+r^2)\sin t]}.$$

Now,

$$|(1-r^2)\cos t + i[2r\sin\varphi - (1+r^2)\sin t]|^2$$
$$= (1-r^2)^2 + 4r^2\sin^2 t - 4r(1+r^2)\sin\varphi\sin t + 4r^2\sin^2\varphi$$
$$= 4r^2\left[\sin t - \frac{(1+r^2)}{2r}\sin\varphi\right]^2 + (1-r^2)^2\cos^2\varphi$$
$$\geq (1-r^2)^2\cos^2\varphi.$$

It now follows that $|f_\alpha(z)| \leq 2r/(1-r^2)$.

With these results we can now prove that Robinson's conjecture is valid when S is replaced by S_p.

THEOREM 2. $r_{S_p}[\Gamma_1(S_p)] = 1/2$.

Proof. Since f is in S_p there exists a real γ such that $H(z) = e^{i\gamma}zf'(z)/f(z)$ has positive real part in U. To show that $\Gamma_1 f = h_1 * f$ is spirallike in $|z| < 1/2$ we define H_1 by

$$H_1(z) = [e^{i\gamma}h_1 * zf'(z)]/[h_1 * f(z)] = [h_1 * f(z)H(z)]/[h_1 * f(z)].$$

Then Lemma 1 assures that $H_1[|z| < 1/2]$ is contained in the convex hull of $H(U)$. The result follows by noting that $k(z) = z(1-z)^{-2}$ is spirallike for $\gamma = 0$, and the radius of spirallikeness of $\Gamma_1(k)$ is $1/2$.

As another example where we can use Lemma A directly we prove Livingston's result given in (2); that is, $r_{K_1}[\Gamma_1(K_1)] = (\sqrt{5} - 1)/2 = r_0$ where f is in K_1 if and only if f' is in P. Since

$$(h_1 * f)'(z) = [h_1(z) * zf'(z)]/z = [h_1(z) * zf'(z)]/[h_1(z) * z]$$

we need only show that $\operatorname{Re}\{[h_1(z) * zf'(z)]/[h_1(z) * z]\} > 0$ for $|z| < r_0$. By Lemma A it suffices to show that

$$H(z) = h_1(z) * z(1+\alpha\sigma z)/(1-\sigma z) \neq 0 \text{ for } 0 < |z| < r_0,\ |\alpha| = |\sigma| = 1.$$

However from the definition of h_1 we have that

$$2H(z) = \frac{1+\alpha\sigma z}{1-\sigma z}2z + \frac{(1+\alpha)\sigma z^2}{(1-\sigma z)^2} = z\frac{1+\alpha\sigma z}{1-\sigma z}\left[2 + \frac{(1+\alpha)\sigma z}{(1-\sigma z)(1+\alpha\sigma z)}\right]$$

By using Lemma 2 we obtain

$$\left|2+\frac{(1+\alpha)\sigma z}{(1-\sigma z)(1+\alpha\sigma z)}\right| \geq 2-\left|\frac{(1+\alpha)\sigma z}{(1-\alpha z)(1+\alpha\sigma z)}\right| \geq 2-2r/(1-r^2)>0,$$
$0 \leq r < r_0$.

The result then follows by considering the function f_1 defined by

$$f_1(z) = -2\log(1-z) - z.$$

Let T be Rogosinbski's class of typically real functions on U, noting that functions in T need not be univalent. Let C_I be Robertson's (see [15]) class of functions in T that have their images convex in the direction of the imaginary axis. Recall Fejer's observation that h is in C_I if and only if zh' is in T. We include a new proof of Robertson's result in [16] showing that T is invariant under convolution with functions in C_I. We then give a corollary showing its application to Robinson's 1/2 conjecture.

