PSEUDOCONVEXITY AND VALUE DISTRIBUTION FOR SCHUBERT ZEROES

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The distribution of zeroes of holomorphic sections in a Hermitian vector bundle was first studied using characteristic forms by Bott and Chern [2], and later by Cowen [5], Griffiths-King [8] and Stoll [15] [17]. In the general setting, let $f\colon X \to Y$ be a holomorphic map (where X, Y are complex spaces); assume in Y a reasonable set of subvarieties, $\mathfrak{A} = \{S_b\}_{b \in N}$, is given. One wishes to describe the typical behavior of the fiber $S_{b,f} = f^{-1}(S_b)$, $b \in N$. Assume X carries a pseudoconvex (respectively, pseudoconcave) exhaustion function; *i.e.*, a proper, C^∞ map $\varphi\colon X \to \mathbb{R}$ whose Levi form $L(\varphi) = dd^c\varphi \geq 0$ (respectively, $L(\varphi) \leq 0$) off a compact set. If $\{S_{b,f}\}$ is zero dimensional, suitable growth conditions or geometric properties of f imply that $S_{b,f} \neq \emptyset$ for almost all $S_b \in \mathfrak{A}$ (*e.g.* [3] [5] [6] [7] [14] [20]). If $\{S_{b,f}\}$ has positive dimension, in order to prove the same an additional closed, nonnegative form measuring the volume of $S_{b,f}$ was usually required ([9] [14] [17] [19]). In place of the latter hypothesis, one may assume there is a closed form $\theta \in A_2^{1,1}(X)$ such that outside a compact set, $\theta \geq 0$, $\theta \geq L(\varphi)$ and $\theta^m \not\equiv 0$ ($m = \dim X$). In terms of this θ the Casorati-Weierstrass type theorems can be established even in the case $L(\varphi)$ has eigenvalues of different signs. It is unknown, however, if such a θ exists for a given φ. If φ is strongly logarithmic pseudoconvex (in the sense of Griffiths-King-Stoll [8] [15]), the natural choice of θ is of course $L(\varphi)$. In this case, (under certain conditions) one can prove the equidistribution property: the valence of a generic S_b grows to infinity over suitable sequences of open sets at the same rate as the characteristic of f ([19,4.9]). Taking into account also the 0-convex exhaustion function of Andreotti-Grauert [1], a unified notion of pseudoconvexity which admits equidistribution seems to be of interest. To this end, the g-pseudoconvex, (g,y)-pseudoconvex as well as the g-pseudoconcave exhaustion functions are introduced in Section 1.

The equidistribution theorems are first proved for an admissible family $\mathfrak{A}$ in Y (Section 2). These can be applied to the case of Schubert zeroes of sections in a semi-ample vector bundle over Y (Section 3). The results obtained generalize those of Chern [3, p. 537] [4, 4.8], Cowen [5,7.1], Stoll [15,13.3,13.4] [17,4.6] and Wu [20, pp. 86–88].

1. EXHAUSTION FUNCTION AND G-PSEUDOCONVEXITY

For the basic notations the reader is referred to [19]. All complex spaces are assumed reduced, pure dimensional and countable at infinity. Let X be a complex space of dimension $m > 0$. Let $\varphi\colon X \to \bar{\mathbb{R}}\,[-\infty,\infty)$ be an exhaustion function; *i.e.*,

Received May 15, 1978. Revision received August 1, 1978.
Partially supported by NSF Grant MCS 76-08478.

Michigan Math. J. 26 (1979).

an upper-semicontinuous map such that the sets $X[r] = \{x \in X : \varphi(x) \leq r\}$ are compact for all $r \geq 0$ and φ is C^∞ off some $X[c_0]$. The exhaustion function φ is called *semi-regular* if φ is unbounded and the set of critical points of $\varphi | X_{reg} - X[r_0]$ has measure zero for some $r_0 \geq c_0$. For example, if X is non-compact and there is a compact set $K \subseteq X$ such that either φ has isolated critical points in $X_{reg} - K$ or φ is real analytic on $X_{reg} - K$, then φ is semi-regular.

LEMMA 1.1. *Assume φ is semi-regular. Let $\zeta \in A_0^{2m}(X)$ with $\bar{\zeta} = \zeta$. Assume $h: \mathbb{R}[c_0, \infty) \to \mathbb{R}$ is continuous. Set $h_\varphi = h \circ \varphi$. Then for large $r_0 \geq c_0$, the function $A(r) = \int_{X(r)} \zeta$ is absolutely continuous on $\mathbb{R}[r_0, \infty)$ and*

$$\int_{X[r_0, r)} h_\varphi \zeta = \int_{r_0}^{r} h(t) A'(t)\, dt \qquad (r > r_0). \tag{1.1}$$

Proof. Take a positive form $\chi \in A_0^{m-1,m-1}(X)$. At first assume $\zeta \geq 0$. There is a measurable function $Q: X - X[c'] \to \mathbb{R}[0, \infty)$, for some $c' \geq c_0$, such that

$$\zeta = Q\, d\varphi \wedge d^c\varphi \wedge \chi \qquad \text{almost everywhere on } X - X[c']$$

(*cf.* [12, 5.37]). For $r > r_0 > c'$, Fubini's Theorem implies

$$\int_{X[r_0, r]} \zeta = \int_{r_0}^{r} \left(\int_{dX(t)} Q\, d^c\varphi \wedge \chi \right) dt.$$

This proves that A is absolutely continuous on $\mathbb{R}[r_0, \infty)$. Now the Jordan decomposition of $\zeta | X_{reg}$ yields the general case. The absolute continuity of A implies (1.1).

