# A CRITICAL GROWTH RATE FOR FUNCTIONS REGULAR IN A DISK

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## 1. INTRODUCTION

Suppose that $k(r)$ is a positive continuous non-decreasing function of $r$ on $0 \leq r < 1$, which satisfies

$$k(r) \to \infty \text{ as } r \to 1. \tag{1}$$

We consider the class $A^{(k(r))}$ of functions $f(z)$ regular in

$$U = \{z \mid |z| < 1\}$$

and which satisfy there

$$\log |f(z)| \leq k(|z|). \tag{2}$$

We start by remarking that $A^{(k(r))}$ always contains functions of unbounded characteristic in $U$. Thus BAGEMIHL, ERDÖS and SEIDEL [1, Theorem 5] have shown that, if $a$ is a small positive constant and the sequence $n_i$ of positive integers grows sufficiently rapidly,

$$f(z) = a \prod_{n=1}^{\infty} \left\{1 - \left(\frac{z}{1 - 1/n_i}\right)^{n_i}\right\} \in A^{(k(r))}.$$

On the other hand $f(z)$ has $n_i$ zeros on the circle $|z| = 1 - 1/n_i$, and so, if $\rho_n$ are the moduli of the zeros of $f(z)$, the $\rho_n$ do not satisfy the BLASCHKE condition

$$\sum (1 - \rho_n) < \infty. \tag{3}$$

Hence $f(z)$ cannot have bounded characteristic.

Nevertheless H. S. SHAPIRO and A. L. SHIELDS [3] proved that if $f(z) \not\equiv 0$ and $f \in A^{(k(r))}$, with $k(r) = (1 - r)^{-b}$, for $0 < b < 1$, then (3) holds for the real positive zeros $\rho_n$ of $f(z)$, and hence for the moduli of the zeros on a fixed radius $\arg z = \theta$. In this paper we obtain the precise condition on $k(r)$ for this latter result to hold.

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## 2. STATEMENT OF MAIN RESULT

*Definition 1.* If $k(r)$ is a positive continuous nondecreasing function of $r$ for $0 \leq r < 1$, then the class $A^{\langle k(r) \rangle}$ will be called *S.-S.* (SHAPIRO-SHIELDS) if and only if *(3)* holds for the positive zeros $\rho_n$ of any $f(z)$ in the class.

We proceed to prove the following

THEOREM. *A necessary and sufficient condition for the class $A^{\langle k(r) \rangle}$ to be S.-S. is that*

$$J = \int_0^1 \sqrt{\frac{k(r)}{1-r}}\, dr < \infty. \tag{4}$$

## 3. PROOF OF THE SUFFICIENCY

The sufficiency of (4) is actually an almost immediate consequence of an earlier result by the authors [2]. We need to recall [2, p. 179]

*Definition 2.* Suppose that there exists a function $z = \phi(w)$, which is regular in $U$, real and increasing on the segment [0,1] of the real axis and $\phi(0) = r_0 \geq 0$, $\phi(1) = 1$ and

$$|\phi(w)| < 1 \text{ in } U.$$

Suppose further that

$$|1 - \phi(w)| < C_1\, |1 - w|, \quad 0 \leq |w| < 1.$$

Suppose next that $f(z)$ is meromorphic in $U$ and that

$$T(r, f\{e^{i\theta}\, \phi(w)\}) \leq C_2, \quad 0 \leq \theta \leq 2\pi \;\; 0 < r < 1$$

where $T(r, \psi)$ denotes the *NEVANLINNA* characteristic of $\psi(w)$ in $|w| < r$, and $C_1$, $C_2$ are positive constants. Then we say that $f(z)$ has locally bounded characteristic, (l.b.c.) in $U$.

We proved [2, Theorem 5].