THEOREM 3. *If h is in C_I and f is in T then $h * f$ is in T.*

Proof. It is a standard result that f is in T if and only if $f(z) = z(1-z^2)^{-1}p(z)$, where p is in P and has real coefficients. Also, for any function g in T, there exists a nondecreasing function μ_g on $[0,\pi]$ with $\mu_g(\pi) - \mu_g(0) = \pi$ and such that

$$g(z) = \frac{1}{\pi}\int_0^\pi \frac{z}{1-2z\cos t+z^2}\,d\mu_g(t).$$

For each t in $[0,\pi]$ the function g_t defined by

$$zg_t'(z) = z(1-2z\cos t+z^2)^{-1}, \qquad g(0)=0,$$

is convex because zg_t' is starlike. Thus, for t and s in $[0,\pi]$, $g(z,s,t) = zg_s'(z) * g_t(z)$ is starlike by Theorem A and has real coefficients so that $g(z,t,s)$ is in T. Hence $g(z,s,t) = z(1-z^2)^{-1}p(z,s,t)$ where p is in P. By using these facts and the properties of convolution, we obtain for h in C_I that

$$\begin{aligned}(h*f)(z) &= \frac{1}{\pi}\int_0^\pi h * \frac{z}{1-2z\cos t+z^2}\,d\mu_f(t)\\ &= \frac{1}{\pi}\int_0^\pi zh'(z) * \int_0^\pi \frac{d\zeta}{1-2\zeta\cos t+\zeta^2}\,d\mu_f(t)\\ &= \frac{1}{\pi}\int_0^\pi \frac{1}{\pi}\left[\int_0^\pi \frac{z}{1-2z\cos s+z^2}\,d\mu_{zh'}(s)\right] * g_t(z)\,d\mu_f(t)\\ &= \frac{1}{\pi^2}\int_0^\pi\int_0^\pi zg_s'(z) * g_t(z)\,d\mu_{zh'}(s)\,d\mu_f(t)\\ &= \frac{z}{1-z^2}\frac{1}{\pi^2}\int_0^\pi\int_0^\pi p(z,s,t)\,d\mu_{zh'}(s)\,d\mu_f(t) = \frac{z}{1-z^2}p_1(z)\end{aligned}$$

where p_1 has real coefficients and is in P from the properties of $\mu_{zh'}$ and μ_f. Thus, from the characterization of T the theorem is proved.

COROLLARY. *If f is in S and has real coefficients then $h_1 * f$ is typically real for $|z| < 1/2$.*

Proof. This follows from Theorem 3 since $r_C(h_1) = r_{C_I}(h_1) = 1/2$ and any function in S with real coefficients is in T.

We now give a generalization of Bernardi's results in [5]. Throughout Bernardi's paper he considered c a positive integer. We shall consider c a complex number with $c \neq -1$. In each case when c is considered a positive integer we obtain Bernardi's results. We define h_c by

$$h_c(z) = \sum_{n=1}^{\infty} \frac{n+c}{1+c} z^n = \frac{z - [c/(1+c)]\, z^2}{(1-z)^2}.$$

For f in A let the operator $\Gamma_c\colon A \to A$ be defined by $\Gamma_c(f) = h_c * f$.

THEOREM 4. (i) *If* $\operatorname{Re}\{c\} > 0$ *then* $r_K[\Gamma_c(C)] = 1$.

(ii) *If*

$$\operatorname{Re}\{c\} > \frac{4r - 1 - r^2}{1 - r^2} \tag{4}$$

and f is in C, S^* or K then $h_c * f$ is convex, starlike, or close-to-convex, respectively, for $|z| < r$. If c is real and greater than -1 then

$$r_C[\Gamma_c(C)] = r_{S^*}[\Gamma_c(S^*)] = r_K[\Gamma_c(K)] = r_0 \tag{5}$$

where $r_0 = \{2 - (3 + c^2)^{1/2}\}/(1 - c)$ for $c \neq 1$ and for $c = 1$, $r_0 = 1/2$.

(iii) *If*

$$|c + 1| > 2r/(1 - r^2) \tag{6}$$

and f is in K_1, then $\operatorname{Re}\{(h_c * f)'(z)\} > 0$ *for $|z| < r$. If c is real and greater than -1, then $r_{K_1}[\Gamma_c(K_1)] = r_1$ where $r_1 = [-1 + (2 + 2c + c^2)^{1/2}]/(1 + c)$.*

Proof. Part (i) follows from Theorem 1 by the easily proved fact that h_c is in K if and only if $|c/(1+c) - 1/2| \le 1/2$ which is equivalent to $\operatorname{Re}\{c\} > 0$.