Assume $g: \mathbb{R}[c_0, \infty) \to \mathbb{R}$ ($c_0 \geq 0$) is of class C^3 with $\|e^g\|_{c_0}^r = \|e^g\|_{L^1[c_0, r]} \to \infty$. Define

$$\xi_g = L(\varphi) + g'_\varphi\, d\varphi \wedge d^c\varphi \qquad \text{on } X - X[c_0]. \tag{1.2}$$

The exhaustion φ of X is called *g-pseudoconvex* (respectively, *g-pseudoconcave*) if and only if $\xi_g \geq 0$ (respectively, $\xi_g \leq 0$) on $U_0 = X_{reg} - X[c_0]$. There exists a closed form $\theta_g^* \in A_2^{1,1}(X)$ such that

$$\theta_g^* | X - X[c_1] = u_\varphi \xi_g \tag{1.3}$$

for some $c_1 \geq c_0$. Here $u = e^g$ is uniquely determined. Without loss of generality assume $c_1 = c_0$. If φ is g-pseudoconvex, define $\theta_g = \theta_g^*$; if φ is g-pseudoconcave, define $\theta_g = -\theta_g^*$. The exhaustion φ is called *strongly g-pseudoconvex* if and only if $\xi_g > 0$ at almost all points of U_0.

The following example shows that the g-pseudoconvexity generalizes the 0-convexity of Andreotti-Grauert [1]. For $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, let $\|z\|^2 = \sum_{1=1}^{n} z_j \bar{z}_j$. Define $\varphi(z) = \log(\log\|z\|^2)$ if $\|z\| > 1$, and $\varphi = -\infty$ otherwise. Assume $n > 1$. Let

$E = \{z \in \mathbb{C}^n : z_1 = \dots = z_{n-1} = 0\}$, $B = \{z \in \mathbb{C}^n : \|z\| \le 1\}$, and $\iota : E \to \mathbb{C}^n$ be the inclusion. Then $\iota^* L(\varphi) < 0$ on $E - B$. With $g(r) = r + e^r$, (1.2) implies

$$\xi_g = \frac{dd^c\|z\|^2}{\|z\|^2 \log\|z\|^2} > 0 \qquad \text{on } \mathbb{C}^n - B.$$

Thus the exhaustion φ of $\mathbb{C}^n$ is (strictly) g-pseudoconvex but not 0-convex.

A complex space X (of dimension m) is said to be *rational relative to* (φ, g) if and only if φ is a g-pseudoconvex exhaustion of X such that $\theta_g^m \not\equiv 0$ on U_0 and

$$A_g(r) = \int_{X(r)} \theta_g^m = 0(1) \qquad (r \to \infty).$$

For instance, if $\pi: X \to \mathbb{C}^p$ is a proper holomorphic map of strict rank m with algebraic image, then X is rational relative to $\varphi = 1 + \|\pi\|^2$, $g(r) = -\log r$.

If $y, z: \mathbb{R}[c_0, \infty) \to \mathbb{R}$ are of class C^1 and if θ_y^*, θ_z^* are defined on $X - X[c_1]$ by (1.3), then for each $q \in \mathbb{Z}[1, m]$,

$$\theta_y^* \wedge (\theta_z^*)^{q-1} = (\theta_v^*)^q \qquad \text{off } X[c_1] \tag{1.4}$$

where $v = (y + (q-1)z)/q$. A g-pseudoconvex exhaustion φ of X is called (g,y)-*pseudoconvex of degree* q if $y: \mathbb{R}[c_0, \infty) \to \mathbb{R}$ is of class C^1 such that

$$\theta_y^* \wedge \theta_g^{q-1} \ge 0 \qquad \text{off a compact set.} \tag{1.5}$$

An exhaustion function φ of X is called c. g-*convex* ([19]) if $g: \mathbb{R}(0, \infty) \to \mathbb{R}$ is increasing of class C^1, if $\xi_{-g} \ge 0$ on $X - X[0]$, and if $X[0]$ has measure zero. If φ is c.g-convex, define $u_1 = e^{-g}$, and

$$\omega_{u_1} = (u_1)_\varphi \xi_{-g} \qquad \text{on } X - X[0].$$

A c.g-convex exhaustion φ is $(0, y_q)$-pseudoconvex of degree q with $y_q = -qg$, for each $q \in \mathbb{Z}[1, m]$; in fact, (1.4) implies

$$\theta_0^{q-1} \wedge \theta_{-qg}^* = \omega_{u_1}^q \ge 0 \qquad \text{on } X - X[c_1]. \tag{1.6}$$

An exhaustion φ of X is called g-*semiparabolic* if φ is a c.g-convex with $\theta_0^m \not\equiv 0$ and if $\omega_{u_1}^m \equiv 0$ off a compact set.

2. EQUIDISTRIBUTION FOR ADMISSIBLE FAMILIES

The following result generalizes the calculus lemma of Wu [20, II, p. 379]; the proof draws on an idea of Dektyarev [6, p. 69].

LEMMA 2.1. *Let* $h, q_j: \mathbb{R}[r_0, \infty) \to \mathbb{R}[0, \infty)$, $j = 1,2$, *where* h *is positive, increasing, absolutely continuous, and* q_j, hq_j *are locally integrable. Let* $E \subseteq \mathbb{R}[r_0, \infty)$

be a set of measure zero. Assume

$$\liminf_{r\to\infty, r\notin E} \|q_1\|_{r_0}^r / \|q_2\|_{r_0}^r = 0. \tag{2.1}$$

Then

$$\liminf_{r\to\infty, r\notin E} \|hq_1\|_{r_0}^r / \|hq_2\|_{r_0}^r = 0.$$

Proof. Assume $\|q_1\|_{r_0}^r \not\equiv 0$ (otherwise the lemma is trivial). Then (2.1) implies that $\|q_2\|_{r_0}^r \to \infty$. Define

$$G(r,c) = \int_{r_0}^{r} (q_1 - cq_2)(t)\, h(t)\, dt \qquad (r > r_0,\ c > 0).$$

Suppose there exist $c > 0$ and $r' \geq r_0$ such that $G(r,c) > 0$ for all $r > r'$, $r \notin E$. Then for such r,

$$\int_{r'}^{r} (q_1 - cq_2)(t)\, dt + O(1) = \frac{G(r,c)}{h(r)} + \int_{r'}^{r} G(t,c) \frac{h'(t)}{h(t)^2}\, dt > 0.$$

This clearly contradicts (2.1).