THEOREM A. *If $f \in A^{\langle k(r) \rangle}$ and (4) holds then $f$ has l.b.c. and we may take $r_0 = 1/e$, and $C_1$, $C_2/J^2$ absolute constants.*

We take this opportunity to close a slight gap in our proof of Theorem *A*. With a suitable $\phi$ satisfying the above conditions we actually showed [2, p. 137] that

$$\frac{1}{2\pi} \int_0^{2\pi} \log^+ |f\{\phi(e^{i\lambda})\}|\, d\lambda \leq \frac{1}{2\pi} \int_0^{2\pi} k(|\phi(e^{i\lambda})|)\, d\lambda \leq C_2. \tag{5}$$

What needs to be proved however is

$$(6)\qquad \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f\{\phi(re^{i\lambda})\}|\,d\lambda \le C_2, \qquad 0<r<1.$$

If $f\{\phi(w)\}$ is regular in $|w| \le 1$, (6) is a consequence of (5). In the general case we apply (5) to $f(tz)$ instead of $f(z)$, where $0<t<1$. Clearly, since $f(z) \in A^{(k(r))}$ and $k(r)$ is increasing, $f(tz) \in A^{k(r)}$. We deduce from (5) that

$$\frac{1}{2\pi}\int_0^{2\pi} \log^+ |f\{t\phi(e^{i\lambda})\}|\,d\lambda \le C_2,$$

and since $f\{t\phi(z)\}$ is regular in $|z| \le 1$, we deduce

$$(7)\qquad \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f\{t\phi(re^{i\lambda})\}|\,d\lambda \le C_2, \qquad 0<r<1, 0<t<1.$$

If $|w| \le r$, where $r<1$, then $|\phi(w)| \le r_0$, where $r_0<1$. Also $\log^+ |f(z)|$ is continuous for $|z| \le r_0$. Thus $\log^+ |f\{t\phi(re^{i\lambda})\}|$ is a continuous function jointly in $t$, $0 \le t \le 1$, and $\lambda$, $0 \le \lambda \le 2\pi$ when $r$ is fixed, $0<r<1$. Thus we may allow $t$ to tend to one under the integral sign in (7) and deduce (6). By applying the above argument with $f(ze^{i\theta})$ instead of $f(z)$, we deduce that $f(z)$ has l.b.c. with $C_1$, $C_2$, as in Theorem A.

We now deduce the sufficiency of (4) for (3) to hold. Let $\rho_n$ be those zeros of $f(z)$ which satisfy $\frac{1}{e} < \rho_n < 1$. Since $\phi(w)$ increases from $r_0 = 1/e$ to 1 as $w$ increases from 0 to 1, there exists $r_n$ such that $0<r_n<1$ and $\phi(r_n) = \rho_n$. Thus if $\psi(w) = f\{\phi(w)\}$, then $T(1,\psi(w)) \le C_2$, and $\psi(r_n) = f(\rho_n) = 0$. Thus

$$\sum (1-r_n) < \infty.$$

Also in view of Definition 2 we have

$$1-\rho_n < C_1(1-r_n),$$

and we deduce (3).

With a little effort one can by the same method prove a corresponding result for those zeros $z_n$ of $f(z)$ which lie in a STOLZ angle $\arg(1-z) < \frac{\pi}{2} - \delta$ for a fixed $\delta$. It is a consequence of the conditions on $\phi(w)$ in Definition 2 and follows in particular from the explicit construction of $\phi(w)$ in [2, pp. 192–193], that the image of $\phi(w)$ covers a sector $S(r,\delta) = \{z \mid 0 < |1-z| < r, \arg(1-z) < \frac{\pi}{2} - \delta\}$, for any positive $\delta$, where $r = r(\delta)$ depends on $\delta$. Further $\phi(w)$ is univalent in the inverse image of $S(r,\delta)$ and

$$\frac{1-\phi(w)}{1-w} \to C, \qquad \text{as } \phi(w) \to 1 \text{ in } S(r,\delta).$$