The first part of (ii) will follow from Theorem 1 upon showing that whenever inequality (4) holds then $r \le r_C(h_c)$. Consider, for $k(z) = z/(1-z)^2$, that

$$1 + \frac{z h_c''(z)}{h_c'(z)} = \frac{h_c * k * k}{h_c * k}(z) = \frac{h_c * k\,[(k * k)/k]}{h_c * k}(z). \tag{7}$$

Since $(k * k)/k(z) = (1 + z)/(1 - z)$, Lemma A assures that the term on the right hand side of inequality (4) has positive real part whenever

(8) $$h_c * [k(1+\alpha\sigma z)/(1-\sigma z)] \neq 0 \qquad \text{for all } \alpha,\sigma, |\alpha| = |\sigma| = 1$$

and $0 < |z| < r$. Thus we need only show that this r is determined by the condition (4). From the definition of h_c and the comparison of their corresponding Taylor series we have that

(9) $$h_c * k\frac{1+\alpha\sigma z}{1-\sigma z} = \frac{1}{1+c}k\frac{1+\alpha\sigma z}{1-\sigma z}\left[c + \frac{zk'}{k} + \frac{(1+\alpha)\sigma z}{(1-\sigma z)(1+\alpha\sigma z)}\right].$$

Thus, it suffices to show that the bracketed term in (9) is nonzero for $|z| < r$ determined by (4). From the definition of $k(z)$ and Lemma 2 we have that

$$c + \frac{zk'(z)}{k(z)} + \frac{(1+\alpha)\sigma z}{(1-\sigma z)(1+\alpha\sigma z)} \neq 0$$

whenever $\text{Re}\,\{c + (1+z)/(1-z)\} > 2r/(1-r^2)$. This holds when

(10) $$\text{Re}\,\{c\} > \frac{2r}{1-r^2} - \frac{1-r}{1+r} = \frac{4r-1-r^2}{1-r^2}$$

as claimed. For real $c > -1$, (10) is equivalent to $c + 1 - 4r + (1-c)r^2 > 0$, which holds whenever $0 \leq r < r_0$. In order to complete the verification of (5) we consider the cases of sharpness. For the convex case we convolute $h_c(z)$ with $z/(1-z)$ to obtain

$$1 + \frac{zh_c''(z)}{h_c'(z)} = \frac{(1+c)+4z+(1-c)z^2}{(1-z)[(1+c)+(1-c)z]} = J(z).$$

It easily follows that $J(-r_0) = 0$. For the starlike and close-to-convex case we convolute h_c with $k(z)$ to obtain

$$\frac{z(h_c * k)'}{h_c * k} = 1 + \frac{zh_c''(z)}{h_c'(z)} = J(z).$$

So that again $J(-r_0) = 0$. Since for $c = 1$, $h_c * f = h_1 * f = \Gamma_1 f$ the case $c = 1$ follows from our previous results.

We prove part (iii) by using Theorem 1 and noting as before that

$$(h_c * f)'(z) = \frac{h_c(z) * zf'(z)}{h_c(z) * z}$$

Since f is in K_1, Lemma A assures that $\text{Re}\,\{(h_c * f)'\} > 0$ whenever

(11) $$h_c * [z(1+\alpha\sigma z)/(1-\sigma z)] \neq 0, \qquad |\alpha| = |\sigma| = 1$$

and $0 < |z| < r$. We need only show that this r is determined by condition (6). We have

$$h_c * z\frac{1+\alpha\sigma z}{1-\sigma z} = \frac{1+\alpha\sigma z}{1-\sigma z}\left[1+c+\frac{(1+\alpha)\sigma z}{(1-\sigma z)(1+\alpha\sigma z)}\right]\frac{z}{1+c}.$$

Thus applying Lemma 2 we have that inequality (11) holds whenever inequality (6) holds as claimed. For $c > -1$, inequality (6) is equivalent to

$$|z| < r_1 = [-1 + (2 + 2c + c^2)^{1/2}]/(1+c).$$

That $r_1 = r_{K_1}[\Gamma_c(K_1)]$ follows by convoluting $h_c(z)$ with $f_1(z) = -z - 2\log(1-z)$ to obtain

$$(h_c * f_1)'(z) = [(1+c) + 2z - (1+c)z^2]/(1-z)^2 = g(z)$$

where $g(-r_1) = 0$.