Throughout this section, the general assumptions (I)–(V) shall be in force. (I) X is a complex space of dimension m with at least one non-compact branch. (II) The family $\mathfrak{A} = \{S_b\}_{b\in N}$ is strictly admissible in a complex space Y and is defined by $Y \overset{\tau}{\leftarrow} M \overset{\pi}{\rightarrow} N$ ([19]); the index set N is a compact, connected complex manifold of dimension $k > 0$. (III) $f: X \rightharpoonup Y$ is a meromorphic map almost adapted to $\mathfrak{A}$ ([19]). Let $s = \operatorname{codim} S_b$, $q = m - s$. (IV) $\varphi: X \to \mathbb{R}\,[-\infty,\infty)$ is an exhaustion function of one of the following types: g-pseudoconvex, g-pseudoconcave, or g-semiparabolic.

Let $'X \subseteq X \times Y$ be the graph of the holomorphic correspondence associated to f ([11]). Let $P: {'X} \to X$, $F: {'X} \to Y$ be the projections. There is a largest open set $X^0 \subseteq X$ such that $P: P^{-1}(X^0) \to X^0$ is bihomorphic. Assume $w: \bar{\mathbb{R}}\,[r_0,\infty] \to \mathbb{R}\,[0,\infty)$, $-\infty \leq r_0 \leq c_0$, is continuous. Take $b \in N_{X[r],f}$, $r > c_0$ ([19, Section 1]). Define the counting function, respectively, valence of S_b ([19, 2.3, 4.1]) for $r \geq t \geq c_0$ by

$$N_f(X(t), S_b, \theta_g^q) = \int_{S_{b,f} \cap {'X(t)}} \nu_F^b({'\theta_g})^q \qquad ({'\theta_g} = P^*\theta_g)$$

$$N_{f,w}(r, c_0, S_b) = \int_{c_0}^{r} N_f(X(t), S_b, \theta_g^q)\, w(t)\, dt.$$

If $\xi \in A_0^{p,p}(Y)$, $0 \leq p \leq m$, and $y: \mathbb{R}\,[r_0,\infty) \to \mathbb{R}$ is of class C^1, define

$$D^p_{f,y}(r,r',\xi) = \int_{X[r',r)} \theta^*_y \wedge \theta_g^{m-p-1} \wedge f^*\xi$$

$$(2.2)\qquad D_f^{p,w}(r,r',\xi) = \int_{X[r',r)} e^{-w_\varphi}\theta_g^{m-p} \wedge f^*\xi \qquad (r > r' \geq r_0)$$

$$D_f^p(r,r',\xi) = D_f^{p,0}(r,r',\xi),$$

where $\theta^*_y \in A_0^{1,1}(X)$ (see (1.2)-(1.3)). If $\eta \in A_0^{p',p'}(N)$ with $p' - k + s = p \in \mathbb{Z}[0,s]$, define $\eta_Y = \tau_*\pi^*\eta$. Let ω_N be a C^∞ volume element on N normalized so that $\int_N \omega_N = 1$. Define $\Omega = (\omega_N)_Y$. For $r > r' \geq c_0$, the integral

$$(2.3)\qquad \begin{aligned} T_{f,w}(r,r',\Omega) &= \int_{r'}^{r} D_f^s(t,-\infty,\Omega)\,w(t)\,dt \\ &= O(1) + \int_{r'}^{r} D_f^s(t,c_0,\Omega)\,w(t)\,dt \quad (r' \text{ fixed}) \end{aligned}$$

exists ([19, 4.1]). $T_{f,w}(r,r',\Omega)$ is called the characteristic of f for $\mathfrak{A}$ in respect to (ω_N,w). By [19, 2.3, 4.1], the Crofton Formula holds:

$$(2.4)\qquad T_{f,w}(r,c_0,\Omega) = \int_N N_{f,w}(r,c_0,S_b)\,\omega_N \qquad (r > c_0).$$

Assume (V) If $q > 0$ and if φ is not g-semiparabolic, either (i) $\theta_g^m \not\equiv 0$ on $X - X[c_0]$ or (ii) $\theta_g^q \wedge f^*\Omega \not\equiv 0$ on $X^0 - X[c_0]$. Observe that if $\theta_g^m \not\equiv 0$ on an open set $U_1 \subseteq U_0$, then $\theta_g > 0$ on U_1. Hence (i) implies (ii) by [19, 2.4]; if $q = 0$, then $f^*\Omega \not\equiv 0$ on $U_0 \cap X^0$ ([19, 2.4]).

Let $\xi \in A_0^{p,p}(Y)$ be nonnegative and $w > 0$. Then w is called *a growth function for* (f,ξ) if and only if there exists a continuous $\alpha: \mathbb{R}[c_0,\infty) \to \mathbb{R}(0,\infty)$ such that $\|\alpha w\|_{c_0}^r \to \infty$ and $D_f^p(r,c_0,\xi)/\alpha(r)$ is increasing in r. If $p = s$, for such w, α [19, 4.3] yields

$$(2.5)\qquad \frac{T_{f,w}(r,c_0,\xi)}{\|\alpha w\|_{c_0}^r} \to \lim_{r\to\infty} \frac{D_f^s(r,c_0,\xi)}{\alpha(r)}.$$

Let $\sigma = \{r_j\}$ be a φ-admissible sequence, that is, a strictly increasing sequence tending to infinity such that each $X(r_j)$ is a Stokes domain ([19, Section 4]). For each $b \in N^{[\sigma]} = \bigcap_{j=1}^{\infty} N_{X[r_j],f}$, $S_{b,F}$ has pure dimension q (if not empty) ([19, Section 1]). Suppose φ is either g-pseudoconvex or g-pseudoconcave. Let

$$\Delta = \Delta(\Omega,\sigma,g,w)$$

be the set of all $b \in N - N^{[\sigma]}$ and all $b \in N^{[\sigma]}$ for which there exists no subsequence $\{r_j'\}$ of σ with $r_j' \to \infty$ such that (in terms of θ_g) $N_{f,w}(r_j', c_0, S_b) \to \infty$ and

$$\lim_{j\to\infty} \frac{N_{f,w}(r_j', c_0, S_b)}{T_{f,w}(r_j', c_0, \Omega)} = 1.$$

The map f is said to have zero $\mathfrak{A}$-defect, $d(f, \mathfrak{A}) = 0$, relative to (Ω,σ,g,w), if the defect set Δ has measure zero.