Here $0 < C \le C_1$. Thus, for large $n$, $f\{\phi(w)\}$ has a zero $w_n$, in $U$, where

$$1 - z_n = (C + o(1))(1 - w_n), \qquad \text{as } n \to \infty.$$

In particular

$$|\arg(1 - w_n)| < \frac{\pi}{2} - \frac{1}{2}\delta, \qquad |1 - z_n| < 2C|1 - w_n|$$

for large $n$. We deduce that for large $n$

$$1 - |z_n| < |1 - z_n| < 2C|1 - w_n| < 3C \operatorname{cosec} \frac{1}{2}\delta\,(1 - |w_n|).$$

Since $f\{\Phi(w)\}$ has bounded characteristic, we deduce again that

$$\sum (1 - |z_n|) < \infty,$$

where the sum is taken over those zeros lying in a STOLZ angle bisected by the positive axis, or by any radius.

## 4. PROOF OF THE NECESSITY

*Preliminary Results.* In order to prove the necessity we need to construct a conformal mapping which is a slight adaptation of one used by us in our previous paper [2]. The main difference is that we need at present a symmetric map, whereas for [2] an unsymmetric map was necessary. We shall indicate the differences which this involves in our definitions. The arguments are almost identical. Suppose then that $k(r)$ is again positive, continuous and increasing in $[0,1)$, that $k(0) = 1$ and

(8)
$$\int_0^1 \sqrt{\frac{k(r)}{1-r}}\, dr = +\infty.$$

We define a function $\varepsilon(\xi)$ for $\xi \ge 0$ as follows. Let $r$ be the unique number, such that

(9)
$$e^{\xi} = \sqrt{\frac{k(r)}{1-r}} \quad \left(\text{instead of } e^{2\xi} = \sqrt{\frac{k(r)}{1-r}} \text{ in [2, (3.13)]}\right)$$

and set

$$\varepsilon_1(\xi) = \sqrt{\{(1-r)k(r)\}}\,.$$

We then define $\varepsilon_2(\xi)$, $\varepsilon_3(\xi)$ and $\varepsilon(\xi)$ in terms of $\varepsilon_1(\xi)$ just as in [2, pp. 188 and 189], namely

$$\varepsilon_2(\xi) = \min(1, \varepsilon_1(\xi)),$$

(10) $$\varepsilon_3(\xi) = a\varepsilon_2(\xi) \Big/ \left\{ 1 + \int_0^{\xi} \varepsilon_2(t)\, dt \right\}, 0 < a < 1.$$

$$\varepsilon_n = \inf_{(n-2)A_2 < t \le (n-1)A_2} \varepsilon_3(t), \quad n \ge 2; \quad \varepsilon_n = 0, n < 2,$$

where $A_2 = 16{,}000\pi$, and

(11) $$\begin{aligned} \varepsilon(x) &= \varepsilon_n, \quad (n-1)A_2 < x < nA_2 \\ \varepsilon(x) &= \min(\varepsilon_n, \varepsilon_{n+1}), \quad x = nA_2. \end{aligned}$$

We then consider as in [2] that map from the domain $\Delta$ in the $\zeta = \xi + i\eta$ plane given by

(12) $$|\eta| < \frac{1}{2}(\pi + \varepsilon(\xi)), \qquad -\infty < \xi < +\infty,$$

onto the strip $S$ in the $s = \sigma + i\tau$ plane given by

(13) $$|\tau| < \frac{\pi}{2}, \qquad -\infty < \sigma < +\infty,$$

which is symmetric, i.e. $\sigma = \pm\infty$ correspond to $\xi = \pm\infty$ and $s(0) = 0$. However the function $u(z)$ which we now consider is given by

(14) $$u(z) = -e^{\xi}\cos\eta, \quad (\text{instead of } u = -e^{2\xi}\sin 2\eta \text{ in } [2, (3\text{-}9)]),$$

where $s$, $\zeta$ are related as above and

(15) $$s = \sigma + i\tau = \log\frac{1+z}{1-z}$$

$$\left(\text{instead of } s = \frac{1}{2}\left(\log\frac{1+z}{1-z} + i\frac{\pi}{2}\right) \text{in } [2, (3.3)]\right).$$