Another type of problem to which we can apply these techniques is the following. Let $G(z) = -f(-z)/f(z)$ and $F(z) = f'(-z)/f'(z)$. In [7] Burdick, Keogh, and Merkes determined the smallest α and β such that $\operatorname{Re}\{G(z)\}^{\alpha} > 0$ and $\operatorname{Re}\{F(z)\}^{\beta} > 0$, z in U, for f in C, S^* and K. Noting that

$$G(z) = [f(z) * z/(1+z)]/[f(z) * z/(1-z)]$$

and that $F(z) = [k(z) * f(z) * z/(1+z)]/[k(z) * f(z) * z/(1-z)]$ we obtain the following generalizations of their work. Given β, $0 \le \beta < 1$, let

$$S^*(\beta) = \{f \in S : \operatorname{Re}[zf'(z)/f(z)] > \beta, z \in U\}.$$

LEMMA 3. *If f is in $S^*(\beta)$ then*

(12) $$\operatorname{Re}\{[f(z) * z/(1+z)]/f(z)\}^{1/2(1-\beta)} > 0, \qquad z \in U.$$

The result is sharp.

This result follows readily from the Herglotz representation for functions in $S^*(\beta)$. Since f is in $S^*(\beta)$ there exists a probability measure μ such that

$$\log[f(z)/z] = 2(1-\beta)\int_{-\pi}^{\pi}\log(1-ze^{it})\,d\mu(t)$$

Thus we obtain

$$\frac{-f(-z)}{f(z)} = \exp\left[2(1-\beta)\int_{-\pi}^{\pi}\log\frac{1-ze^{it}}{1+ze^{it}}\,d\mu(t)\right],$$

and the result follows. The sharpness follows by considering

$$f(z) = k_\beta(z) - z(1-z)^{2(\beta-1)}.$$

THEOREM 5. *If f is in K then*

$$\operatorname{Re}\left\{\left[\frac{k_\beta(z) * f(z) * z/(1+z)}{k_\beta(z) * f(z)}\right]^{1/2(2-\beta)}\right\} > 0, \qquad z \in U. \tag{13}$$

The result is sharp.

Proof. Let $K_\beta(z) = \int_0^z (k_\beta(\zeta)/\zeta)\, d\zeta$. Since f is in K, $zf'(z) = g(z)p(z)$ for g star-like and $\operatorname{Re}\{p(z)\} > 0$, z in U. Let p_i designate a function with $\operatorname{Re}\{p_i(z)\} > 0$, z in U, for $i = 1,2,3$, and 4. Using these notations, Lemma A, and the remark following Lemma A a number of times we have

$$\frac{k_\beta(z) * f(z) * \dfrac{z}{1+z}}{k_\beta(z) * f(z)} = \frac{K_\beta(z) * zf'(z) * \dfrac{z}{1+z}}{K_\beta(z) * zf'(z)}$$

$$\begin{aligned} &= \frac{K_\beta(z) * g(z)p_1(z) * \dfrac{z}{1+z}}{K_\beta(z) * g(z)p_1(z)} = \frac{[K_\beta(z) * g(z)]\,p_2(z) * \dfrac{z}{1+z}}{[K_\beta(z) * g(z)]\,p_2(z)} \\ &= \frac{\left\{[K_\beta(z) * g(z)] * \dfrac{z}{1+z}\right\}p_3(z)}{[K_\beta(z) * g(z)]\,p_2(z)} \\ &= \frac{\left\{\left[k_\beta(z) * \displaystyle\int_0^z \frac{g(\zeta)}{\zeta}\,d\zeta\right] * \dfrac{z}{1+z}\right\}p_3(z)}{\left[k_\beta(z) * \displaystyle\int_0^z \frac{g(\zeta)}{\zeta}\,d\zeta\right]p_2(z)} \\ &= \left\{\frac{\left[k_\beta(z) * \displaystyle\int_0^z \frac{g(\zeta)}{\zeta}\,d\zeta\right] * \dfrac{z}{1+z}}{k_\beta(z) * \displaystyle\int_0^z \frac{g(\zeta)}{\zeta}\,d\zeta}\right\} p_3(z)p_2^{-1}(z). \end{aligned} \tag{14}$$