Take a singular potential $\{\lambda_b\}_{b\in N}$ such that $dd^c\lambda_b = \omega_N$ on $N - \{b\}$ ([19, 2.5]). Let Λ denote the integral average of $\{\lambda_b\}_{b\in N}$; then $\Lambda \in A_0^{k-1,k-1}(N)$ and $\Lambda \geq 0$ ([14, 6.3]).

THEOREM 2.2. *Assume* $w = e^y : \mathbb{R}\,[c_0,\infty) \to \mathbb{R}\,(0,\infty)$ *is a growth function for* (f,Ω) *and one of the following holds:*

(a) φ *is (g,y)-pseudoconvex of degree* $q+1$ (*where* y *is* C^1), *and for* $\eta = \Lambda$ *or for some positive* $\eta \in A_0^{k-1,k-1}(N)$,

$$D_{f,y}^{s-1}(r, c_0, \eta_Y) = o'(T_{f,w}(r, c_0, \Omega)) \qquad (r \to \infty) \tag{2.6}$$

(*Here "o′" means the* o-*relation holds for some* φ-*admissible sequence* σ).

(b) φ *is semi-regular,* $h = g - y$ *is increasing, absolutely continuous, and for some* $\eta \in A_0^{k-1,k-1}(N)$ *as above,*

$$D_f^{s-1,h}(r, c_0, \eta_Y) = o'(T_{f,w}(r, c_0, \Omega)) \qquad (r \to \infty). \tag{2.7}$$

Then $d(f,\mathfrak{A}) = 0$ *relative to* $(\Omega,\sigma,g,\tilde{w})$, *where* $\tilde{w} = w$ *for* (a), $\tilde{w} = u$ *for* (b). (*In the second case,* σ *is a suitable* φ-*admissible sequence.*)

Proof. Since f is almost adapted to $\mathfrak{A}$, the F.M.T. (relative to suitable bumps) ([18, 9.1.5] [19, 2.6]) remains valid for $\chi = \theta_g^q$. Thus [19, 4.4, 4.5] are applicable in the present case. Let $G(r) = \|w\|_{c_0}^r$. Then $L(G_\varphi) = \theta_y^*$ off a compact subset of X. Hence (a), (1.5) and [19, 2.2, 4.5] yield the desired result.

Now assume (b). By (1.1), for $r_0 >> c_0$,

$$D_f^{s-1,h}(r, r_0, \eta_Y) = \int_{r_0}^{r} e^{-h(t)} (D_f^{s-1})'(t, c_0, \eta_Y)\, dt \qquad (r > r_0).$$

Therefore (2.5), (2.7) and Lemma 2.1 imply that

$$D_f^{s-1}(r, c_0, \eta_Y) = o'(T_{f,u}(r, c_0, \Omega)) \qquad (r \to \infty).$$

Then [19, 4.5] concludes the proof.

Remark. In view of (1.6) and (2.5), [19, 4.9] follows from Theorem 2.2 (a) by setting $w = e^{-(q+1)g}$, $\alpha = e^{qg}$ and $\Omega = \Omega_s$.

THEOREM 2.3. *Assume one of the following holds:*

(a) φ *is g-pseudoconcave.*

(b) X *is rational relative to* (φ, g), Y *is compact and* $s = 1$.

(c) φ *is semi-regular,* g*-pseudoconvex with* $g(r) \to \infty$, *and*

$$(D_f^{s-1})'(r, c_0, \eta_Y) = O(D_f^s(r, c_0, \Omega)) \qquad (r \to \infty)$$

for $\eta = \Lambda$ *or some positive* $\eta \in A_0^{k-1,k-1}(N)$.

Let σ *be an arbitrary* φ*-admissible sequence. Then* $d(f, \mathfrak{A}) = 0$ *relative to* (Ω, σ, g, u).

Proof. Let $G(r) = \| e^g \|_{c_0}^r$. Then (a) implies

$$L(G_\varphi) \wedge \theta_g^q = \theta_g^* \wedge \theta_g^q \leq 0 \qquad \text{on } X - X[c_0].$$

Hence [19, 4.4] and (2.5) yield the result. If Y is compact and $s = 1$, Λ_Y is a bounded function on Y. Thus (b) (and also, clearly (c)) implies (2.6) which holds with $y = g$.

A continuous function $\rho: \mathbb{R}[c_0, \infty) \to \mathbb{R}(0, \infty)$ is said to have *weak growth* if ρ is increasing and $\| 1/\rho \|_{c_0}^r \to \infty$ as $r \to \infty$.

COROLLARY 2.4. *Assume* φ *is* g*-pseudoconvex. Assume there exist a constant* $c > 1$ *and a function* ρ *of weak growth such that one of the following holds:*

(a) $$\left[\int_{c_0}^r D_f^{s-1}(t, c_0, \Lambda_Y) \frac{dt}{\rho(t)} \right]^c = O(T_{f,u}(r, c_0, \Omega)).$$

(b) $$\frac{|D_f^{s-1}(r, c_0, \Lambda_Y)|^c}{\rho(r)\, u(r)} = O(D_f^s(r, c_0, \Omega)).$$

Then there exists a φ*-admissible sequence* σ *such that* $d(f, \mathfrak{A}) = 0$ *relative to* (Ω, σ, g, u).