With the above definitions we can prove

LEMMA 1. *If $u(z)$ is the function defined by* (14) *and the constant $a$ in* (10) *is chosen sufficiently small, then we have*

(16) $$u(z) \le k(|z|), z \in U.$$

This result is the analogue of Lemma 7 of [2] and the proof is quite similar and uses the properties of the conformal map from $\Delta$ to $S$. The difference between our present and our former construction are indicated in (9), (14) and (15). We go briefly through the main steps of the argument which leads to (16). We start by proving that for $0 < t_1 < t_2 < \infty$, we have

$$\varepsilon_1(t_2) \geq e^{2(t_1 - t_2)} \varepsilon_1(t_1), \tag{17}$$

and

$$\int_0^\infty \varepsilon_1(\xi)\, d\xi = \infty. \tag{18}$$

The argument is similar to that given in the proof of [2, Lemma 5]. We deduce just as in [2, Lemma 6] that if $t_1 \leq t_2 \leq t_1 + 1$, we have $\varepsilon_3(t_2) \geq \frac{1}{2} e^{-2} \varepsilon_3(t_1)$ and further that $\int_0^\infty \varepsilon_3(t)\, dt = \infty$. These results lead to $\varepsilon_{n+1} > A_4\, \varepsilon_n$, where $A_4$ is an absolute constant, and

$$\int_0^\infty \varepsilon(t)\, dt = \infty. \tag{19}$$

The argument is given in [2, p. 190].

To complete the argument for Lemma 1 we prove the analogues of (4.3) and (4.4) of [2]. Namely if $u(re^{i\theta})$ is given by (14) and $u > 0$, we deduce that

$$u(re^{i\theta}) < \varepsilon_1(\xi)\, e^{\xi}$$

(instead of $u(re^{i\theta}) < \varepsilon_1(\xi) e^{2\xi}$ in [2, (4.3)]) and

$$1 - r < \varepsilon_1(\xi)\, e^{-\xi}$$

(instead of $1 - r < \varepsilon_1(\xi)\, e^{-2\xi}$ in [2, (4.4)]). The proof is then completed as in [2, p.p. 190, 191] with the differences indicated above. In the present case it is sometimes possible to prove somewhat sharper inequalities, but it is probably simplest to stick to the results of [2].

LEMMA 2. *If* $\sigma$, $\xi$ *are related as above with* $\varepsilon(x)$ *satisfying* (11) *and* (19) *then*

$$\sigma - \xi \to -\infty \text{ as } \xi \to +\infty.$$

This is LEMMA 2 of [2].

We note that in view of the above construction, we have

$$(20) \qquad 0 \le \varepsilon(t) \le 1 < \frac{\pi}{2}, \qquad -\infty < t < +\infty.$$

We deduce

LEMMA 3. *Let* $U_1 = \left\{ w: \left| \arg \frac{1+w}{1-w} \right| < \frac{3\pi}{4} \right\}$. *Then there exists a domain* $G$, *such that* $U \subset G \subset U_1$, *which is symmetric with respect to the real axis and such that*

$$(21) \qquad \log |g\{\phi(z)\}| \le k(|z|), \ |z| < 1,$$

*where*

$$(22) \qquad g(w) = \exp \left\{ \frac{w+1}{w-1} \right\}$$

*and* $w = \phi(z)$ *maps* $U$ *conformally onto* $G$, *so that* $\phi(-1) = -1$, $\phi(0) = 0$ *and* $\phi(1) = 1$. *Further*

$$(23) \qquad \lim_{r \to 1-} \frac{1-\phi(r)}{1-r} = 0.$$

We define $w$ by

$$(24) \qquad \zeta = \log \frac{1+w}{1-w}.$$

Evidently $\Delta$, given by (12), contains the strip $|\eta| < \frac{\pi}{2}$, and is contained in $|\eta| < \frac{3}{4}\pi$ in view of (20). Thus if $w$, $\zeta$ are related as in (24), then $\Delta$ corresponds to a domain $G$ in the $w$-plane which contains $U$ and is contained in $U_1$, since $U$ and $U_1$ correspond to $|\eta| < \frac{\pi}{2}$ and $|\eta| < \frac{3\pi}{4}$ respectively by (24).