It is also easy to prove that if h is in C then $h * k_\beta$ is in $S^*(\beta)$. So from the convexity of $\int_0^z [g(\zeta)/\zeta]\, d\zeta$, Lemma 3, and the remark following Lemma A we have that (14) equals $p_4^{2(1-\beta)}(z)p_3(z)p_1^{-1}(z)$. Inequality (13) follows. Sharpness is proved by considering the function $f(z) = h_c(z) = \{z - [c/(1+c)]\, z^2\}/(1-z)^2$. A straightforward calculation gives that

$$|\arg\{[k_\beta(z) * h_c(z) * z(1+z)^{-1}]/[k_\beta(z) * h_c(z)]\}|$$

$$(15)\qquad = \left|\arg\left\{\left(\frac{1-z}{1+z}\right)^{3-2\beta}\left[\frac{1+[1-2\beta/(1+c)]\,z}{1-[1-2\beta/(1+c)]\,z}\right]\right\}\right|.$$

If we let $1 - 2\beta/(1+c) = \mathrm{Re}^{i\varphi}$ and $z = e^{i\theta}$, then, for $\theta = -\varphi + \pi/2$, (15) becomes

$$(16)\qquad |(3-2\beta)\pi/2 + \operatorname{Arcsin}[2R/(1+R^2)]|.$$

Since R approaches 1 as $|c|$ approaches ∞ we have that (16) approaches $(2-\beta)\pi$ and the result follows.

As another application we answer Problem 6.45 posed by Krzyz in *Research Problems in Complex Analysis,* in [2]. For $0 < \alpha \leq 1$ let S_α^* be the class of α-strongly-starlike functions defined by Brannan and Kirwan in [6]; that is, $S_\alpha^* = \{f \in S\colon |\arg[zf'(z)]| < \alpha\pi/2, |z| < 1\}$. Let " $\circledast$ " be defined on $A \times A$ as follows: for $f(z) = \Sigma a^n z^n$ and $g(z) = \Sigma b_n z^n$ define $f \circledast g$ by

$$(f * g)(z) = \Sigma(a_n b_n/n) z^n,$$

Krzyz posed the following problems:

(a) Prove (or disprove) that S_α^* is closed under $\circledast$.

(b) Prove (or disprove) that if f is in S_α^* and g is in S_β^* then $f \circledast g$ in S_γ^* for some $\gamma = \gamma(\alpha,\beta) < 1$.

(c) One could ask the same question for different convolutions in place of $\circledast$.

We first make the observation that $(f \circledast g)(z) = \int_0^z [f(\zeta)/\zeta]\, d\zeta * g(z)$. Since f is in S_α^*, $\int_0^z [f(\zeta)/\zeta]\, d\zeta$ is convex, thus the remark following Lemma A applies. If we let $h(z) = (f \circledast g)(z)$, then

$$\frac{zh'(z)}{h(z)} = \frac{\displaystyle\int_0^z \frac{f(\zeta)}{\zeta}\, d\zeta * [g(z)]\left[\frac{zg'(z)}{g(z)}\right]}{\displaystyle\int_0^z \frac{f(\zeta)}{\zeta}\, d\zeta * g(z)}$$

lies in the convex hull of the image of U under zg'/g; that is,

$$|\arg[zh'(z)/h(z)]| < \alpha\pi/2$$

since g is in S_α^*. Therefore S_α^* is closed under $\circledast$ and part (a) is resolved. For part (b), since $f(z) * \int_0^z [g(\zeta)/\zeta]\, d\zeta = \int_0^z [f(\zeta)/\zeta]\, d\zeta * g(z)$ if follows that $\gamma = \min[\alpha, \beta]$ will work and if $\beta = 1$ and $0 < \alpha \leq 1$ sharpness follows using $f(z) = k(z)$.

For part (c) if we denote a general type of weighted convolution by $\circledast_w$, there are still many interesting open problems. Ideally one would like to "weight it" by a function whose coefficients are of maximum modulus in the class. However, a straightforward, but long, argument using the fact that f in S_α^* has $\operatorname{Re}\{[zf'(z)/f(z)]\}^{1/\alpha} > 0$, shows that there are different extremal functions for different coefficients, hence an inherent difficulty in weighting the convolutions in this manner. It might be of interest to note that if one defines $\circledast_w$ for f and g in S_α^* by

$$(f \circledast_w g)(z) = (f * g * w)(z),$$

then an argument similar to that showing that S_α^* is closed under $\circledast$ shows that a sufficient condition for S_α^* to be closed under this $\circledast_w$ is that $zw'(z)$ be convex.

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Department of Mathematics
Texas Tech University
Lubbock, Texas 79409