Proof. In view of Theorem 2.3, assume $D_f^{s-1}(r_0, c_0, \Lambda_Y) > 0$ for some $r_0 > c_0$. Let $a(r) = \| 1/\rho \|_{c_0}^r$. Define $\tilde{D}(y) = D_f^{s-1}(a^{-1}(y), c_0, \Lambda_Y)$ for $y > 0$, and

$$Q(y) = \tilde{D}(y)(\| \tilde{D} \|_0^y)^{-c}$$

for $y > a(r_0)$. Condition (a) yields

$$D_f^{s-1}(r, c_0, \Lambda_Y) \leq \text{const.}\, Q(a(r))\, T_{f,u}(r, c_0, \Omega) \qquad (r >> r_0).$$

Since $Q \in L^1[y', \infty)$ for large y', there exists a φ-admissible sequence $\sigma = \{r_j\}$ such that $Q(a(r_j)) \to 0$. Thus (2.6) holds (with $y = g$) for σ. The case of (b) is similar (*cf.* [19, 4.11]).

PROPOSITION 2.5. *Assume* $\omega_{N,1}$, $\omega'_{N,1}$ *are cohomologous fundamental forms of Kähler metrics on* N. *Define* $\Omega_s = (\omega_{N,1}^k)_Y$. *Assume one of the following:*

(a) φ *is* g*-pseudoconcave,* $\sigma = \{r_j\}$ *is an arbitrary* φ*-admissible sequence, and* $w = u$.

(b) $y: \mathbb{R}[c_0, \infty) \to \mathbb{R}$ *is of class* C^1, φ *is* (g,y)*-pseudoconvex of degree* $q + 1$, $w = e^y$ *is a growth function for* (f, Ω_s), *and there exist a positive* $\eta \in A_0^{k-1,k-1}(N)$ *and a* φ*-admissible sequence* $\sigma = \{r_j\}$ *for which* (2.6) *holds* (*with* $\Omega = \Omega_s$). *Then*

$$\lim_{j\to\infty} \frac{T_{f,w}(r_j, c_0, \Omega'_s)}{T_{f,w}(r_j, c_0, \Omega_s)} = 1. \tag{2.8}$$

Proof. There exists a singular potential $\{\lambda_b\}_{b\in N}$ such that $\lambda_b > 0$ and $dd^c\lambda_b = \omega^k_{N,1}$ on $N - \{b\}$ (see [19, 2.5]). The proof of [14, AII, 6.8] shows that $\Lambda > 0$ on N. If φ is g-pseudoconcave, [19, (3.3)] yields

$$D_f^{s-1}(r_j, c_0, \Lambda_Y) = O(1) \qquad (j \to \infty).$$

Now under hypothesis (a) or (b), (2.8) can be proved in the same way as in [19, 4.5(2)].

3. EQUIDISTRIBUTION FOR SCHUBERT ZEROES

Let V denote a complex vector space of dimension $n + 1 > 1$. The Grassmann manifold $G_p(V)$ of projective p-planes has dimension $d(p,n) = (p + 1)(n - p)$. To each symbol $\alpha = (a_0, \ldots, a_p) \in \mathfrak{S}(p,n)$ and flag $b \in \mathbb{F}(\alpha)$, the associated Schubert variety $S(b,\alpha)$ has dimension $|\alpha| = \sum_{j=0}^{p} a_j$. Define $s_\alpha = d(p,n) - |\alpha|$. The set $\mathfrak{S}(\alpha) = \{S(b,\alpha)\}_{b\in F(\alpha)}$ is strictly admissible in $G_p(V)$ relative to the projections $G_p(V) \xleftarrow{\tau} S(\alpha) \xrightarrow{\pi} \mathbb{F}(\alpha)$ ([5] [16]). Let ℓ be a positive definite Hermitian form on V. The associated j^{th} universal Chern form on $G_p(V)$ is denoted by $c_j[p]$ ([16]). Then $\omega_p = c_1[p]$ is the 2-form of the Fubini-Study metric $\kappa_{(p)}$ on $G_p(V)$ induced by ℓ such that

$$D(p,n) = \deg G_p(V) = \int_{G_p(V)} \omega_p^{d(p,n)}.$$

On the flag manifold $\mathbb{F}(\alpha)$ there exists a volume element ω_α invariant under the actions of the unitary group such that $\int_{\mathbb{F}(\alpha)} \omega_\alpha = 1$ ([16, 5.1]). Define $c(\alpha) = \tau_*\pi^*\omega_\alpha$. If $s \in \mathbb{Z}[1, n-p]$, let

$$\alpha_p^s = (n - p - s, n - p, \ldots, n - p) \in \mathfrak{S}(p,n).$$

Define $\mathfrak{D}^s_{p,n} = \mathfrak{S}(\alpha^s_p)$. According to [16, 5.4],

$$c(\alpha_p^s) = c_s[p]. \tag{3.1}$$

Let φ be a strongly g-pseudoconvex exhaustion function of X. Let $U_1 \subseteq U_0$ be the largest open set on which θ_g is positive. Then θ_g defines a Kähler metric, κ_g, on U_1. Let (Y, κ) be an n-dimensional Hermitian manifold with associated 2-form ω. Assume $f: X \rightharpoonup Y$ is meromorphic. Let σ_j, $0 \le j \le m$, be the j^{th}-elementary symmetric function of the (continuous) eigenvalues of $f^*\omega \,|\, U_1 - I_f$. Then

(3.2) $$\sigma_j \theta_g^m = \binom{m}{j} \theta_g^{m-j} \wedge f^* \omega^j \qquad \text{on } U_1 - I_f.$$