Let $w = \phi(z)$ denote the correspondence between $w$ and $z$, when $w$ and $\zeta$ are related by (24), $s$ and $\zeta$ by that map from the strip (13) into $\Delta$ given by (12), which was discussed above, and $z$ and $s$ by (15). Then $w = \phi(z)$ clearly maps $U$ onto $G$. Also $z = -1, 0, 1$ correspond respectively to $s = -\infty, 0, \infty$, $\zeta = -\infty, 0, \infty$ and so $w = -1, 0, 1$. Clearly $w = \phi(z)$ is real for real $z$ and so symmetric.

Next, in view of (14), (22) and (24) we have

$$\begin{aligned} u = -e^{\xi} \cos\eta = \operatorname{Re}\{-e^{\zeta}\} &= \log|\exp(-e^{\zeta})| \\ &= \log \left| \exp \frac{w+1}{w-1} \right| = \log |g(w)|. \end{aligned}$$

Thus $u(z) = \log |g\{\phi(z)\}|$, and (21) follows from (16).

Finally (23) follows from Lemma 2, and the proof of Lemma 3 is complete.

Next we need the following elementary

LEMMA 4. *For* $z \in U_1 \setminus U$ *and* $0 < r < 1$ *we have*

$$\log \left| \frac{z-r}{1-rz} \right| \leq 2\frac{1-r}{1+r} \operatorname{Re} \frac{z+1}{z-1}. \tag{25}$$

We write $\zeta = \dfrac{z+1}{z-1}$, $\quad a = \dfrac{1+r}{1-r}$. Then (25) reduces to

$$\log \left| \frac{a+\zeta}{a-\zeta} \right| \leq \frac{2}{a} \operatorname{Re} \zeta, \quad \frac{\pi}{4} < |\arg \zeta| < \frac{\pi}{2}.$$

We write $\zeta = a(x+iy)$, so that $0 < x \leq |y|$. Then our inequality becomes

$$\frac{1}{2} \log \frac{(1+x)^2 + y^2}{(1-x)^2 + y^2} \leq 2x.$$

or

$$\frac{1}{2} \log \frac{1+t}{1-t} \leq 2x$$

where

$$t = \frac{2x}{1+x^2+y^2} \leq \frac{2x}{1+2x^2} \leq \frac{1}{\sqrt{2}}$$

Thus

$$\begin{aligned}
\frac{1}{2} \log \frac{1+t}{1-t} &= t + \frac{1}{3}t^3 + \frac{1}{5}t^5 + \dots \\
&\leq t + t^3 \left( \frac{1}{3} + \frac{1}{5}t^2 + \frac{1}{7}t^4 + \frac{1}{9}\frac{t^6}{1-t^2} \right) \\
&\leq t + t^3 \left( \frac{1}{3} + \frac{1}{10} + \frac{1}{28} + \frac{1}{36} \right) = t + \frac{313}{630}t^3 < t + \frac{1}{2}t^3.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{1}{2} \log \frac{1+t}{1-t} &\leq \frac{2x}{1+2x^2} \left( 1 + \frac{2x^2}{(1+2x^2)^2} \right) \\
&= \frac{2x(1+6x^2+4x^4)}{1+6x^2+12x^4+8x^6} \leq 2x
\end{aligned}$$

as required.