Let $\eta(r) = A_g(r) - A_g(c_0)$. The map f is said to be *balanced in codimension* p *with respect to* (κ_g, κ) if and only if there exist $c \in \mathbb{R}(1,2)$ and a continuous, increasing $\beta: \mathbb{R}[r_1, \infty) \to \mathbb{R}(0,\infty)$ $(r_1 \geq c_0)$ such that $\eta\beta/u$ has weak growth and

(3.3) $$\sigma_{p-1}^c = O(\beta_\varphi \sigma_p) \qquad \text{on } U_1 - X[r_1].$$

The map f is said to have *distortion of type* (ρ,p) *with respect to* (κ_g, κ) if and only if $\rho: \mathbb{R}[c_0, \infty) \to \mathbb{R}(0,\infty)$ is increasing, absolutely continuous such that u/ρ^{p-1} is a growth function for (f, ω^p) and

(3.4) $$\sup_{\|v\|_x = 1} \|df(v)\|_{f(x)} = O(\rho_\varphi) \qquad \text{on } U_1 - I_f.$$

(Here v is a tangent vector to U_1 at x.) As an example, let X be an algebraic variety in $\mathbb{C}^p$, $\varphi(z) = 1 + \|z\|^2$ and $g(r) = -\log r$. Let $f: X \to Y$ be holomorphic. If with respect to (κ_g, κ), f is quasi-conformal, then f is balanced in codimension p $(2 \leq p \leq n)$; if df has bounded norm on $U_1 - X[r_1]$, then f has distortion of type $(1,p')$ for $1 \leq p' \leq n$ (*cf.* Wu [20, III], Griffiths [7, AII]).

THEOREM 3.1. *Take* $s \in \mathbb{Z}[1, n-p]$. *Assume* (I) *and let* $f: X \rightleftharpoons G_p(V)$ *be a meromorphic map. Assume every branch of* X *contains a point* $x \notin I_f$ *such that* $\operatorname{codim}_x \Sigma_{b,f} = s$ *for some* $\Sigma_b \in \mathfrak{D}_{p,n}^s$. *Assume one of the following:*

(a) $s = 1$, *and either* (i) X *is rational relative to* (φ,g) *or* (ii) φ *is* g*-semiparabolic.*

(b) φ *is* g*-pseudoconcave and* $\theta_g^m \not\equiv 0$ *on* U_0.

(c) φ *is semi-regular, strongly* g*-pseudoconvex,* X *is rational relative to* (φ,g) *and* f *has distortion of type* (ρ,s) *with respect to* $(\kappa_g, \kappa_{(p)})$.

Set $(\tilde{g}, \tilde{w}) = (g, u)$ *for* (a)-(i), (b), (c), *and* $(\tilde{g}, \tilde{w}) = (0, u_1^{q+1})$ *for* (a)-(ii) (*where* $q = m - s$). *Then* $d(f, \mathfrak{D}_{p,n}^s) = 0$ *relative to* $(c_s[p], \sigma, \tilde{g}, \tilde{w})$ *for every* φ*-admissible sequence* σ.

Proof. By [19, 1.4], f is almost adapted to $\mathfrak{D}_{p,n}^s$. In view of (3.1), (2.5), and Theorems 2.2, 2.3, it remains to prove the case of (c). Let $h = (s-1)\log \operatorname{Max}(1, \rho)$. Then $w = e^{g-h}$ is a growth function for $(f, c_s[p])$. By (3.2), (3.4), if $r > r' \geq c_0$,

$$D_f^{s-1,h}(r, r', \Lambda_{G_p(V)}) \leq \text{const. } D_f^{s-1,h}(r, r', \omega_p^{s-1})$$
$$= \text{const.} \int_{X[r',r)} e^{-h_\varphi} \sigma_{s-1} \theta_g^m = O(1).$$

Therefore Theorem 2.2 concludes the proof.

Remark. The family $\mathfrak{D}_{p,n}^s$ may also be indexed in the admissible sense by $G_{n-p-s}(V)$ (see [16, p. 29] [19, 1.1]). The above conclusion remains valid for this index set. $\mathfrak{D}_{p,n}^1$ is the set of polar divisors in $G_p(V)$ ([4, p. 10]).

Let $W \to Y$ be a semi-ample holomorphic $\mathbb{C}^k$-bundle over a complex space Y ([16]). Assume $\eta: Y \times V \to W$ is a semiamplification with $\dim V = n + 1 > k$. The meromorphic classification map $\varphi_V: Y \rightleftharpoons G_p(V)$ $(p = n - k)$ is holomorphic on Y_∞

where η is ample and $S_W = Y - Y_\infty$ is thin analytic. In terms of a positive definite Hermitian form ℓ on V, a quotient metric is defined on $W | Y_\infty$. To each $\mathfrak{b} \in \mathfrak{S}(p,n)$, a Chern form $c_W(\mathfrak{b}) \in A_\infty^{p_1,p_1}(Y_\infty)$, $p_1 = s_\mathfrak{b}$, is assigned ([16]). The Giambelli's Theorem ([16, 7.5]) implies that

$$c_W(\mathfrak{b}) = \varphi_V^*(c(\mathfrak{b})) \qquad \text{on } Y_\infty. \tag{3.5}$$

Take $\mathfrak{a} \in \mathfrak{S}(p,n)$. Given $b = (E_0, \ldots, E_p) \in \mathbb{F}(\mathfrak{a})$, the Schubert zero set $S_W(b,\mathfrak{a})$ is defined [5] [17]) by

$$S_W(b,\mathfrak{a}) = \bigcap_{j=0}^{p} \{y \in Y: \dim \eta_y(E_j) \le a_j\}.$$