Finally we use (23) to prove

LEMMA 5. *There exists a sequence* $r_n$, $n = 1, 2, \ldots$, *such that*

(i) $0 < r_n < 1$,

(ii) $\sum_{n=1}^{\infty} (1 - r_n) \leq \frac{1}{2}$,

and

(iii) $\sum_{n=1}^{\infty} (1 - \rho_n) = +\infty$,

*where* $\phi(\rho_n) = r_n$, *and* $\phi(z)$ *is the function occurring in Lemma 3.*

For every positive integer $p$ we choose $t_p$ so that

$$0 < 1 - \phi(t_p) < 2^{-p-1}(1 - t_p) < 2^{-p-1}.$$

Such a choice of $t_p$ is possible in view of (23). We then define $k_p$ to be the integral part of $(1 - t_p)^{-1}$, and define $k_p$ of the numbers $\rho_n$ to be equal to $t_p$ for each $p$. The resulting sequence $\rho_n$ is then rearranged to be non-decreasing. Then

$$\sum_{n=1}^{\infty} (1 - \rho_n) = \sum_{p=1}^{\infty} k_p(1 - t_p) \geq \sum_{p=1}^{\infty} \frac{1}{2} = \infty,$$

which proves (iii). Finally if $r_n = \phi(\rho_n)$, we have (i) and

$$\sum_{n=1}^{\infty} (1 - r_n) = \sum_{p=1}^{\infty} k_p(1 - \phi(t_p)) < \sum_{p=1}^{\infty} 2^{-1-p} k_p(1 - t_p) \leq \sum_{p=1}^{\infty} 2^{-1-p} = \frac{1}{2}.$$

which is (ii). This proves Lemma 5.

## 5. COMPLETION OF PROOF OF THEOREM 1

Let $r_n$ be the sequence constructed in Lemma 5 and let

$$B(w) = \prod_{n=1}^{\infty} \frac{r_n - w}{1 - r_n w}.$$

The product converges in the whole plane apart from poles and zeros and $w = 1$, and $B(w)$ is meromorphic in the closed plane except at $w = 1$. Also we have $|B(w)| < 1$, $w \in U$, and in view of lemma 4 we have for $w \in U_1 \setminus U$

$$\log |B(w)| < 2\mathrm{Re}\, \frac{w + 1}{w - 1} \sum_{n=1}^{\infty} \frac{1 - r_n}{1 + r_n} \leq \mathrm{Re}\, \frac{w + 1}{w - 1}.$$

Thus if $g$, $\phi$ are as in Lemma 3, we have for $z \in U$

$$|B\{\phi(z)\}| \le |g\{\phi(z)\}| \le \exp k(|z|),$$

so that $B\{\phi(z)\} \in A^{(k(r))}$. On the other hand $B\{\phi(z)\}$ has zeros at the points $z = \rho_n$, which do not satisfy (3). Thus $A^{(k(r))}$ is not $S.$-$S.$

We have assumed so far that $k(0) = 1$. In the general case we apply the above construction with $k_1(\rho) = k(\rho) + C$, where $C = 1 - k(0)$. Then if $f_1(z)$ is the corresponding function, we have in $U$

$$\log | f_1(z)| \le k(|z|) + C,$$

and the positive zeros of $f_1(z)$ do not satisfy (3). Thus

$$f(z) = e^{-C} f_1(z)$$

satisfies (2), but the positive zeros of $f(z)$ do not satisfy (3). This completes the proof that (4) is necessary, and the proof of our Theorem.

## REFERENCES

1. F. Bagemihl, P. Erdös and W. Seidel, *Sur quelques propriétés frontières des fonctions holomorphes définies par certains produits dans le cercle-unité.* Ann. Sci. École Norm. Sup. (3) 70 (1953), 135–147.

2. W. K. Hayman and B. Korenblum, *An extension of the Riesz-Herglotz formula.* Ann. Acad. Sci. Fenn. Ser. A. I. Math. 2 (1976), 175–201.

3. H. S. Shapiro and A. L. Shields, *On the zeros of functions with finite Dirichlet integral and some related function spaces.* Math. Z. 80 (1962), 217–229.

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