Let X be a complex space of dimension m and $f: X \rightleftharpoons Y$ a meromorphic map. Assume every branch of X contains a point $x \notin I_f$ such that $f(x) \in Y_\infty$ and, for some $b \in \mathbb{F}(\mathfrak{a})$,

$$q = \dim_x S_W(b,\mathfrak{a})_f = m - s_\mathfrak{a} \ge 0.$$

Then $\varphi_V \cdot f | X^0 - S_{W,f}$ extends meromorphically to X ([17, 4.5]); the extended map $\hat{f}$ is almost adapted to $\mathfrak{S}(\mathfrak{a})$ ([19, 1.4]). Assume (I), (IV), and $\dim S_{W,F} \le q - 1$. Let $w: \mathbb{R}[r_0,\infty) \to \mathbb{R}[0,\infty)$, $-\infty \le r_0 \le c_0$, be continuous. In view of [17, 4.5], define the valence of $S_W(b,\mathfrak{a})$ for $b \in \mathbb{F}(\mathfrak{a})_{X[r],\hat{f}}$ (where $r > c_0$) by

$$N_{f,w}^{\mathfrak{a}}(r,c_0,b) = \int_{c_0}^{r} N_{\hat{f}}(X(t), S(b,\mathfrak{a}), \theta_g^q)\, w(t)\, dt.$$

Let $\Delta(\mathfrak{a}) = \{\mathfrak{b} \in \mathfrak{S}(p,n): a_j \le b_j,\ j = 0, \ldots, p,\ |\mathfrak{a}| + 1 = |\mathfrak{b}|\}$, $s = s_\mathfrak{a}$. For $\mathfrak{b} \in \Delta(\mathfrak{a})$, $p_1 = s_\mathfrak{b}$, define $D_f^{p_1,w}(r,r',\mathfrak{b}) = D_f^{p_1,w}(r,r',c_W(\mathfrak{b}))$, etc., as in (2.2). Define

$$T_{f,w}(r,r',\mathfrak{a}) = \int_{r'}^{r} D_f^s(t,-\infty,\mathfrak{a})\, w(t)\, dt \qquad (r > r' \ge c_0).$$

(The existence of the above integrals follow from (3.5) and (2.3)). Then (2.4) and (3.5) yield the Crofton Formula for $\mathfrak{S}_W(\mathfrak{a}) = \{S_W(b,\mathfrak{a})\}_{b \in \mathbb{F}(\mathfrak{a})}$:

$$T_{f,w}(r,c_0,\mathfrak{a}) = \int_{\mathbb{F}(\mathfrak{a})} N_{f,w}^{\mathfrak{a}}(r,c_0,b)\, \omega_\mathfrak{a} \qquad (r > c_0).$$

THEOREM 3.2. *Assume one of the following holds; in* (b)-(d) *assume* $(\theta_g, c_W(\mathfrak{A}))$ *satisfies* (V) *with* $q = m - s$, *and in* (c)-(d) $w = e^y: \mathbb{R}[c_0,\infty) \to \mathbb{R}(0,\infty)$ *is a growth function for* $(f, c_W(\mathfrak{A}))$:

(a) $s = 1$, *and either* (i) X *is rational relative to* (φ, g) *or* (ii) φ *is g-semiparabolic.*

(b) φ *is g-pseudoconcave.*

(c) *φ is semi-regular, g-pseudoconvex, $h = g - y$ is increasing, absolutely continuous, and along some φ-admissible sequence σ,*

$$D_f^{s-1,h}(r,c_0,\mathfrak{b}) = o'(T_{f,w}(r,c_0,\mathfrak{a})) \qquad \text{for each } \mathfrak{b} \in \Delta(\mathfrak{a}).$$

(d) *φ is (g,y)-pseudoconvex of degree $q+1$ (where y is C^1), and along some φ-admissible σ,*

$$D_{f,y}^{s-1}(r,c_0,\mathfrak{b}) = o'(T_{f,w}(r,c_0,\mathfrak{a})) \qquad \text{for each } \mathfrak{b} \in \Delta(\mathfrak{a}).$$

Define $(\tilde{g},\tilde{w})$ for (a)-(c) as in Theorem 3.1; set $(\tilde{g},\tilde{w}) = (g,w)$ for (d). Then $d(f,\mathfrak{S}_W(\mathfrak{a})) = 0$ relative to $(c_W(\mathfrak{a}),\sigma,\tilde{g},\tilde{w})$. (In the cases (a), (b), σ is an arbitrary φ-admissible sequence.)

Proof. As in [17], let $\{\lambda_b\}_{b\in \mathbb{F}(\mathfrak{a})}$ be an invariant singular potential such that $dd^c\lambda_b = \omega_a$ on $\mathbb{F}(\mathfrak{a}) - \{b\}$ ([9]). The associated integral average, Λ_a, is invariant under the actions of the unitary group. It follows from Matsushima's Theorem ([10] [17]) that

$$\tau_*\pi^*\Lambda_a = \sum_{\mathfrak{b}\in\Delta(\mathfrak{a})} \gamma_{\mathfrak{a}\mathfrak{b}}\, c(\mathfrak{b}) \qquad (\gamma_{\mathfrak{a}\mathfrak{b}} = \text{const.})$$

Thus (3.5), (2.5) and Theorems 2.2, 2.3 conclude the proof.

In the next two theorems assume the following:

(A): $L \to Y$ is a holomorphic line bundle over a complex space Y; (V,η) is a semi-amplification of L with $\dim V = n+1 > 1$. Take $p \in \mathbb{Z}[0,n-1]$. Let $\mathfrak{E}_p = \{E_L[b]\}_{b\in G_p(V)}$ ([15]). Assume (I), (IV). Let $f: X \rightleftharpoons Y$ be a meromorphic map. Assume every branch of X contains a point $x \notin I_f$ with $f(x) \in Y_\infty$ such that $q = \dim_x E_L[b]_f = m - p - 1 \geq 0$ for some $b \in G_p(V)$.

Let $c(L,\ell)$ denote the Chern form defined by a quotient metric on $L|Y_\infty$. Let $\Phi: Y \to \mathbb{P}(V^*)$ be the dual classification map associated to (V,η) ([15, p. 28]). Then

$$c(L,\ell) = \Phi^*\omega_0 \qquad \text{on } Y_\infty. \tag{3.6}$$

Let $w: \mathbb{R}[r_0,\infty) \to \mathbb{R}[0,\infty)$, $-\infty \leq r_0 \leq c_0$, be continuous. For $p_1 \in \mathbb{Z}[0,m]$, define $D_f^{p_1,w}(r,r',L) = D_f^{p_1,w}(r,r',c(L,\ell)^{p_1})$, etc., as in (2.2). Assume f is safe of order $p+1$ ([15, p. 38]. Define

$$T_{f,w}^p(r,r',L) = \int_{r'}^{r} D_f^{p+1}(t,-\infty,L)\,w(t)\,dt \qquad (r > r' \geq c_0).$$

Let $\mathfrak{a}_{0,p'}$ be the Schubert family defined by $\mathbb{P}(V^*) \xleftarrow{\tau} \mathbb{F}_{0,p'} \xrightarrow{\pi} G_{p'}(V^*)$, $p' = n - p - 1$. Then $\Phi \circ F$ is almost adapted to $\mathfrak{a}_{0,p'}$; and for a generic $b \in G_p(V)$, $S_{L,F}$ contains no branch of $E_L[b]_F$. Thus, in view of [15, 4.21], for each fixed $r > c_0$, define the valence of $E_L[b]$ by

$$N_{f,w}(r,c_0,E_L[b]) = \int_{c_0}^{r} {}_{\iota}N_{\phi\circ F}('X(t),\ddot{E}[b],'\theta_g^q)\,w(t)\,dt$$

for almost all $b \in G_p(V)$.

According to [14, p. 132], there exist constants $d_{p'j} \geq 0$ $(0 \leq j \leq p+1)$ such that

$$(3.7) \qquad \Omega_{p'j} := (\omega_{p'}^{d(p',n)+j-p-1})_{P(V^*)} = d_{p'j}\omega_0^j$$

where $d_{p',p+1} = D(p',n)$. Therefore (2.4), (3.6), (3.7) and [13, 2.7] yield the Crofton Formula for $\mathfrak{E}_p$:

$$T_{f,w}^p(r,c_0,L) = \frac{1}{D(p,n)} \int_{G_p(V)} N_{f,w}(r,c_0,E_L[b])\omega_p^{d(p,n)} \qquad (r > c_0).$$

Theorems 2.2, 2.3 imply the following:

THEOREM 3.3. *Assume one of the following holds; in* (b)-(d) *assume* $(\theta_g, c(L,\ell)^{p+1})$ *satisfies* (V) *with* $q = m - p - 1$, *and in* (c)-(d)

$$w = e^y : \mathbb{R}[c_0,) \to \mathbb{R}(0,\infty)$$

is a growth function for $(f, c(L,\ell)^{p+1})$:

(a) $p = 0$, *and either* (i) X *is rational relative to* (φ, g) *or* (ii) φ *is g-semiparabolic.*

(b) φ *is g-pseudoconcave.*

(c) φ *is semi-regular, g-pseudoconvex,* $h = g - y$ *is increasing, absolutely continuous, and along some* φ*-admissible sequence* σ,

$$D_f^{p,h}(r,c_0,L) = o'(T_{f,w}^p(r,c_0,L)) \qquad (r \to \infty).$$

(d) φ *is* (g,y)*-pseudoconvex of degree* $q+1$ (*where* y *is* C^1), *and along some* φ*-admissible sequence* σ,

$$D_{f,y}^p(r,c_0,L) = o'(T_{f,w}^p(r,c_0,L)) \qquad (r \to \infty).$$

Define $(\tilde{g},\tilde{w})$ *as in Theorem* 3.2. *Then* $d(f,\mathfrak{E}_p) = 0$ *relative to* $(c(L,\ell)^{p+1}, \sigma, \tilde{g}, \tilde{w})$. (*In the cases* (a), (b), σ *is an arbitrary* φ*-admissible sequence.*)

THEOREM 3.4. *In addition to* (A), *assume* Y *is non-singular,* (V,η) *is an amplification,* $\omega = c(L,\ell) > 0$, *and* $p > 0$. *Let* κ_y *be the Hermitian metric on* Y *defined by* ω. *Assume* φ *is strongly* g*-pseudoconvex, and one of the following holds with respect to* (κ_g, κ_y):

(a) f *is balanced in codimension* $p + 1$.

(b) X *is rational relative to* (φ, g), φ *is semi-regular, and* f *has distortion of type* $(\rho, p+1)$.

Then $d(f, \mathfrak{E}_p) = 0$ *relative to* $(c(L, \ell)^{p+1}, \sigma, g, u)$ *for some* φ*-admissible* σ. (*In the case* (b) *this* σ *is arbitrary.*)

Proof. By (3.2) and (3.3), (a) implies that

$$[D_f^p(r,r',L)]^c \leq \text{const.}\,\eta(r)\beta(r)D_f^{p+1}(r,r',L)$$

for $r > r' >> c_0$. Hence (3.6), (3.7) and Corollary 2.4 yield the desired result. The case of (b) follows from Theorem 3.3 (as in Theorem 3.1).

The above proof yields:

THEOREM 3.5. *Let* (Y, κ) *be an* n*-dimensional compact, Hermitian manifold with* 2*-form* ω. *Assume* (I), (IV), *and* $f: X \rightharpoonup Y$ *is a meromorphic map such that every branch of* X *contains a point* $x \notin I_f$ *with* $\text{rank}_x f = n$. *If* $n > 1$, *assume* φ *is strongly* g*-pseudoconvex and* f *is balanced in codimension* n *with respect to* (κ_g, κ). *If* $n = 1$, *assume* X *is rational relative to* (φ, g). *Then there is a* φ*-admissible* σ *such that relative to* (ω^n, σ, g, u), $d(f, \mathfrak{A}) = 0$ *for the family* $\mathfrak{A}$ *of points in* Y.

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